Алгебра и анализ
Том 35 (2023), 5

NORMALIZED GROUND STATE SOLUTIONS FOR THE FRACTIONAL SOBOLEV CRITICAL NLSE WITH AN EXTRA MASS SUPERCRITICAL NONLINEARITY

© JIABIN ZUO, YUYOU ZHONG, DUŠAN D. REPOVŠ

This paper is concerned with the existence of normalized ground state solutions for the mass supercritical fractional nonlinear Schrödinger equation involving a critical growth in the fractional Sobolev sense. The compactness of Palais–Smale sequences will be obtained by a special technique, which borrows from the ideas of Soave (J. Funct. Anal. 279 (6) (2020), art. 1086102020). This paper represents an extension of previously known results, both in the local and the nonlocal cases.

§1. Introduction

This paper is devoted to the following fractional Sobolev critical nonlinear Schrödinger equation (NLSE) in $\mathbb{R}^N$ $(N \geqslant 2)$:

$$\begin{cases} (-\Delta)^s u = \mu u + |u|^{2_s^*-2}u + \eta |u|^{p-2}u, \\ \|u\|_{L^2}^2 = m^2, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $\mu \in \mathbb{R}$ is an unknown real number (which will appear as a Lagrange multiplier), 2_s^* is the fractional Sobolev critical exponent, $\eta > 0$, $p \in (2 + \frac{4s}{N}, 2_s^*)$, $m > 0$ is a finite parameter, and $(-\Delta)^s$ is the fractional Laplace operator defined by

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where $C(N,s)$ is the dimensional constant, which depends on N and s (for more details we refer the interested reader to Di Nezza et al. [6]).

The fractional Schrödinger equation originated from Laskin's paper [10], and in recent years, the interest in its study has grown considerably. It is worthwhile and very interesting to look for normalized solutions to such equations that have a prescribed L^2-norm, because they represent the physical view of the conservation of mass.

Ключевые слова: normalized solutions; fractional Schrödinger equation; mass supercritical; Sobolev critical.

The existence and properties of normalized solutions for certain problems strongly depend on the behavior of the combined nonlinearity $g(u) = |u|^{q-2}u + \eta|u|^{p-2}u$, where $2 < p < q < 2_s^*$. Zhang et al. (see [13]) investigated a class of Sobolev subcritical fractional NLSEs where the parameters p and q are in different order, and they obtained several interesting results concerning the existence of normalized solutions. In particular, when the degree of nonlinearity $g(u)$ exceeds the mass critical index $2 + 4s/N$, the functional turns out to be unbounded from below, which makes it impossible to adopt the direct variational method.

Secchi and Appolloni [2] studied the existence and multiplicity of ground state normalized solutions for the fractional mass supercritical NLSEs by using the min-max theory under more general assumptions, but in the Sobolev subcritical sense. Under certain conditions on the potential, Peng et al. (see [14]) showed that NLSE has at least one normalized solution, with the help of a new min-max argument and the splitting lemma for nonlocal version also in the case when the mass is supercritical and Sobolev subcritical.

However, to the best of our knowledge, there are very few papers on normalized solutions of fractional NLSEs. Moreover, they only consider the Sobolev subcritical case. Therefore it is natural to inquire what difficulties appear when a fractional Sobolev critical nonlinearity is considered. For example, Zhen and Zhang [18] investigated a critical fractional NLSE with an L^2-supercritical perturbation, but their coefficient of perturbation was not allowed to be sufficiently large. While in the present paper, η can be large because $\eta \in [\eta^*, +\infty)$. Very recently, almost at the time of our study, Li and Zou [11] considered the same problem using the concentration compactness principle for overcoming the lack of compactness. With the above methods and techniques, Zuo and Rădulescu [20] investigated the existence and nonexistence of normalized solutions for a class of fractional mass supercritical nonlinear Schrödinger coupled systems with Sobolev critical nonlinearities. In response to this difficulty, we consider a technical analysis method combined with the Brézis-Lieb lemma, which comes from the ideas of Soave [16].

Of course, in the case when $s \to 1$, the fractional Laplacian $(-\Delta)^s$ reduces to the classical Laplace operator $-\Delta$, the literature on the relevant problem (1.1) is very large. Here we shall only mention some key papers, which are relevant to our study. Brézis and Nirenberg [4] presented a pioneering work. Later, many researchers made important progress in this field. For L^2-supercritical perturbation $\eta|u|^{p-2}u$, Soave [16] made the first contribution concerning the existence of normalized solutions for NLSEs in the Sobolev critical case. Next, Alves et al. in [1] obtained a similar result for this kind of NLSEs when dimension N is at least 5, and η is sufficiently large. In particular, under weaker, more general conditions, Jeanjean and Lu (see [9]) proved the existence of ground states and established the asymptotic behavior of the ground state energy with mass change. They also obtained infinitely many radial solutions when $N \geqslant 2$, and established the existence and multiplicity of nonradial sign-changing solutions for every $N \geqslant 4$.

