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Fractional Sobolev Spaces with Kernel Function on Compact Riemannian Manifolds

Ahmed Aberqi, Abdesslam Ouaziz and Dušan D. Repovš

Abstract. In this paper, a new class of Sobolev spaces with kernel function satisfying a Lévy-integrability-type condition on compact Riemannian manifolds is presented. We establish the properties of separability, reflexivity, and completeness. An embedding result is also proved. As an application, we prove the existence of solutions for a nonlocal elliptic problem involving the fractional $p(\cdot,\cdot)$-Laplacian operator. As one of the main tools, topological degree theory is applied.

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Keywords. Nonlinear elliptic problem, fractional Sobolev space, kernel function, Lévy-integrability condition, compact Riemannian manifold, existence of solutions, topological degree theory.

1. Introduction

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension N. The purpose of this paper is to present fundamental properties of a new class of Sobolev spaces with general kernel on $(\mathcal{M}, g)$. In addition, we shall solve the following equation:

$$\begin{cases} (\mathcal{L}_g^K)_{p(y,\cdot)}w(y) = \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y,w(y)) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathcal{M}\backslash\mathcal{U}. \end{cases} \tag{1.1}$$

Here, $s \in (0,1)$ is fixed, $r \in C(\mathcal{U}, (1,\infty))$, $\lambda > 0$, $\mathcal{U} \subset \mathcal{M}$ is an open bounded subset of $\mathcal{M}$, β is a suitable potential function in $\mathbb{R}^+$ with $\beta \in L^\infty(\mathcal{U})$, $p \in \mathcal{C}(\mathcal{U}\times\mathcal{U}, (1,\infty))$ satisfies the following conditions:

$$p(z,a) = p(a,z), \quad \text{for every } (z,a) \in \mathcal{M}^2, \tag{1.2}$$

$$1 < p^- = \min_{(y,z)\in\mathcal{M}^2} p(y,z) \le p(y,z) < p^+ = \sup_{(y,z)\in\mathcal{M}^2} p(y,z), \tag{1.3}$$

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and $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that:

$(\mathcal{B}_1)$ There exist $\alpha > 0$ and a continuous function $q : \mathcal{M} \to (1, +\infty)$, such that

$$1 < q(y) < p_s^\star(y) = \frac{Np(y,y)}{N - sp(y,y)}$$

and

$$f(y,z) \le \alpha \left(1 + |z|^{q(y)-1}\right), \quad \text{a.e. } \ y \in \mathcal{M}, \ z \in \mathbb{R},$$
$$r^+ = \sup_{y\in\mathcal{M}} r(y) \le q^- = \min_{y\in\mathcal{M}} q(y) \le p_s^\star(y)$$

and the operator $(\mathcal{L}_g^K)_{p(y,\cdot)}$ is defined by

$$(\mathcal{L}_g^K)_{p(y,\cdot)} w(y) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathcal{M}\setminus \mathfrak{B}_\varepsilon(y)} |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))K(y,z)\mathrm{d}v_g(z),$$

for every $z \in \mathcal{M}$, where

$$\mathfrak{B}_\varepsilon(y) = \{z \in \mathcal{M} : d_g(y,z) < \varepsilon\} \quad \text{and} \quad \mathrm{d}v_g(z) = |g_{ij}|^{\frac{1}{2}} \mathrm{d}z.$$

(see Definition 2.1). Furthermore, $K : \mathcal{U}\times\mathcal{U} \to (0, +\infty)$ is a symmetric kernel function satisfying the following variant of Lévy-integrability type condition:

$$gK \in L^1 \left(\mathcal{M} \times \mathcal{M}, \mathrm{d}v_g(y)\mathrm{d}v_g(z)\right), \quad \text{where } g(y,z) = \min\{d_g(y,z), 1\} \tag{1.4}$$

and the following coercivity condition for some $\alpha_0 > 0$

$$\frac{\alpha_0}{d_g(y,z)^{N+sp(y,z)}} \le K(y,z), \quad \text{a.e. } (y,z) \in \mathcal{M}^2, \ y \ne z. \tag{1.5}$$

We give examples of symmetric kernel functions that satisfy Lévy-integrability and coercivity type conditions.

Example. The following functions satisfy conditions (1.4)–(1.5).

- ♣ $K(y,z) = d_g(y,z)^{-N-sp(y,z))}$.
- ♣ $K(y,z) = \frac{\alpha_0}{d_g(y,z)^{N+sp(y,z))}}$, where α_0 is a positive real.
- ♣ $K(y,z) = \exp\left(\frac{1}{d_g(y,z)^{N+sp(y,z))}}\right)$.
- ♣ $K(y,z) = \exp\left(-\delta d_g(y,z)^2\right)$, where δ is a positive real.

