ON GEOMETRIC REPRESENTATION OF $\mathbb{L}$-HOMOLOGY CLASSES

FRIEDRICH HEGENBARTH AND DUŠAN D. REPOVŠ

Abstract. In this chapter we give a geometric representation of $H_n(B;\mathbb{L})$ classes, where $\mathbb{L}$ is the 4-periodic surgery spectrum, by establishing a relationship between the normal cobordism classes $\mathcal{N}_n^H(B,\partial)$ and the n-th $\mathbb{L}$-homology of B, representing the elements of $H_n(B;\mathbb{L})$ by normal degree one maps with a reference map to B. More precisely, we prove that for every $n \geq 6$ and every finite complex B, there exists a map $\Gamma : H_n(B;\mathbb{L}) \longrightarrow \mathcal{N}_n^H(B,\partial)$.

1. Introduction

Giving a meaning to algebraic objects (e.g., homology classes of generalized homology theories) in terms of geometric objects, is a fundamental task in algebraic topology. The relation between singular homology and classical cobordism theory is a well-known example.

In this paper we shall consider normal degree one maps $X^n \to M^n$ with a reference map $q : M^n \to B$, where X^n is either a generalized or a topological n-manifold, M^n is a topological n-manifold, and B is a finite complex. We shall denote normal cobordism classes of such objects $X^n \to M^n$ with $\mathcal{N}_n^H(B)$ (resp., $\mathcal{N}_n(B)$). We emphasize that $\mathcal{N}_n^H(B)$ (resp., $\mathcal{N}_n(B)$) should not be confused with the structure set in the case when B is a topological n-manifold (or a PD_n-complex). It is easy to see that in our definition,

$$\mathcal{N}_n(B) \cong \Omega_n(B \times G/TOP),$$

where $\Omega_n(\cdot)$ denotes n-dimensional cobordisms.

A *generalized n-manifold X^n (with boundary)* is an n-dimensional Euclidean neighborhood retract (ENR) with the local homology

$$H_*(X^n, X^n \setminus \{x\};\mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\};\mathbb{Z}), \quad \text{for every } x \in X^n.$$

The *boundary* ∂X^n of X^n is defined by

$$\partial X^n = \{x \in X^n : H_n(X^n, X^n \setminus \{x\};\mathbb{Z}) \cong 0\}.$$

Then ∂X^n is a generalized $(n-1)$-manifold without boundary (see Mitchell [5]). Also, X^n (resp. ∂X^n) is an n-dimensional (resp. $(n-1)$-dimensional) Poincaré

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duality space. We shall consider generalized manifolds with boundary as pairs of $(n+1)$-dimensional (resp. n-dimensional) generalized manifolds $(W^{n+1}, \partial W^{n+1})$.

For every $n \geq 6$, QUINN [8] has constructed an obstruction $i(X^n) \in \mathbb{Z}$, which vanishes if there exists a cell-like map $M^n \to X^n$, of M^n onto X^n where M^n is a closed topological n-manifold (for more on cell-like maps see the survey MITCHELL-REPOVŠ [6]). The construction is done locally, in particular $i(U) = i(X^n)$ for every open subset $U \subset X^n$. Because of its nice properties, one considers $I(X^n) = 1 + 8i(X^n)$, which we shall hereafter call the *Quinn index* of X^n. In particular, $I(W^{n+1}) = I(\partial W^{n+1})$, which follows from the product property of $I(X^n)$. For more on this topic see the monograph CAVICCHIOLI-HEGENBARTH-REPOVŠ [2]).

Our main result (to be proved in Section 3) establishes a relationship between $\mathcal{N}_n^H(B)$ and the n-th $\mathbb{L}$-homology of a finite complex B, i.e. $H_n(B;\mathbb{L})$, where $\mathbb{L}$ denotes the 4-periodic surgery spectrum, i.e.

$$\mathbb{L}_0 \cong \mathbb{Z} \times G/TOP$$

(see RANICKI [9]). Namely, we represent the elements of $H_n(B;\mathbb{L})$ by normal degree one maps with a reference map to B, i.e. as the objects of the following type

$$\begin{array}{ccc} X^n & \longrightarrow & N^n \\ & & \Big\downarrow q \\ & & B \end{array} \tag{1.1}$$

where X^n is a generalized n-manifold with boundary and N^n is a topological n-manifold with boundary, and normal cobordism classes of such objects are denoted by $\mathcal{N}_n^H(B,\partial)$ (resp. $\mathcal{N}_n(B,\partial)$).