Inspired by the work mentioned above, in this paper we consider the problem of existence for ground state normalized solutions of fractional Sobolev critical NLSEs with a mass supercritical nonlinearity. Compactness can be restored by combining some of the main ideas of Brézis and Nirenberg [4] and Jeanjean [8]. In order to introduce the main result of this paper, we first define a fractional Sobolev space:

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid [u]^2_{H^s} = \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty \right\},$$

in which the norm is defined by

$$\|u\| = (\|u\|^2_{L^2} + [u]^2_{H^s})^{\frac{1}{2}}.$$

For convenience, we shall simply denote the norm of the Lebesgue space $L^p(\mathbb{R}^N)$ by $\|u\|_p$ for $p \in [1, \infty)$. A standard method for investigating problem (1.1) is to find critical points of the following energy functional:

$$I_\eta(u) = \frac{1}{2}[u]^2_{H^s} - \frac{1}{2^*_s}\|u\|^{2^*_s}_{2^*_s} - \frac{\eta}{p}\|u\|^p_p,$$

restricted to the set

$$S(m) = \left\{ u \in H^s(\mathbb{R}^N) \middle| \|u\|^2_2 = m^2 \right\}.$$

Obviously, I_η is of class C^1 in $H^s(\mathbb{R}^N)$.

Now, we can state our main result.

Theorem 1.1. *Assume that $N \geqslant 2$ and $p \in (2 + \frac{4s}{N}, 2^*_s)$. Then for every $m > 0$, there exists $\eta^* = \eta^*(m) > 0$ such that for every $\eta \geqslant \eta^*$, problem* (1.1) *admits a radial normalized solution $\widetilde{u}$ whose associated Lagrange multiplier μ is negative.*

Remark 1.1. Our conclusion can be regarded as an extension of Alves et al. [1, Theorem 1.1].

Remark 1.2. According to Theorem 1.1 and Zhen and Zhang [18, Theorem 1.3:(1)–(2)], we know that the existence of a radial normalized ground state is possible when η is sufficiently small or sufficiently large, however it remains an open problem for the rest of the range of η.

Remark 1.3. If $N = 4s$ or $\frac{p}{p-1}2s < N < 4s$ or $2s < N < \frac{p}{p-1}2s$, we can get our conclusion without any restriction on η, see Zhen and Zhang [18, heorem 1.3(3)]. If we further assume that $N^2 > 8s^2$, then we can also arrive at a similar conclusion, but in this case we view the mass m as the parameter instead of η, see Zhang and Han [17, Theorem 1.3].

The paper is organized as follows. §2 contains the proofs of some important lemmas, which play a key role in the proof of the compactness condition. In §3, we prove the strong convergence of the Palais-Smale sequence at some level set, using a special technique. In §4, we prove Theorem 1.1.

§2. Main lemmas

Although the study of normalized solutions is convenient for applications, it also presents some difficulties. For example, the Nehari manifold method cannot be used because the constant μ is unknown. This also makes it difficult to verify the boundedness of Palais–Smale sequences by employing some common methods.

To this end, following Soave [16], let

$$\zeta_p = (Np - 2N)/2ps, \text{ for every } p \in (2, 2_s^*], \tag{2.1}$$

(it is easy to see that $\zeta_p \in (0, 1])$ and define the Pohozaev manifold

$$\mathcal{P}_{\eta,m} = \{u \in S(m) | P_\eta(u) = 0\}, \text{ where } P_\eta(u) = [u]_{H^s}^2 - \|u\|_{2_s^*}^{2_s^*} - \eta\zeta_p\|u\|_p^p,$$

where the definition of ζ_p is related to (2.4). It is well known that any critical point of $I_\eta|_{S(m)}$ stays in $\mathcal{P}_{\eta,m}$, as a consequence of Zhen and Zhang [18, Proposition 2.1 and Remark 2.1].

In order to get the mountain pass geometry, we are going to introduce a scaling transformation. For $u \in S(m)$ and $\xi \in \mathbb{R}$, we let

$$(\xi \star u)(x) = e^{\frac{N\xi}{2}} u(e^\xi x) = v(x), \text{ for a.e. } x \in \mathbb{R}^N,$$

which is based on a very interesting idea from Jeanjean [8]. A careful analysis shows that the transformed functional $\widetilde{I}_\eta = I_\eta(\xi \star u)$ has the same mountain pass geometry and mountain pass level as the original functional $I_\eta(u)$.

For the reader's convenience, we give the proof of the following lemma, which can also be found in Li and Zou [11].

Lemma 2.1. *Assume that* $u \in S(m)$ *is arbitrary but fixed. Then we have*:
(1) $[\xi \star u]_{H^s}^2 \to 0$ *and* $I_\eta(\xi \star u) \to 0$, *as* $\xi \to -\infty$;
(2) $[\xi \star u]_{H^s}^2 \to +\infty$ *and* $I_\eta(\xi \star u) \to -\infty$, *as* $\xi \to +\infty$.

Proof. By direct calculation, we get

$$\begin{aligned}
&[\xi \star u]_{H^s}^2 = e^{2\xi s} \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy = e^{2\xi s}[u]_{H^s}^2, \\
&\|\xi \star u\|_\beta^\beta = e^{\frac{(\beta-2)N\xi}{2}} \|u\|_\beta^\beta, \text{ for every } \beta \geqslant 2.
\end{aligned} \tag{2.2}$$

On the basis of (2.2), we have

$$[\xi \star u]_{H^s}^2 \to 0, \quad \text{as } \xi \to -\infty,$$
$$I_\eta(\xi \star u) = \frac{1}{2}[\xi \star u]_{H^s}^2 - \frac{1}{2_s^*}\|\xi \star u\|_{2_s^*}^{2_s^*} - \frac{\eta}{p}\|\xi \star u\|_p^p \to 0, \quad \text{as } \xi \to -\infty,$$

thereby demonstrating (1).