Recently, results on fractional Sobolev spaces and problems involving the $p(y,\cdot)$-operator and their applications have received a lot of attention. For example, Kaufmann et al. [27] first introduced the new class $W^{s,q(y),p(y,z)}(\mathcal{U})$ defined by

$$W^{s,q(y),p(y,z)}(\mathcal{U}) = \{w \in L^{q(y)}(\mathcal{U}) : \int_{\mathcal{M}\times\mathcal{M}} \frac{|w(y) - w(z)|^{p(y,z)}}{K(y,z)} \mathrm{d}y\mathrm{d}z < +\infty\},$$

where $q \in C(\overline{\mathcal{U}}, (1,\infty))$ and $K(y,z) = |y - z|^{N+sp(y,z)}$, and they proved the existence of a compact embedding

$$W^{s,q(y,p(y,z)}(\mathcal{U}) \hookrightarrow L^{r(y)}(\mathcal{U}), \quad \text{for every } r \in C(\mathcal{U})$$

such that $1 < r(y) < p_s^\star(y)$, for every $y \in \overline{\mathcal{U}}$. They also studied solvability of the following fractional $p(y,\cdot)$-Laplacian problem:

$$\begin{cases} \mathcal{L}w(y) + |w(y)|^{q(y)-2}w(y) = h(y) & \text{in } \mathcal{Q}, \\ w = 0 & \text{in } \partial\mathcal{Q}, \end{cases} \tag{1.6}$$

with $h \in L^{a(y)}(\mathcal{Q}), a(y) > 1$.

For more results on the functional framework, we refer to Bahrouni and Rădulescu [9] who proved the solvability of the following problems:

$$\begin{cases} \mathcal{L}w(y) + |w(y)|^{q(y)-2}w(y) = \lambda|w(y)|^{r(y)-2}w(y) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathbb{R}^N\backslash\mathcal{U}, \end{cases} \tag{1.7}$$

using Ekeland's variational method, where $\mathcal{U}$ is an open bounded subset of $\mathbb{R}^N, \lambda > 0$, $r(y) < p^- = \min_{(y,z)\in\mathcal{U}\times\mathcal{U}} p(y,z)$, and $\mathcal{L}w$ is the fractional $p(y,\cdot)$ Laplacian operator.

Bahrouni [8] continued to study the space $W^{s,q(y,p(y,z)}(\mathcal{U})$. More specially, he proved the strong comparison principle for $(-\Delta)^s_{p(y,.)}$ and by using the sub-supersolution method, he showed the solvability of the following nonlocal equation:

$$\begin{cases} (-\Delta)^s_{p(y)}\, w(\cdot) = h(y, w(\cdot)) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathbb{R}^n\backslash\mathcal{U}, \end{cases} \tag{1.8}$$

where $\mathcal{U}$ is an open bounded domain, $s \in (0,1)$, p is a continuous function, and h satisfies the following growth:

$$|h(y,z)| \leq A_1|z|^{r(y)-1} + A_2, \quad \text{for every } (y,z) \in \mathbb{R}^{N+1},$$

where $r \in C(\mathbb{R}^N, \mathbb{R}),\ 1 < r(y) < p_s^\star(y),$ for every $y \in \mathbb{R}^N$.

The generalized fractional Sobolev space was studied in [8,9,27] and further developed in [26]. They proved a fundamental compact embedding for this space and investigated the multiplicity and boundedness of solutions to the following problem:

$$\begin{cases} (-\Delta)^s_{p(y)}\, w(\cdot) = f(\cdot, w(\cdot)) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathbb{R}^N\backslash\mathcal{U}, \end{cases} \tag{1.9}$$

where $p \in C(\mathbb{R}^N \times \mathbb{R}^N, (1,+\infty))$ is such that p satisfies the following conditions:

$$p(y,z) = p(z,y), \quad \text{for every } (y,z) \in \mathbb{R}^{2N}, \tag{1.10}$$

$$1 < \inf_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,z) \leq p(y,z) < \sup_{(y,z)\in\mathbb{R}^N\times\mathbb{R}^N} p(y,z) < \frac{N}{s}, \tag{1.11}$$

$f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $\left(-\Delta_{p(y)}\right)^s$ is an operator defined by

$$(-\Delta)^s_{p(y)}\, w(y) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\backslash B_\varepsilon(y)} \frac{|w(y)-w(z)|^{p(y,z)-2}(w(y)-w(z))}{|y-z|^{N+sp(y,z)}} \mathrm{d}z,$$

where $B_\varepsilon(y) = \{z \in \mathbb{R}^N : |z-y| < \varepsilon\}$.

The approaches for ensuring the existence of weak solutions for a class of nonlocal fractional problems with variable exponents were addressed in greater depth in [1,2,6,8–11,17,18,26,27,29,31,33] and the references therein.

In the non-Euclidean case, classical Sobolev spaces on Riemannian manifolds have been investigated for more than seventy years [5,25,30]. The theory of these spaces has been applied to isoperimetrical inequalities [25] and the Yamabe problem [35]. In [22] the authors investigated the theory of generalized Sobolev spaces on compact Riemannian manifolds. Moreover, they proved the compact embeddability of these spaces into the Hölder space. They also studied a PDE problem involving $p(\cdot)$-Laplacian operator.

In addition, the authors in [21] studied variable exponent function spaces on complete non-compact Riemannian manifolds. They used classical assumptions on the geometry to establish compact embeddings between Sobolev spaces and the Hölder function space. Finally, they also showed the existence of solutions to the $p(\cdot)$-Laplacian problem. The authors in [24] introduced the fractional Sobolev spaces on Riemannian manifolds. As a consequence, they investigated fundamental properties, such as compact embeddings, completeness, density, separability, and reflexivity. They also investigated the existence of solutions to the following equation:

$$\begin{cases} (-\mathcal{L}_g)_p^s\, w(y) = f(y, w(y)) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathcal{M}\backslash\mathcal{U}, \end{cases} \tag{1.12}$$

where $\mathcal{M}$ is a compact manifold of dimension d, $\mathcal{U} \subset \mathcal{M}$ is an open bounded subset of $\mathcal{M}$, $s \in (0,1)$, $p > 1$ with $d > ps$, $f : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and the operator $(-\mathcal{L}_g)_p^s\, w(y)$, $y \in \mathcal{M}$ is defined by

$$(-\mathcal{L}_g)_p^s\, w(y) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathcal{M}\backslash\mathfrak{B}_\varepsilon(y)} \frac{|w(y) - w(z)|^{p-2}(w(y) - w(z))}{(d_g(y,z))^{d+ps}} \mathrm{d}v_g(z).$$