Theorem 1.1. *For every $n \geq 6$ and every finite complex B, there exists a map*

$$\Gamma : H_n(B;\mathbb{L}) \longrightarrow \mathcal{N}_n^H(B,\partial).$$

2. GEOMETRIC REPRESENTATION OF THE ELEMENTS OF $H_n(B;\mathbb{L})$

Let B be a finite complex, $n \geq 6$, and let $H_n(B;\mathbb{L})$ be the Steenrod homology of B with respect to the 4-periodic surgery spectrum $\mathbb{L}$. The Steenrod homology behaves well on the category of all compact metric spaces - see KAHN-KAMINKER-SCHOCHET [4]. The spectrum $\mathbb{L}$ is algebraically defined. It is an Ω-spectrum, i.e. it is a sequence of Δ-sets $\mathbb{L}_q$, where $\mathbb{L}_q$ is homotopy equivalent to $\Omega\mathbb{L}_{q-1}$ with $\mathbb{L}_0 \cong \mathbb{Z} \times G/TOP$ as Δ-sets. Each $\mathbb{L}_q$ consists of a sequence $\mathbb{L}_q < j >$.

We shall follow RANICKI [9, Chapter 12] to define elements of $H_n(B;\mathbb{L})$. To this end, we have to embed $B \subset S^m$, where m is sufficiently large, and consider the dual complexes of B and S^m, which will be denoted respectively as $\bar{B}$ and Σ^m. However, we shall keep the notation B and S^m also for the duals.

Let (V^m, W^m) be a pair of simplicial complexes homotopy equivalent to the pair $(S^m, S^m \setminus B)$. An element $x \in H_n(B;\mathbb{L})$ is then given by a simplicial map

$$u : (V^m, W^m) \longrightarrow (\mathbb{L}_{n-m}, *),$$

which sends W^m to 0 and satisfies a certain cycle condition. Clearly, u is well-defined by x, up to some equivalence (i.e. boundary) condition. Moreover, if $\sigma \in B$, then

$$u(\sigma) \in \mathbb{L}_{n-m} < m - |\sigma| >,$$

where $|\sigma|$ denotes the dimension of σ, i.e. $m - |\sigma|$ is the dimension of its dual cell $D(\sigma, V^m)$ in V^m.

As described in Nicas [7, pp. 25-26], the Ω-spectrum property implies the equivalence

$$\mathbb{L}_q < j > \longrightarrow (\Omega \mathbb{L}_{q-1}) < j-1 > \cong \mathbb{L}_{q-1} < j+1 > .$$

By iteration, one obtains the equivalence

$$\mathbb{L}_0 < n - |\sigma| > \longrightarrow \mathbb{L}_{n-m} < m - |\sigma| >$$

as Δ-sets. Note that

$$\mathbb{L}_0 < j > \cong \mathbb{Z} \times (G/TOP < j >),$$

where $G/TOP < j >$ denotes the singular complex of j-simplices of G/TOP.

The face maps

$$\partial_0, ..., \partial_j : \mathbb{L}_0 < j > \longrightarrow \mathbb{L}_0 < j-1 >$$

leave the $\mathbb{Z}$-components invariant. Since we are using dual complexes, the face maps can be written as follows

$$\mathbb{L}_{n-m} < m - j > \longrightarrow \mathbb{L}_{n-m} < m - j - 1 > .$$

Lemma 2.1. *Suppose that*

$$u : (V^m, W^m) \longrightarrow (\mathbb{L}_{n-m}, *)$$

represents the element $x \in H_n(B; \mathbb{L})$ *as above, where* $(S^m, S^m \setminus B)$ *is homotopy equivalent to the simplicial model for* (V^m, W^m)*. Then under the equivalence*

$$\mathbb{L}_{n-m} < m - j > \simeq \mathbb{L}_0 < n - j > \cong \mathbb{Z} \times (G/TOP < n - j >),$$

$u(\sigma), u(\tau) \in \mathbb{L}_{n-m}$ *determine the same* $\mathbb{Z}$*-component.*

Proof. The subface $\sigma \prec \tau \in B$ is a certain composition of face maps denoted by ∂, with the dual δ,

$$D(\tau, V^m) \prec D(\sigma, V^m).$$

The assertion of Lemma 2.1 now follows from the commutativity of the following diagram

$$\begin{array}{ccccc} u(\tau) \in \mathbb{L}_{n-m} < m-|\tau| > & \longrightarrow & \mathbb{L}_0 < n-|\tau| > & & \\ \uparrow \delta & & & \searrow^{pr} & \\ & & & & \mathbb{Z} \\ & & & \nearrow_{pr} & \\ u(\sigma) \in \mathbb{L}_{n-m} < m-|\sigma| > & \longrightarrow & \mathbb{L}_0 < n-|\sigma| > & & \end{array} \tag{2.1}$$

□

Corollary 2.2. *Consider any pair of simplices $\sigma, \tau \in B$ of the simplicial complex B such that $\sigma \cap \tau \neq \emptyset$. Then $u(\sigma), u(\tau) \in \mathbb{L}_{n-m}$ determine the same $\mathbb{Z}$-component.* □

Moreover, Lemma 2.1 also implies the following corollary.

Corollary 2.3. *Suppose that*

$$u, u' : (V^m, W^m) \longrightarrow (\mathbb{L}_{n-m}, *)$$

represent the same element $x \in H_n(B; \mathbb{L})$, where $(S^m, S^m \backslash B)$ is homotopy equivalent to the simplicial model for (V^m, W^m). Then for every simplex $\sigma \in B$, the elements $u(\sigma), u'(\sigma) \in \mathbb{L}_{n-m}$ determine the same $\mathbb{Z}$-component.