On the other hand, from (2.2) it follows that $[\xi \star u]_{H^s}^2 \to +\infty$, as $\xi \to +\infty$. Moreover

$$I_\eta(\xi \star u) = \frac{1}{2}e^{2\xi s}[u]_{H^s}^2 - \frac{1}{2_s^*}e^{\frac{(2_s^*-2)N\xi}{2}}\|u\|_{2_s^*}^{2_s^*} - \frac{\eta}{p}e^{\frac{(p-2)N\xi}{2}}\|u\|_p^p \to -\infty,$$

as $\xi \to +\infty$, because $p \in (2 + \frac{4s}{N}, 2_s^*)$, which in turn demonstrates (2). This completes the proof of Lemma 2.1. □

The following two inequalities (the fractional Sobolev inequality (2.3) and the fractional Gagliardo–Nirenberg inequality (2.4)) play an important role in our proof of the main result in §4.

Thanks to Servadei and Valdinoci [15], there exists a optimal fractional critical Sobolev constant $\mathcal{S} > 0$ such that

$$\mathcal{S}\|u\|_{2_s^*}^2 \leqslant [u]_{H^s}^2, \text{ for every } u \in H^s(\mathbb{R}^N). \tag{2.3}$$

Also, according to Frank et al. [7], there exists an optimal constant $C(N,p,s)$ such that for every $p \in (2, 2_s^*)$, we have

$$\|u\|_p^p \leqslant C^p(N,p,s)[u]_{H^s}^{p\zeta_p}\|u\|_2^{p(1-\zeta_p)} \text{ for every } u \in H^s(\mathbb{R}^N), \tag{2.4}$$

where ζ_p is given by (2.1).

In the following lemma, we give a specific value of $\rho(m,\eta)$ and analyze the asymptotic behavior of $\rho(m,\eta)$ when η is sufficiently large, which is more detailed than in Luo and Zhang [13, Lemma 5.2].

Let

$$S_r(m) = S(m) \bigcap H_{\text{rad}}^s(\mathbb{R}^N) = \{u \in S(m) : u(x) = u(|x|)\}.$$

Lemma 2.2. *There exists a sufficiently small $\rho(m,\eta) > 0$ such that*

$$0 < \inf_{u \in X} I_\eta(u) \leqslant \sup_{u \in X} I_\eta(u) < \inf_{u \in Y} I_\eta(u),$$

with

$$X = \left\{u \in S_r(m), [u]_{H^s}^2 \leqslant \rho(m,\eta)\right\}, \ Y = \left\{u \in S_r(m), [u]_{H^s}^2 = 2\rho(m,\eta)\right\}.$$

Moreover, $\rho(m,\eta) \to 0$ *as* $\eta \to \infty$.

Proof. In view of (2.3) and (2.4), we get

$$\frac{1}{2_s^*}\|v\|_{2_s^*}^{2_s^*} + \frac{\eta}{p}\|v\|_p^p \leqslant \frac{1}{2_s^*\mathcal{S}^{\frac{2_s^*}{2}}}([v]_{H^s}^2)^{\frac{2_s^*}{2}} + \frac{\eta C^p(N,p,s)}{p}([v]_{H^s}^2)^{\frac{Np-2N}{4s}} m^{\frac{2sp-Np+2N}{2s}}. \tag{2.5}$$

Then for every $u \in S_r(m)$, fixing $[u]_{H^s}^2 \leqslant \rho(m,\eta)$ and $[v]_{H^s}^2 = 2\rho(m,\eta)$, where $\rho(m,\eta)$ is a positive number that depends on m and η, one has

$$\begin{aligned} I_\eta(v) - I_\eta(u) &\geqslant \frac{1}{2}[v]_{H^s}^2 - \frac{1}{2}[u]_{H^s}^2 - \frac{1}{2_s^*}\|v\|_{2_s^*}^{2_s^*} - \frac{\eta}{p}\|v\|_p^p \\ &\geqslant \frac{1}{2}\rho(m,\eta) - \frac{2^{\frac{2_s^*}{2}}}{2_s^*\mathcal{S}^{\frac{2_s^*}{2}}}(\rho(m,\eta))^{\frac{2_s^*}{2}} \\ &\quad - \frac{\eta C^p(N,p,s)}{p} 2^{\frac{Np-2N}{4s}}(\rho(m,\eta))^{\frac{Np-2N}{4s}} m^{\frac{2sp-Np+2N}{2s}}. \end{aligned}$$

Thus, choosing

$$\begin{aligned} &\rho(m,\eta) \\ &= \min\left\{\left(\frac{p}{8\eta C^p(N,p,s)2^{\frac{Np-2N}{4s}} m^{\frac{2sp-Np+2N}{2s}}}\right)^{\frac{4s}{Np-2N-4s}}, \left(\frac{2_s^*}{8}\right)^{\frac{N-2s}{2s}}\left(\frac{\mathcal{S}}{2}\right)^{\frac{N}{2s}}\right\}, \end{aligned} \tag{2.6}$$

we can deduce that

$$\begin{aligned} &\frac{1}{2}\rho(m,\eta) - \frac{2^{\frac{2_s^*}{2}}}{2_s^*\mathcal{S}^{\frac{2_s^*}{2}}}(\rho(m,\eta))^{\frac{2_s^*}{2}} \\ &\qquad - \frac{\eta C^p(N,p,s)}{p} 2^{\frac{Np-2N}{4s}}(\rho(m,\eta))^{\frac{Np-2N}{4s}} m^{\frac{2sp-Np+2N}{2s}} > 0. \end{aligned}$$