Aberqi et al. [2] introduced the space $W^{s,p(y,z)}(\mathcal{M})$ and proved some important properties of this space and studied the following problem:

$$(\mathcal{P}) \begin{cases} (-\mathcal{L}_g)_{p(y,.)}^s\, w(y) + \mathcal{V}(y)|w(y)|^{q(y)-2} w = h(y, w(y)) & \text{in } \mathcal{U} \\ w|_{\partial\mathcal{U}} = 0. \end{cases}$$

Fractional Sobolev spaces and problems involving the $p(\cdot,\cdot)$-Laplacian operator have attracted significant attention in recent decades. This class of operators appears rather naturally in a variety of applications, including optimization and financial mathematics, we cite the well-known example by Carbotti et al. [16] who obtained the following equation:

$$\frac{\partial V}{\partial t}(S_t, t) + \mathcal{A}V(S_t, t) = rV(S_t, t) - r\frac{\partial V}{\partial t}(0,t)S_t,$$

where $\mathcal{A} := a\partial^2 - b(-\Delta)^s$ with $a, b \geqslant 0$ and $r \in \mathbb{R}$. Here, S_t is the price at time t and V the value of option. They are also useful in optimal control, engineering, quantum mechanics, obstacle problems, elasticity, image processing, minimal surfaces, stabilization of Lévy processes, game theory, population dynamics, fluid filtration, and stochastics, see for example [4,7,15,17,19,20,23,32,33] and the references therein.

Our work's novelty is extending general Sobolev spaces to Sobolev spaces $W_K^{q(y),p(y,z)}(\mathcal{U})$ with kernel function K on $\mathcal{M}$. We shall prove important properties of this new class of spaces. In particular, we shall investigate the existence of solutions to problem (1.1) using the topological degree method. This work generalizes previous results [1,2,8,9,12,24,26,27]. However, the main difficulty is presented by the fact that the $p(\cdot,\cdot)$-Laplacian operator has a more complicated nonlinearity than the p-Laplacian operator. For example, it is non-homogeneous. Other complications are due to the non-Euclidean framework of our problem. Also checking for example the density of the space $C^\infty(\mathcal{M})$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$, because the notion the translation in Riemannian manifolds is not defined. To the best of our knowledge, there were no such results prior to this work.

Our first major result is the following theorem.

Theorem 1.1. *Suppose that $(\mathcal{M}, g)$ is a compact N-dimensional Riemannian manifold, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U}\times\mathcal{U} \to (0,+\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, $p : \mathcal{U}\times\mathcal{U} \to (1,+\infty)$ satisfies conditions (1.2)–(1.3) and*

$$sp(y,z) < N,\ p(y,y) < q(y), \quad \textit{for every } (y,z) \in \mathcal{U}^2,$$

and $\ell : \overline{\mathcal{M}} \to (1,+\infty)$ is a continuous variable exponent, such that

$$p_s^*(y) = \frac{Np(y,y)}{N - sp(y,y)} > \ell(y) \geq \ell^- = \min_{y\in\overline{\mathcal{M}}} \ell(y) > 1.$$

Then, the space $W_K^{q(y),p(y,z)}(\mathcal{U})$ is continuously embeddable in $L^{\ell(y)}(\mathcal{U})$ and there exists a positive constant $C = C(N,s,p,q,\mathcal{U})$ such that

$$|w|_{L^{l(y)}(\mathcal{U})} \leq \|w\|_{W_K^{q(y),p(y,z)}(\mathcal{U})}, \quad \textit{for every } w \in W_K^{q(y),p(y,z)}(\mathcal{U}).$$

Moreover, this embedding is compact.

Our second main result is related to the investigation of the following fractional $p(y,\cdot)$-Laplacian problem with a general kernel K:

$$(1.1)\begin{cases} (\mathcal{L}_g^K)_{p(y,\cdot)}w(y) = \lambda\beta(y)|w(y)|^{r(y)-2}w(y) + f(y,w(y)) & \text{in } \mathcal{U}, \\ w = 0 & \text{in } \mathcal{M}\backslash\mathcal{U}. \end{cases}$$

Using Berkovits' topological degree, we study the existence of solutions and prove the following theorem.

Theorem 1.2. *Suppose that $(\mathcal{M}, g)$ is a compact N-dimensional Riemannian manifold, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U}\times\mathcal{U} \to (0,+\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions. Assume that assumption $(\mathcal{B}_1)$ holds. Then, problem (1.1) has at least one weak solution $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$.*

This paper is organized as follows: in Sect. 2, we collect the main definitions and properties of generalized Lebesgue spaces and generalized Sobolev spaces on compact manifolds and provide crucial background on recent Berkovits degree theory. In Sect. 3, we establish completeness, separability, and reflexivity properties of our spaces (Lemmas 3.2, 3.3, and 3.5). In Sect. 4, we prove our first main result (Theorem 1.1). In Sect. 5, we prove our second main result (Theorem 1.2). Finally, in Appendix, we prove some lemmas needed for the proofs of our main results.