Proof. By hypothesis, $u \sim u'$, so there is a cobordism map

$$v : \Delta^1 \times (V^m, W^m) \longrightarrow (\mathbb{L}_{n-m}, *),$$

i.e.

$$v(\Delta^1 \times \sigma) \in \mathbb{L}_{n-m} < m - |\sigma| - 1 >,$$

where

$$v(\partial_0 \Delta^1 \times \sigma) = u(\sigma) \quad \text{and} \quad v(\partial_1 \Delta^1 \times \sigma) = u'(\sigma).$$

Therefore under the face maps, we get the following diagram
(2.2)

$$\begin{array}{ccccc} \mathbb{L}_{n-m} < m-|\sigma| > & & & & \\ & \searrow^{\delta_0} & & & \\ & & \mathbb{L}_{n-m} < m-|\Delta^1 \times \sigma| > & \longrightarrow & \mathbb{L}_0 < n-|\Delta^1 \times \sigma| > \\ & \nearrow_{\delta_1} & & & \downarrow pr \\ \mathbb{L}_{n-m} < m-|\sigma| > & & & & \mathbb{Z} \end{array}$$

which proves the assertion of Corollary 2.3. □

3. Proof of Theorem 1.1

It follows from Lemma 2.1 and Corollaries 2.2 and 2.3 that the $\mathbb{Z}$-components depend only on $x \in H_n(B;\mathbb{L})$, and that $u(\sigma), u(\sigma') \in \mathbb{L}_{n-m}$ define the same element if σ and σ' can be connected by a chain $\sigma_1, ..., \sigma_r$ of simplices such that

$$\sigma_j \cap \sigma_{j+1} \neq \emptyset, \text{ for every } j \in \{1, ..., r-1\}.$$

This leads to the following theorem.

Theorem 3.1. *Suppose that B is connected. Then the construction described above defines a map*

$$I : H_n(B;\mathbb{L}) \longrightarrow L_0(\mathbb{Z}) = 1 + 8\mathbb{Z}.$$

Remark 3.2. *The $\mathbb{Z}$-component coming from the identification*

$$\mathbb{L}_0 \simeq \mathbb{Z} \times G/TOP$$

is the 0-dimensional signature invariant defind by Quinn [8, Section 3.2].

The second component of the identification

$$\mathbb{L}_{n-m} < m-|\sigma| > \simeq \ \mathbb{L}_0 < n-|\sigma| > \simeq \ \mathbb{L}_0(\mathbb{Z}) \times (G/TOP < n-|\sigma| >)$$

associates to $x \in H_n(B;\mathbb{L})$, represented by the map, where (V^m, W^m) is a pair of simplicial complexes homotopy equivalent to the pair $(S^m, S^m \setminus B)$,

$$u : (V^m, W^m) \longrightarrow (\mathbb{L}_{n-m}, *),$$

a family of adic normal degree one maps given by $u(\sigma)$

(3.1)

$$\begin{array}{ccccccc} & & & & & & V^m \\ & & & & & \overset{incl.}{\nearrow} & \uparrow incl. \\ M^n_\sigma & \longrightarrow & N^{n-|\sigma|}_\sigma & \longrightarrow & D(\sigma, V^m) & & \\ & & & & & \searrow & \\ & & & & & & B \end{array}$$

therefore it defines a map

$$\Gamma : H_n(B; \mathbb{L}) \longrightarrow \mathcal{N}^H_n(B, \partial).$$

Similarly, one obtains a map

$$\Gamma^+ : H_n(B; \mathbb{L}^+) \longrightarrow \mathcal{N}_n(B, \partial).$$

Clearly, the following diagram commutes

(3.2)

$$\begin{array}{ccc} H_n(B; \mathbb{L}^+) & \xrightarrow{\Gamma^+} & \mathcal{N}_n(B, \partial) \\ \downarrow & & \downarrow \\ H_n(B; \mathbb{L}) & \xrightarrow{\Gamma} & \mathcal{N}^H_n(B, \partial) \end{array}$$

Summary 3.3. *For every $n \geq 6$, one can construct the maps*

$$I : H_n(B; \mathbb{L}) \longrightarrow L_0(\mathbb{Z}) = 1 + 8\mathbb{Z}$$

and

$$\Gamma : H_n(B; \mathbb{L}) \longrightarrow \mathcal{N}^H_n(B, \partial)$$

via the simplicial equivalence

$$\mathbb{L}_{n-m} < m - j > \simeq \ \mathbb{L}_0 < n - j > \simeq \ \mathbb{L}_0(\mathbb{Z}) \times (G/TOP < n - j >).$$

This completes the proof of Theorem 1.1. □

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Dipartimento di Matematica "Federigo Enriques", Università degli studi di Milano, 20133 Milano, Italy

Email address: friedrich.hegenbarth@unimi.it

Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

Email address: dusan.repovs@guest.arnes.si