Now, by (2.5) and the definition of $\rho(m,\eta)$ in (2.6), we get

$$I_\eta(u) \geqslant \frac{1}{2}[u]_{H^s}^2 - \frac{2^{\frac{2_s^*}{2}}}{2_s^*\mathcal{S}^{\frac{2_s^*}{2}}}[u]_{H^s}^{2_s^*} - \frac{\eta C^p(N,p,s)}{p} 2^{\frac{Np-2N}{4s}} m^{\frac{2sp-Np+2N}{2s}} [u]_{H^s}^{\frac{Np-2N}{2s}} > 0,$$

which means that the inequality in Lemma 2.2 holds true. Finally, the relation $\lim_{\eta\to\infty}\rho(m,\eta) = 0$ also follows from (2.6). This completes the proof of Lemma 2.2. □

Next, fix $u_0 \in S_r(m)$. It follows from Lemma 2.1 and Lemma 2.2 that there exist numbers $\xi_1 = \xi_1(m, \eta, u_0) < 0$ and $\xi_2 = \xi_2(m, \eta, u_0) > 0$ such that the functions $u_{1,\eta} = \xi_1 \star u_0, u_{2,\eta} = \xi_2 \star u_0$ satisfy

$$[u_{1,\eta}]^2_{H^s} < \frac{\rho(m,\eta)}{2}, [u_{2,\eta}]^2_{H^s} > 2\rho(m,\eta), \ I_\eta(u_{1,\eta}) > 0, \text{ and } \quad I_\eta(u_{2,\eta}) < 0.$$

Now, in a way similar to the discussion in Jeanjean [8] or Luo and Zhang [13, Proposition 5.3], we fix the minimax

$$E_\eta(m) = \inf_{\psi\in\Gamma} \max_{t\in(0,1]} I_\eta(\psi(t)),$$

where

$$\Gamma = \{\psi \in C([0,1], S_r(m)) : [\psi(0)]^2_{H^s} < \rho(m,\eta)/2, \ I_\eta(\psi(1)) < 0\}.$$

By virtue of Lemma 2.2, we know that

$$[\psi(1)]^2_{H^s} > \rho(m,\eta), \text{ for every } \psi \in \Gamma.$$

Therefore there exists $t_0 \in (0,1)$ such that

$$[\psi(t_0)]^2_{H^s} = \rho(m,\eta)/2 \text{ and } \max_{t\in[0,1]} I_\eta(\psi(t)) \geqslant I_\eta(\psi(t_0)) \geqslant \inf_{u\in X} I_\eta(u) > 0,$$

therefore $E_\eta(m) > 0$.

The following lemma is a key step to analyze the level value of the mountain pass, so we present a more detailed calculation process (in comparison with Li and Zou [11]).

Lemma 2.3. $\lim\limits_{\eta\to\infty} E_\eta(m) = 0$.

Proof. Fix $u_0 \in S_r(m)$, and consider the path $\psi_0(t) = [(1-t)\xi_1 + t\xi_2]\star u_0 \in \Gamma$. We have

$$E_\eta(m) \leqslant \max_{t\in[0,1]} I_\eta(\psi_0(t)) \leqslant \max_{r\geqslant 0} \left\{ \frac{1}{2} r^2 [u_0]^2_{H^s} - \frac{\eta}{p} r^{\frac{Np-2N}{2s}} \|u_0\|_p^p \right\}.$$

Thus, setting $C_1 = [u_0]^2_{H^s}$ and $C_2 = \|u_0\|_p^p$, we consider the maximum value of the following function

$$f(r) = \frac{1}{2} C_1 r^2 - \frac{\eta}{p} C_2 r^{\frac{Np-2N}{2s}}, \text{ for any } r \geqslant 0.$$

Letting

$$f'(r) = C_1 r - \left(\frac{Np-2N}{2s}\right) \frac{\eta}{p} C_2 r^{\frac{Np-2N-2s}{2s}} = 0,$$

we get the maximum of $f(r)$ at

$$r_{\max} = \left(\frac{2spC_1}{(Np-2N)\eta C_2}\right)^{\frac{2s}{Np-2N-4s}}.$$

Hence,

$$\begin{aligned}\max_{r\geqslant 0}&\left\{\frac{1}{2}r^2[u_0]^2_{H^s}-\frac{\eta}{p}r^{\frac{Np-2N}{2s}}\|u_0\|_p^p\right\}\\&=\frac{1}{2}\left(\frac{2spC_1}{(Np-2N)\eta C_2}\right)^{\frac{4s}{Np-2N-4s}}C_1-\frac{\eta}{p}\left(\frac{2spC_1}{(Np-2N)\eta C_2}\right)^{\frac{Np-2N}{Np-2N-4s}}C_2\\&\leqslant\frac{1}{2}\left(\frac{2spC_1}{(Np-2N)\eta C_2}\right)^{\frac{4s}{Np-2N-4s}}C_1,\end{aligned}$$

so there exists $C>0$ that is independent of $\eta>0$ such that

$$E_\eta(m)\leqslant C\left(\frac{1}{\eta}\right)^{\frac{4s}{Np-2N-4s}}\to 0 \text{ as } \eta\to\infty,$$

because $p>2+4s/N$. This completes the proof of Lemma 2.3. □

In accordance with the pattern of Luo and Zhang [13, Propositions 5.3–5.4], for $\{\xi_n\}\subset\mathbb{R}$ we know that $I_\eta(u_n)$ and $I_\eta(\xi_n\star u_n)$ have the same mountain pass level value. Moreover, there is a certain relationship between their Palais–Smale sequences.