2. Preliminaries

2.1. Generalized Lebesgue Spaces on Compact Manifolds

Throughout this section, $(\mathcal{M}, g)$ will be a compact Riemannian manifold of dimension N. To start, we briefly review some fundamental Riemannian geometry concepts that will be needed. For more details see [5,24,25].

A local chart on $\mathcal{M}$ is a pair $(\mathcal{U}, \varphi)$, where $\mathcal{U}$ is an open subset of $\mathcal{M}$ and φ is a homeomorphism of $\mathcal{U}$ onto an open subset of $\mathbb{R}^N$. Furthermore, a collection $(\mathcal{U}_i, \varphi_i)_{i\in J}$ of local charts, such that $\mathcal{M} = \bigcup_{j\in J} \mathcal{U}_j$, is called an atlas of manifold $\mathcal{M}$. For some atlas $(\mathcal{U}_j, \varphi_j)_{j\in J}$of $\mathcal{M}$, we say that a family $(\mathcal{U}_j, \varphi_j, \beta_j)_{j\in J}$ is a partition of unity subordinate to the covering $(\mathcal{U}_j, \varphi_j)_{j\in J}$ if the following holds:

(1) $\sum_{j\in J} \beta_j = 1$,
(2) $(\mathcal{U}_j, \varphi_j)_{j\in J}$ is an atlas of $\mathcal{M}$,
(3) $\operatorname{supp}(\beta_j) \subset \mathcal{U}_j$, for every $j \in J$.

Definition 2.1. (*See* [25]) Suppose that $w : \mathcal{M} \to \mathbb{R}$ is a continuous function with compact support, $(\mathcal{U}_j, \varphi_j)_{j\in J}$ is an atlas of $\mathcal{M}$, and $(\mathcal{U}_j, \varphi_j, \beta_j)_{j\in J}$ is a partition of unity subordinate to $(\mathcal{U}_j, \varphi_j)_{j\in J}$. We define the Riemannian measure of w in $\mathcal{M}$ as follows:

$$\int_{\mathcal{M}} w(y)\mathrm{d}v_g(y) = \sum_{j\in J} \int_{\varphi_j(\mathcal{U}_j)} (|g_{ij}|^{\frac{1}{2}} \beta_j w) \circ \varphi_j^{-1}(y)\mathrm{d}y,$$

where $\mathrm{d}v_g(y) = |g_{ij}|^{\frac{1}{2}}\mathrm{d}y$ is the Riemannian volume element on $(\mathcal{M}, g)$, g_{ij} are the components of the metric g in the local chart $(\mathcal{U}_j, \varphi_j)_{j\in J}$, and $\mathrm{d}y$ is the Lebesgue volume of $\mathbb{R}^N$.

Definition 2.2. (*See* [5]) Let $\gamma : [a, b] \to \mathcal{M}$ be a differentiable curve in $\mathcal{M}$ such that $\gamma \in C^1([a, b], \mathcal{M})$. Then the length of γ is given by

$$L(\gamma) = \int_a^b (g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}}\mathrm{d}t.$$

Definition 2.3. (*See* [5]) For any $(y, z) \in \mathcal{M}^2$, we define the distance $d_g(y, z)$ between y and z as follows

$$d_g(y, z) = \inf\{L(\gamma) : \gamma(a) = y,\ \gamma(b) = z\}.$$

Theorem 2.4. (Stine's theorem [5]) *For any $(a, b) \in \mathcal{M}^2$, $d_g(a, b)$ defines a distance on $(\mathcal{M}, g)$, and the topology determined by $d_g(a, b)$ is equivalent to the topology of $\mathcal{M}$ as a manifold.*

Next, we recall basic definitions and preliminary facts on the generalized Lebesgue spaces $L^{q(x)}(\mathcal{U})$ on compact manifolds, where $\mathcal{U}$ is an open subset of manifold $\mathcal{M}$. For more background, we refer to [2,5,25]. We need to recall the notion of the covariant derivative.

Definition 2.5. (*See* [25]) Let ∇ be the Levi–Civita connection. For $w \in C^\infty(M)$, $\nabla^k w$ denotes the kth covariant derivative of w. In local coordinates, the pointwise norm of $\nabla^k w$ is given by

$$\left|\nabla^k w\right| = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k w)_{i_1 i_2 \ldots i_k} (\nabla^k w)_{j_1 j_2 \ldots j_k}.$$

When $k = 1$, the components of ∇w in local coordinates are given by $(\nabla w)_i = \nabla^i w$. By definition, one has that

$$|\nabla w| = \sum_{i,j=1}^{\infty} g^{ij} \nabla^i \; w \nabla^j w.$$

We consider the set:

$$C^+(\mathcal{M}) = \{q : \mathcal{M} \to \mathbb{R}^+ : q \text{ is continuous and } 1 < q^- < q(y) < q^+ < +\infty\},$$

for every $y \in \mathcal{M},\; q^- = \min_{y\in\mathcal{M}} q(y),\; q^+ = \max_{y\in\mathcal{M}} q(y)$.

Definition 2.6. (*See* [21]) Let $q \in C^+(\mathcal{M})$ and $k \in \mathbb{N}$. We define the Sobolev space $L_k^{q(y)}(\mathcal{M})$ as the completion of $C_k^{q(y)}(\mathcal{M})$ with respect to the norm $|w|_{L_k^{q(y)}}(\mathcal{M})$, where

$$C_k^{q(y)}(\mathcal{M}) = \{w \in C^\infty(\mathcal{M}) : \left|\nabla^j w\right| \in L^{q(y)}(\mathcal{M}),\; \text{for every } j = 1, 2, \ldots, k\},$$

and

$$|w|_{L^{q(y)}(\mathcal{M})} = \sum_{j=0}^{k} |\nabla^j w|_{L^{q(y)}(\mathcal{M})},$$

where $|\nabla^j w|$ is the kth covariant derivative of w.

Lemma 2.7. (See [1]) *For every* $w \in L^{q(y)}(\mathcal{M})$, *the following properties hold*:

(i) *If* $\quad |w|_{L^{q(y)}(\mathcal{M})} < 1,$ *then* $|w|^{q^-}_{L^{q(y)}(\mathcal{M})} \leq \rho_{q(y)}(w) \leq |w|^{q^+}_{L^{q(y)}(\mathcal{M})}$.

(ii) *If* $\quad |w|_{L^{q(y)}(\mathcal{M})} > 1,$ *then* $|w|^{q^+}_{L^{q(y)}(\mathcal{M})} \leq \rho_{q(y)}(w) \leq |w|^{q^-}_{L^{q(y)}(\mathcal{M})}$.

(iii) $|w|_{L^{q(y)}(\mathcal{M})} < 1, = 1, > 1$ *if only if* $\rho_{q(y)}(w) < 1, = 1, > 1,$

where $\rho_{q(y)} : L^{q(y)}(\mathcal{M}) \to \mathbb{R}$ *is the mapping defined as follows*

$$\rho_{q(y)}(w) = \int_{\mathcal{M}} |w(y)|^{q(z)} \mathrm{d}v_g(z).$$

Proposition 2.8. (See [2]) *For every* w *and* $w_n \in L^{q(y)}(\mathcal{M})$, *the following statements are equivalent*:

(i) $\lim_{n\to+\infty} |w_n - w|_{L^{q(y)}(\mathcal{M})} = 0,$

(ii) $\lim_{n\to+\infty} \rho_{q(y)}(w_n - w) = 0,$

(iii) $w_n \to w$ *in measure on* $\mathcal{M}$ *and* $\lim_{n\to+\infty} \rho_{q(y)}(w_n) - \rho_{q(y)} w) = 0.$

Lemma 2.9. (Hölder's inequality, see [1]) *For every* $q \in C^+(\mathcal{M})$, *the following inequality holds*:

$$\left|\int_{\mathcal{M}} v(y) w(y) \mathrm{d}v_g(y)\right| \leq \left(\frac{1}{q^-} + \frac{1}{q'^-}\right) |v|_{L^{q(y)}(\mathcal{M})} |w|_{L^{q'(y)}(\mathcal{M})},$$

for every $(v, w) \in L^{q(y)}(\mathcal{M}) \times L^{q'(y)}(\mathcal{M})$, *where* $\frac{1}{q(y)} + \frac{1}{q'(y)} = 1$.

Lemma 2.10. (Simon's inequality, see [34]) *For every $y, z \in \mathbb{R}^N$, the following holds*:

$$\begin{cases} |y-z|^n \le c_n \left(|y|^{n-2}y - |z|^{n-2}z\right)\cdot(y-z), & n \ge 2 \\ |y-z|^n \le C_n \left[\left(|y|^{n-2}y - |z|^{n-2}z\right)\cdot(y-z)\right]^{\frac{n}{2}} \left(|y|^n + |z|^n\right)^{\frac{2-n}{2}}, & 1 < n < 2 \end{cases}$$

where $c_n = \left(\frac{1}{2}\right)^{-n}$ *and* $C_n = \frac{1}{n-1}$.

2.2. Topological Degree Theory

Let E be a real separable Banach space and E^* its dual. Given a non-empty set $U \subset E$, denote by $\bar{U}$ and by ∂U its closure and boundary, respectively.

Definition 2.11. (*See* [13]) Let $f : U \subset E \to E^*$ be an operator.

(1) We say that f is an (S_+)-map if for $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{\text{weakly}}{\rightharpoonup} z \text{ weakly and } \limsup_{n\to\infty} \langle f z_n, z_n - z\rangle \le 0 \;\Rightarrow\; z_n \to z\cdot$$

(2) We say that f is a quasi-monotone operator if for every $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{\text{weakly}}{\rightharpoonup} z \Rightarrow \limsup_{n\to\infty} \langle f z_n, z_n - z\rangle \ge 0.$$

Definition 2.12. (*Condition* $(S_+)_{\mathfrak{B}}$, *see* [28]) Suppose that $U_1 \subset E$ is such that $U \subset U_1$, $\mathfrak{B} : U_1 \to E^*$ is a bounded operator, and $f : U \subset E \to E$ is an operator.

(1) We say that f satisfies condition $(S_+)_{\mathfrak{B}}$ if for every $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, the following combined properties

$$\begin{cases} z_n \overset{\text{weakly}}{\rightharpoonup} z \\ a_n = \mathfrak{B}(z_n) \overset{\text{weakly}}{\rightharpoonup} a \\ \limsup_{n\to+\infty} \langle f z_n, a_n - a\rangle \ge 0 \end{cases} \tag{2.1}$$

imply $z_n \to z$.