Lemma 2.4. *Let $\{\xi_n\star u_n\}\subset S_r(m)$ be a Palais–Smale sequence for I_η at the level $E_\eta(m)$, i.e.,*

$$I_\eta(\xi_n\star u_n)\to E_\eta(m)>0 \text{ and } I'_\eta(\xi_n\star u_n)\to 0 \text{ as } n\to\infty.$$

Then

$$\lim_{n\to\infty}P_\eta(\xi_n\star u_n)=0.$$

Proof. We first have

$$\begin{aligned}I_\eta(\xi_n\star u_n)&=\frac{1}{2}[\xi_n\star u_n]^2_{H^s}-\frac{1}{2_s^*}\|\xi_n\star u_n\|_{2_s^*}^{2_s^*}-\frac{\eta}{p}\|\xi_n\star u_n\|_p^p\\&=\frac{1}{2}e^{2\xi_n s}[u_n]^2_{H^s}-\frac{1}{2_s^*}e^{\frac{(2_s^*-2)N\xi_n}{2}}\|u_n\|_{2_s^*}^{2_s^*}-\frac{\eta}{p}e^{\frac{(p-2)N\xi_n}{2}}\|u_n\|_p^p,\end{aligned}$$

and $I_\eta(\xi_n\star u_n)$ is C^1 with respect to ξ_n. Now, by taking the derivative

$$\frac{\partial}{\partial\xi_n}I_\eta(\xi_n\star u_n)=2se^{2\xi_n s}[u_n]^2_{H^s}-se^{\frac{(2_s^*-2)N\xi_n}{2}}\|u_n\|_{2_s^*}^{2_s^*}-s\eta\zeta_p e^{\frac{(p-2)N\xi_n}{2}}\|u_n\|_p^p,$$

and observing that

$$P_\eta(\xi_n\star u_n)=2e^{2\xi_n s}[u_n]^2_{H^s}-e^{\frac{(2_s^*-2)N\xi_n}{2}}\|u_n\|_{2_s^*}^{2_s^*}-\eta\zeta_p e^{\frac{(p-2)N\xi_n}{2}}\|u_n\|_p^p,$$

we see that

$$\frac{\partial}{\partial\xi_n}I_\eta(\xi_n\star u_n)=sP_\eta(\xi_n\star u_n).$$

Thus, the conclusion of Lemma 2.4 is a consequence of the following limit

$$\lim_{n\to\infty} \frac{\partial}{\partial \xi_n} I_\eta(\xi_n \star u_n) = 0,$$

because $I_\eta'(\xi_n \star u_n) \to 0$ as $n \to \infty$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let $\{u_n\} \subset S_r(m)$ be a Palais–Smale sequence for I_η with the level $E_\eta(m)$. If $\lim\limits_{n\to\infty} P_\eta(u_n) = 0$, then $\{u_n\}$ is bounded in $S_r(m)$.*

Proof. We note that $\zeta_p p > 2$, because $p > 2 + 4s/N$. From the relation $\lim\limits_{n\to\infty} P_\eta(u_n) = 0$, it follows that

$$I_\eta(u_n) = \frac{\eta}{2p}(\zeta_p p - 2)\|u_n\|_p^p + \frac{s}{N}\|u_n\|_{2_s^*}^{2_s^*} + o(1),$$

and from the boundedness of $I_\eta(u_n)$, it follows that $\{\|u_n\|_p^p\}$ and $\{\|u_n\|_{2_s^*}^{2_s^*}\}$ are both bounded, therefore $\{[u_n]_{H^s}^2\}$ is bounded. This completes the proof of Lemma 2.5. □

§3. Compactness condition

In this section we give a very important proof of the compactness conditions, inspired by the ideas of Soave [16].

Proposition 3.1. *Let $\{u_n\} \subset S_r(m)$ be a Palais–Smale sequence for I_η with the level*

$$0 < E_\eta(m) < \frac{s\mathcal{S}^{\frac{N}{2s}}}{N},$$

where $\mathcal{S}$ is the best fractional Sobolev constant defined in (2.3). *If $\lim\limits_{n\to\infty} P_\eta(u_n) =$ 0, then one of the following properties holds*:

(1) *either up to a subsequence, $u_n \rightharpoonup \widetilde{u}$ converges weakly in $H^s(\mathbb{R}^N)$ but not strongly, where $\widetilde{u} \not\equiv 0$ is a solution of the first equation of* (1.1) *for some $\mu < 0$, and*

$$I_\eta(\widetilde{u}) < E_\eta(m) - \frac{s\mathcal{S}^{\frac{N}{2s}}}{N};$$

(2) *or up to a subsequence, $u_n \to \widetilde{u}$ converges strongly in $H^s(\mathbb{R}^N)$, $I_\eta(\widetilde{u}) = E_\eta(m)$, and $\widetilde{u}$ is a solution of* (1.1) *for some $\mu < 0$.*