(2) (Property $(QM)_{\mathfrak{B}}$). We say that f satisfies condition $(QM)_{\mathfrak{B}}$ if for every $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$, we have

$$z_n \overset{\text{weakly}}{\rightharpoonup} z \quad \text{and} \quad a_n = \mathfrak{B} z_n \overset{\text{weakly}}{\rightharpoonup} a \Rightarrow \limsup_{n\to\infty} \langle f z_n, a - a_n\rangle \ge 0\cdot$$

We consider the following sets

$$\mathcal{F}_0^\star(U) = \{g : U \subset E \to E^* \mid \text{ g is demi-continuous, of type } (S_+), \text{and is bounded}\}.$$

$$\mathcal{F}_{\mathfrak{B},1}(U) = \{g : U \subset E \to E \mid g \text{ is demi-continuous, bounded, of type } (S_+)_{\mathfrak{B}}\}.$$

$$\mathcal{F}_{\mathfrak{B}}(U) = \{f : U \subset E \to E \mid f \text{ is demi-continuous and satisfies condition } (S_+)_{\mathfrak{B}}\}.$$

Let $U \subset D_f$ and $\mathfrak{B} \in \mathcal{F}_0^\star(U)$, where D_f denotes the domain of f. We denote by $\mathcal{N}$ the collection of all bounded open sets in E. The following operators will be considered:

$$\mathcal{F}_{S_+}(E) = \{f \in \mathcal{F}_0^\star(\overline{\omega}) : \omega \in \mathcal{N}\},$$
$$\mathcal{F}_B(E) = \{f \in \mathcal{F}_{\mathfrak{B},1}(\overline{\omega}) : \omega \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_0^\star(\overline{\omega})\},$$
$$\mathcal{F}(E) = \{f \in \mathcal{F}_{\mathfrak{B}}(\overline{\omega}) : \omega \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_0^\star(\overline{\omega})\} \cdot$$

Lemma 2.13. (See [28]) *Let ω be a bounded open set in uniformly convex Banach space E, $\mathfrak{B} : \overline{\omega} \to E^*$ a bounded operator, and $f : \overline{\omega} \to E$. Then, we have*

(1) *If f is locally bounded and satisfies condition $(S_+)_{\mathfrak{B}}$ and $\mathfrak{B}$ is continuous, then f has the property $(QM)_{\mathfrak{B}}$.*
(2) *The operator f has the property $(QM)_{\mathfrak{B}}$, if for all $\{\{z_n\}_{n\in\mathbb{N}}, z\} \subset U$*

$$z_n \rightharpoonup z \quad \textit{and} \quad a_n = \mathfrak{B}z_n \rightharpoonup a \ \Rightarrow \liminf \langle fz_n, a_n - a\rangle \geq 0.$$

(3) *If operators $f_1, f_2 : \overline{\omega} \to E$ satisfy $(QM)_{\mathfrak{B}}$ condition, then $f_1 + f_2$ and αf_1 also satisfy $(QM)_{\mathfrak{B}}$ condition, for every positive numbers α.*
(4) *Let $f_1 : \overline{\omega} \to E$ be an operator of the type $(S_+)_{\mathfrak{B}}$ and $f_2 : \overline{\omega} \to E$ an operator satisfying the property $(QM)_{\mathfrak{B}}$. Then, $f_1 + f_2$ satisfies condition $(S_+)_{\mathfrak{B}}$.*

Lemma 2.14. (See [11]) *Let B be a bounded open set in E, $\mathfrak{B} \in \mathcal{F}_0^\star(\overline{B})$ continuous, and $g : D_g \subset E^* \to E$ a demi-continuous operator, such that $\mathfrak{B}(\bar{B}) \subset D_g$. Then, the following properties hold*:

(a) *If g is quasi-monotone operator, then $I + g \circ \mathfrak{B} \in \mathcal{F}_{\mathfrak{B}}(\bar{B})$, where I denotes the identity operator.*
(b) *If g is an operator of type (S_+), then $g \circ \mathfrak{B} \in \mathcal{F}_{\mathfrak{B}}(\overline{B})$.*

Definition 2.15. (*See* [14]) Let $B \subset E$ be a bounded open set, $\mathfrak{B} \in \mathcal{F}_0^\star(\overline{B})$ continuous, and $f, g \in \mathcal{F}_{\mathfrak{B}}(\overline{E})$. Then, the map $H : [0,1] \times \overline{E} \to E$ given by

$$H(s, w) = (1 - s)fw + sgw, \quad \text{for every } (s, w) \in [0,1] \times \overline{B},$$

is called an admissible affine homotopy.

Lemma 2.16. (See [11]) *Let $B \subset E$ be a bounded open set, $\mathfrak{B} \in \mathcal{F}_0^\star(\overline{B})$ continuous, and $f, g \in \mathcal{F}_{\mathfrak{B}}(\overline{E})$. Then, the homotopy $H(s, \cdot)$ satisfies condition $(S_+)_{\overline{B}}$.*

Theorem 2.17. (See [11]) *There exists a unique degree function*

$$d : \left\{(f, F, a) : F \in \mathcal{N}, \mathfrak{B} \in \mathcal{F}_0^\star(\overline{B}), f \in \mathcal{F}_{\mathfrak{B},1}(\overline{E}), a \notin f(\partial E)\right\} \to \mathbb{Z}$$

satisfying the following properties:

(1) *If $a \in F$, then $d(I, F, a) = 1$.*
(2) *If $G : [0,1] \times \overline{B} \to F$ is a bounded admissible affine homotopy with a common continuous essential inner map and $b : [0,1] \to F$ is a continuous mapping in E, then $d(G(x, \cdot), F, b(x))$ is constant for every $x \in [0,1]$ and $b(x) \notin G(t, \partial F)$.*

(3) *If* F_1 *and* F_2 *are disjoint open subsets of* E *with* $a \notin f\left(\overline{F} \backslash (E_1 \cup E_2)\right)$, *then*

$$d(f, F, a) = d\left(f, F_1, a\right) + d\left(f, F_2, a\right).$$

(4) *If* $d(f, F, a) \neq 0$, *then* $fu = a$ *has a solution in* F.

3. Fractional Sobolev Spaces with a General Kernel on Compact Riemannian Manifolds

In this section, we shall introduce fractional Sobolev spaces with a general kernel and prove several qualitative lemmas.

Definition 3.1. Suppose that $(\mathcal{M}, g)$ is a Riemannian compact manifold of dimension N, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q \in C^+(\mathcal{M})$, and $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ satisfying conditions (1.2)–(1.3). We define fractional Sobolev space $W_K^{q(y),p(y,z)}(\mathcal{U})$ with general kernel $K(y, z)$ on compact manifold $\mathcal{M}$ as the set of all measurable functions $w \in L^{q(y)}(\mathcal{U})$, such that $\int_{\mathcal{M}^2} \frac{|w(y)-w(z)|^{p(y,z)}}{\lambda^{p(y,z)}} K(y, z) \mathrm{d}v_g(y) \mathrm{d}v_g(z) < \infty$, for some $\lambda > 0$ and endow it with the natural norm:

$$\|w\|_K^{q(y),p(y,z)}(\mathcal{U}) = [w]_{K,p(y,z)} + |w|_{q(y)},$$

where

$$[w]_{K,p(y,z)} = \inf \left\{ \lambda > 0 : \int_{\mathcal{M}^2} \frac{|w(y) - w(z)|^{p(y,z)}}{\lambda^{p(y,z)}} K(y, z) \mathrm{d}v_g(y) \mathrm{d}v_g(z) < 1 \right\},$$

is the Gagliardo seminorm of u and $(L^{q(y)}(\mathcal{U}), |.|_{q(y)})$ is a variable exponent Lebesgue space.

Lemma 3.2. *Suppose that* $(\mathcal{M}, g)$ *is a Riemannian compact manifold of dimension* N, $\mathcal{U}$ *is a smooth open subset of* $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ *is a symmetric function satisfying Lévy-integrability and coercivity conditions,* $q \in C^+(\mathcal{M})$, *and* $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ *satisfies conditions* (1.2)–(1.3). *Then,* $(W_K^{q(y),p(y,z)}(\mathcal{U}), \| \cdot \|)$ *is a Banach space.*

Proof. Let $\{w_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W_K^{q(y),p(y,z)}(\mathcal{U})$. For any $\varepsilon > 0$, there exists $N_\varepsilon \geq 0$, such that for every $n, m \in \mathbb{N}$, $n, m \geq N_\varepsilon$,

$$|w_n - w_m|_{q(y)} \leq \|w_n - w_m\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq \varepsilon. \tag{3.1}$$

Since $(L^{q(y)}(\mathcal{U}), |.|_{q(y)})$ is a Banach space, there exists $w \in L^{q(y)}(\mathcal{U})$, such that $w_n \to w$ strongly in $L^{q(y)}(\mathcal{U})$ as $n \to +\infty$. Thanks to the converse of the dominated convergence theorem, it follows that for a subsequence still denoted $\{w_n\}$, we have that $w_n \to w$ as $n \to +\infty$ a.e on $\mathcal{U}$.

Let ℓ be an integer, such that $\ell \geq N_\varepsilon$ and $w_\ell \in W_K^{q(y),p(y,z)}(\mathcal{U})$. We use Fatou's lemma and inequality (3.1) to get

$$\begin{aligned}
&\int_{\mathcal{U}\times\mathcal{U}} |w(y)-w(z)|^{p(y,z)}K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\
&\quad\le \liminf_{n\to+\infty}\int_{\mathcal{U}\times\mathcal{U}} |w_n(y)-w_n(z)|^{p(y,z)}K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\
&\quad\le 2^{p^+-1}\liminf_{n\to+\infty}(|w_k(y)-w_\ell(y)-(w_k(z)-w_\ell(z)|^{p(y,z)}K(y,z)\\
&\qquad+|w_\ell(y)-w_\ell(z)|^{p(y,z)}K(y,z))\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\
&\quad\le 2^{p^+}\liminf_{n\to+\infty}(\|w_\ell-w_k\|^{p^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})}+\|w_\ell-w_k\|^{p^-}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\\
&\qquad+\|w_\ell\|^{p^-}_{W_K^{q(y),p(y,z)}(\mathcal{U})}+\|w_\ell\|^{p^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})})<+\infty.
\end{aligned}$$

Thus, $w\in W_K^{q(y),p(y,z)}(\mathcal{U})$. We combine Fatou's lemma and inequality (3.1), and obtain

$$\|w_n-w\|^{p^-}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\le\liminf_{k\to+\infty}\|w_n-w_k\|^{p^-}_{W_K^{q(y),p(y,z)}}(\mathcal{U})\le\varepsilon.$$

That is, $w_n\to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$ as $n\to+\infty$. □