Proof. In general, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is not compact for any $p \in (2, 2_s^*)$, so we need to restore compactness in the radial function space. According to Lemma 2.5, we know that the sequence $\{u_u\}$ is bounded and the embedding $H^s_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2_s^*)$ (see Lions [12, Proposition I.1]). Therefore there exists $\widetilde{u} \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$ such that up to a subsequence, $u_n \rightharpoonup \widetilde{u}$ converges weakly in $H^s(\mathbb{R}^N)$, $u_n \to \widetilde{u}$ converges strongly in $L^p(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$. Since $\{u_n\}$ is a Palais–Smale sequence for $I_\eta\,|_{S(m)}$, by the Lagrange multipliers rule there exists $\{\mu_n\} \subset \mathbb{R}$ such that for every $\phi \in H^s(\mathbb{R}^N)$, we have

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dx\,dy - \int\limits_{\mathbb{R}^N} \Big(\mu_n u_n\phi + |u_n|^{2_s^*-2}u_n\phi + \eta|u_n|^{p-2}u_n\phi\Big)\,dx = o(1)\|\phi\|, \quad (3.1)$$

as $n\to\infty$. Setting $\phi = u_n$, we infer that $\{\mu_n\}$ is also bounded, and therefore up to a subsequence, $\mu_n \to \mu \in \mathbb{R}^N$. By invoking the relation $\lim\limits_{n\to\infty} P_\eta(u_n) = 0$, the compactness of the embedding $H^s_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, and the fact that $\zeta_p < 1$, we get

$$\begin{aligned}\mu m^2 &= \lim_{n\to\infty} \mu_n\|u_n\|_2^2 = \lim_{n\to\infty}\Big([u_n]_{H^s}^2 - \|u_n\|_{2_s^*}^{2_s^*} - \eta\|u_n\|_p^p\Big)\\ &= \lim_{n\to\infty}\eta(\zeta_p-1)\|u_n\|_p^p = \eta(\zeta_p-1)\|\widetilde{u}\|_p^p \leqslant 0, \end{aligned} \quad (3.2)$$

where $\mu = 0$ if and only if $\widetilde{u} \equiv 0$.

Now, we show that

$$\widetilde{u} \not\equiv 0. \quad (3.3)$$

Suppose to the contrary that $\widetilde{u} \equiv 0$. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, it follows that up to a subsequence, $[u_n]_{H^s}^2 \to \gamma \in \mathbb{R}$. Since $P_\eta(u_n) \to 0$ and u_n converges strongly to 0 in $L^p(\mathbb{R}^N)$, it follows that

$$\|u_n\|_{2_s^*}^{2_s^*} = [u_n]_{H^s}^2 - \eta\zeta_p\|u_n\|_p^p \to \gamma,$$

therefore by (2.3), $\gamma \geqslant \mathcal{S}\gamma^{\frac{2}{2_s^*}}$. Furthermore, we can infer that

$$\text{either } \gamma = 0 \text{ or } \gamma > \mathcal{S}^{\frac{N}{2s}}.$$

If $\gamma > \mathcal{S}^{\frac{N}{2s}}$, then due to $I_\eta(u_n) \to E_\eta(m)$ and $\lim\limits_{n\to\infty} P_\eta(u_n) = 0$, we get

$$\begin{aligned} E_\eta(m) + o(1) = I_\eta(u_n) &= \frac{s}{N}[u_n]_{H^s}^2 - \frac{\eta}{p}\left(1 - \frac{\zeta_p p}{2_s^*}\right)\|u_n\|_p^p + o(1)\\ &= \frac{s}{N}[u_n]_{H^s}^2 + o(1) = \frac{\gamma s}{N} + o(1), \end{aligned}$$

so $E_\eta(m) = \frac{\gamma s}{N}$, thereby $E_\eta(m) \geqslant \frac{s\mathcal{S}^{\frac{N}{2s}}}{N}$, which contradicts our conditions.

If instead, we have $\gamma = 0$, we note that $[u_n]^2_{H^s} \to 0$, $\|u_n\|^{2^*_s}_{2^*_s} \to 0$ and $\|u_n\|^p_p \to 0$. Therefore $I_\eta(u_n) \to 0$, which is a contradiction as well. So (3.3) is proved. Furthermore, from (3.2) and (3.3) it follows that $\mu < 0$. Invoking the limit weak convergence in (3.1), we get

$$(-\Delta)^s \widetilde{u} = \mu \widetilde{u} + |\widetilde{u}|^{2^*_s - 2}\widetilde{u} + \eta|\widetilde{u}|^{p-2}\widetilde{u} \quad \text{in } \mathbb{R}^N, \tag{3.4}$$

and thus by the Pohozaev identity (see Chang and Wang [5, Proposition 4.1]) and related explanations in Zhen and Zhang [18, Proposition 2.1 and Remark 2.1], we have $P_\eta(\widetilde{u}) = 0$. We know that $w_n = u_n - \widetilde{u} \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, and according to Zuo et al. [19, Lemma 2.4] and the Brézis–Lieb lemma [3], we have

$$\begin{aligned} [u_n]^2_{H^s} &= [\widetilde{u}]^2_{H^s} + [w_n]^2_{H^s} + o(1), \\ \|u_n\|^{2^*_s}_{2^*_s} &= \|\widetilde{u}\|^{2^*_s}_{2^*_s} + \|w_n\|^{2^*_s}_{2^*_s} + o(1). \end{aligned} \tag{3.5}$$

Thus, by $\lim\limits_{n\to\infty} P_\eta(u_n) = 0$ and since $u_n \to \widetilde{u}$ converges strongly in L^p, we obtain

$$[\widetilde{u}]^2_{H^s} + [w_n]^2_{H^s} = \eta\zeta_p \|\widetilde{u}\|^p_p + \|\widetilde{u}\|^{2^*_s}_{2^*_s} + \|w_n\|^{2^*_s}_{2^*_s} + o(1).$$

In view of $P_\eta(\widetilde{u}) = 0$, we also have

$$[w_n]^2_{H^s} = \|w_n\|^{2^*_s}_{2^*_s} + o(1).$$

We claim that up to a subsequence

$$\lim_{n\to\infty} [w_n]^2_{H^s} = \lim_{n\to\infty} \|w_n\|^{2^*_s}_{2^*_s} = \gamma \geqslant 0, \quad \Rightarrow \quad \gamma \geqslant \mathcal{S}\gamma^{\frac{2}{2^*_s}}$$

thanks to (2.3). Hence, either $\gamma = 0$ or $\gamma > \mathcal{S}^{\frac{N}{2s}}$.

If $\gamma > \mathcal{S}^{\frac{N}{2s}}3$ then from (3.5), we obtain

$$\begin{aligned} E_\eta(m) = \lim_{n\to\infty} I_\eta(u_n) &= \lim_{n\to\infty} \left(I_\eta(\widetilde{u}) + \frac{1}{2}[w_n]^2_{H^s} - \frac{1}{2^*_s}\|w_n\|^{2^*_s}_{2^*_s} \right) \\ &= I_\eta(\widetilde{u}) + \frac{s\gamma}{N} \geqslant I_\eta(\widetilde{u}) + \frac{s\mathcal{S}^{\frac{N}{2s}}}{N}, \end{aligned}$$

whence the alternative (1) in the assertion of the proposition follows, i.e., up to a subsequence, u_n converges weakly to $\widetilde{u}$ in $H^s(\mathbb{R}^N)$ but not strongly, where $\widetilde{u} \not\equiv 0$ is a solution of the first equation of (1.1) for some $\mu < 0$, and

$$I_\eta(\widetilde{u}) < E_\eta(m) - \frac{s\mathcal{S}^{\frac{N}{2s}}}{N}.$$

If instead, we have $\gamma = 0$, then we claim that $u_n \to \widetilde{u}$ in $H^s(\mathbb{R}^N)$. Indeed, we have $\lim\limits_{n\to\infty} [w_n]^2_{H^s} = 0$, so from $w_n = u_n - \widetilde{u}$ it follows that $[u_n - \widetilde{u}]^2_{H^s} \to 0$.

Next, it suffices to verify that $u_n \to \widetilde{u}$ in L^2. Choosing $\phi = u_n - \widetilde{u}$ in (3.1), invoking (3.4) with $u_n \to \widetilde{u}$, and subtracting, we get

$$\begin{aligned}[u_n - \widetilde{u}]^2_{H^s} - \int\limits_{\mathbb{R}^N} (\mu_n u_n - \mu\widetilde{u})(u_n - \widetilde{u})\, dx \\ = \int\limits_{\mathbb{R}^N} \left(|u_n|^{2_s^*-2}u_n - |\widetilde{u}|^{2_s^*-2}\widetilde{u}\right)(u_n - \widetilde{u})\, dx \\ + \int\limits_{\mathbb{R}^N} \left(|u_n|^{p-2}u_n - |\widetilde{u}|^{p-2}\widetilde{u}\right)(u_n - \widetilde{u})\, dx + o(1).\end{aligned}$$

We note that $\lim\limits_{n\to\infty} \|w_n\|^{2_s^*}_{2_s^*} = 0$. From (3.5), it follows that $\|u_n\|^{2_s^*}_{2_s^*} \to \|\widetilde{u}\|^{2_s^*}_{2_s^*}$, therefore, in the formula above, the first term, the third term, and the fourth term converge to 0. As a result,

$$0 = \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} (\mu_n u_n - \mu\widetilde{u})(u_n - \widetilde{u})\, dx = \lim_{n\to\infty} \mu \int\limits_{\mathbb{R}^N} (u_n - \widetilde{u})^2\, dx.$$

Thus also assertion (2) of Proposition 3.1 has been established, i.e., up to a subsequence, u_n converges strongly to $\widetilde{u}$ in $H^s(\mathbb{R}^N)$, $I_\eta(\widetilde{u}) = E_\eta(m)$, and $\widetilde{u}$ is a solution of (1.1) for some $\mu < 0$. The proof of Proposition 3.1 is complete. □

§4. Proof of Theorem 1.1

Lemma 2.4 and [13, Propositions 5.3–5.4] imply that for a given Palais–Smale sequence $\{u_n\} \subset S_r(m)$ for I_η with the level $E_\eta(m)$, if $\lim\limits_{n\to\infty} P_\eta(\xi_n \star u_n) = 0$, then the sequence $\{\xi_n \star u_n\} \subset S_r(m)$ is also a Palais–Smale sequence for I_η with the same level, thus we can apply Lemma 2.5. In order to prove our main result, it remains to verify the condition $E_\eta(m) < \frac{s\mathcal{S}^{\frac{N}{2s}}}{N}$ of Proposition 3.1, which is a consequence of Lemma 2.3.

Therefore, we know that one of the two conclusions of Proposition 3.1 must be true. We show that the conclusion (1) fails. Indeed, otherwise, $\widetilde{u}$ would be a nontrivial solution of (1.1), i.e., up to a subsequence u_n converges weakly to $\widetilde{u}$ in $H^s(\mathbb{R}^N)$ but not strongly, where $\widetilde{u} \not\equiv 0$ is a solution of the first equation of (1.1) for some $\mu < 0$, and

$$I_\eta(\widetilde{u}) < E_\eta(m) - \frac{s\mathcal{S}^{\frac{N}{2s}}}{N} < 0.$$

However, since $P_\eta(\widetilde{u}) = 0$ by the Pohozaev identity and $\zeta_p p > 2$, we also get

$$I_\eta(\widetilde{u}) = \frac{\eta}{2p}(\zeta_p p - 2)\|u_n\|_p^p + \frac{s}{N}\|u_n\|_{2_s^*}^{2_s^*} > 0,$$

which is a contradiction.

Therefore, the conclusion (2) must be true and $\widetilde{u}$ is a radial normalized solution of (1.1) for some $\mu < 0$. This completes the proof of Theorem 1.1.

Acknowledgments

Jiabin Zuo was supported by the Guangdong Basic and Applied Basic Research Foundation (2022A1515110907) and the Project funded by China Postdoctoral Science Foundation (2023M730767). Yuyou Zhong was supported by the Innovative Research Funding Program for Graduate Students of Guangzhou University (2022GDJC-D09). Dušan D. Repovš was supported by the Slovenian Research Agency grants P1-0292, N1-0278, N1-0114, N1-0083, J1-4031, and J1-4001. We thank the referee for many important comments and suggestions, which have significantly improved the presentation.

References

[1] Alves C. O., Chao J., Miyagaki O. H., *Normalized solutions for a Schrödinger equation with critical growth in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations **61** (2022), no. 1, Paper 18, 24 pp.

[2] Appolloni l., Secchi S., *Normalized solutions for the fractional NLS with mass supercritical nonlinearity*, J. Differential Equations **286** (2021), 248-283.

[3] Brézis H., Lieb E., *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.

[4] Brézis H., Nirenberg L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[5] Chang X., Wang Z.-Q., *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity*, Nonlinearity **26** (2013), no. 2, 479–494.

[6] Di Nezza E., Palatucci G., Valdinoci E., *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[7] Frank R. L., Lenzmann E., Silvestre L., *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. **69** (2016), no. 9, 1671–1726.

[8] Jeanjean L., *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28** (1997), no. 10, 1633–1659.

[9] Jeanjean L., Lu S.-S., *A mass supercritical problem revisited*, Calc. Var. Partial Differential Equations **59** (2020), no. 5, Paper 174, 43 pp.

[10] Laskin N., *Fractional Schrödinger equation*, Phys. Rev. E **66** (2002), no. 5, 056108, 7 pp.

[11] Li Q., Zou W., *The existence and multiplicity of the normalized solutions for fractional Schrödinger equations involving Sobolev critical exponent in the* L^2*-subcritical and* L^2*-supercritical cases*, Adv. Nonlinear Anal. **11** (2022) 1531–1551.

[12] Lions P.-L., *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), no. 3, 315–334.
[13] Luo H., Zhang Z., *Normalized solutions to the fractional Schrödinger equations with combined nonlinearities*, Calc. Var. Partial Differential Equations **59** (2020), no. 4, Paper 143, 35 pp.
[14] Peng S., Xia A., *Normalized solutions of supercritical nonlinear fractional Schrödinger equation with potential*, Commun. Pure Appl. Anal. **20** (2021), no. 11, 3723–3744.
[15] Servadei R., Valdinoci E., *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.
[16] Soave N., *Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case*, J. Funct. Anal. **279** (2020), no. 6, 108610, 43 pp.
[17] Zhang P., Han Z., *Normalized solutions to a kind of fractional Schrödinger equation with a critical nonlinearity*, Z. Angew. Math. Phys. **73** (2022), no. 4, Paper 149, 23 pp.
[18] Zhen M., Zhang B., *Normalized ground states for the critical fractional NLS equation with a perturbation*, Rev. Mat. Complut. **35** (2022), 89–132.
[19] Zuo J., An T., Fiscella A., *A critical Kirchhoff-type problem driven by a $p(\cdot)$-fractional Laplace operator with variable $s(\cdot)$-order*, Math. Methods Appl. Sci. **44** (2021), 1071–1085.
[20] Zuo J., Rădulescu V. D., *Normalized solutions to fractional mass supercritical NLS systems with Sobolev critical nonlinearities*, Anal. Math. Phys. **12** (2022), no. 6, Paper 140, 20 pp.

Поступило 22 августа 2022 г.

School of Mathematics
and Information Science,
Guangzhou University,
510006 Guangzhou, P.R. China
E-mail: zuojiabin88@163.com
E-mail: zhongyy@e.gzhu.edu.cn

Faculty of Education,
University of Ljubljana,
1000 Ljubljana, Slovenia;
Faculty of Mathematics and Physics,
University of Ljubljana,
1000 Ljubljana, Slovenia;
Institute of Mathematics, Physics and Mechanics,
1000 Ljubljana, Slovenia
E-mail: dusan.repovs@guest.arnes.si