Lemma 3.3. *Suppose that $(\mathcal{M},g)$ is a compact Riemannian manifold with $\dim\mathcal{M}=N$, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K:\mathcal{U}\times\mathcal{U}\to(0,+\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions, $q\in C^+(\mathcal{M})$, and $p:\mathcal{U}\times\mathcal{U}\to(1,+\infty)$ satisfies conditions (1.2)–(1.3). Then, $(W_K^{q(y),p(y,z)}(\mathcal{U}),\|\cdot\|)$ is a uniformly convex space.*

Proof. Let $w,v\in W_K^{q(y),p(y,z)}(\mathcal{U})$, and $\eta\in(0,2)$, such that

$$1=\|v\|_{W_K^{q(y),p(y,z)}(\mathcal{U})}=\|w\|_{W_K^{q(y),p(y,z)}(\mathcal{U})}\text{ and }\|w-v\|_{W_K^{q(y),p(y,z)}(\mathcal{U})}\ge\eta.$$

Case 1: $p^-\ge2$. Thanks to [3, Inequality 28], we get

$$\begin{aligned}
&\left\|\frac{w-v}{2}\right\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}+\left\|\frac{w+v}{2}\right\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\\
&\quad\le\frac{1}{2}\left(\|w\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}+\|v\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\right).
\end{aligned}\tag{3.2}$$

Thanks to (3.2), it follows that

$$\left\|\frac{w+v}{2}\right\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\le1-\left(\frac{1}{\eta}\right)^{p^+}.$$

We take $\delta=\delta(\eta)$, such that

$$1-\left(\frac{\eta}{2}\right)^{p(y,z)}=(1-\delta)^{p(y,z)}$$

and get

$$\left\|\frac{w+v}{2}\right\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}\le1-\eta.$$

Case 2: $1 < p(y,z) < 2$. Note that

$$\|w\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} = \left[\int_{\mathcal{M}^2} (|w(y)-w(z)|K(y,z))^{p'(y,z)(p(y,z)-1)} \mathrm{d}v_g(y)\mathrm{d}v_g(z)\right]^{\frac{1}{p(y,z)-1}},$$

with, $\frac{1}{p(y,z)} + \frac{1}{p'(y,z)} = 1$. By the reverse Minkowski's inequality [3, Theorem 2.13] and [3, Inequality 27], we have that

$$\begin{aligned}
&\left\|\frac{w+v}{2}\right\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + \left\|\frac{w-v}{2}\right\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \\
&\leq \Big\{ \int_{\mathcal{M}^2} \Big[|w(y)-w(z)+(v(y)-v(z))|^{p'(y,z)} + |w(y) \\
&\quad -w(z)+(v(y)-v(z))|^{p'(y,z)}\Big] \\
&\quad \times \left(\frac{1}{2}K(y,z)\right)^{p'(y,z)(p(y,z)-1)} \mathrm{d}v_g(y)\mathrm{d}v_g(z)\}^{\frac{1}{p(y,z)-1}}\Big\} \\
&\leq \frac{1}{2}\|w\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})}) + \frac{1}{2}\|v\|^{p(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})})^{p'(y,z)-1} = 1,
\end{aligned}$$

therefore

$$\left\|\frac{w+v}{2}\right\|^{p'(y,z)}_{W_K^{q(y),p(y,z)}(\mathcal{U})} \leq 1 - \left(\frac{1}{\varepsilon}\right)^{p'(y,z)}.$$

To complete the argument, choose $\delta = \delta(\eta)$, such that $1 - (\frac{\eta}{2})^{p'(y,z)} = (1-\delta)^{p'(y,z)}$. □

Remark 3.4. According to the Milman–Petits theorem [3], $W_K^{q(y),p(y,z)}(\mathcal{U})$ is a reflexive space.

Lemma 3.5. *Suppose that* $(\mathcal{M}, g)$ *is a compact Riemannian manifold with* $\dim \mathcal{M} = N$, $\mathcal{U}$ *is a smooth open subset of* $\mathcal{M}$, $K : \mathcal{U} \times \mathcal{U} \to (0, +\infty)$ *is a symmetric function satisfying conditions Lévy-integrability and coercivity conditions,* $q \in C^+(\mathcal{M})$, *and* $p : \mathcal{U} \times \mathcal{U} \to (1, +\infty)$ *satisfies conditions* (1.2)–(1.3). *Then,* $(W_K^{q(y),p(y,z)}(\mathcal{U}), \|\cdot\|)$ *is a separable space.*

Proof. Let $L : W_K^{q(y),p(y,z)}(\mathcal{U}) \to L^{q(y)}(\mathcal{M}) \times L^{p(y,z)}(\mathcal{M}\times\mathcal{M})$ be the operator defined by

$$L(w) = \big(w(y), (w(y)-w(z))K^{\frac{1}{p(y,z)}}(y,z)\big).$$

L is clearly well-defined and is an isometry. By [13, Proposition 3.17], the space $(W_K^{q(y),p(y,z)}(\mathcal{U}), \|\cdot\|)$ is indeed separable. □

4. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1, establishing an embedding of $W_K^{q(y),p(y,z)}(\mathcal{U})$ into $L^{\ell(y)}(\mathcal{U})$.

Proof. Let p, q, and ℓ be continuous functions and $\mathcal{U}$ an open subset of $\mathcal{M}$. There exist two positive constants α_1 and α_2, such that

$$q(y) \geq p(y,y) + \alpha_1 > 0 \tag{4.1}$$

and

$$\frac{Np(y,y)}{N - sp(y,y)} \geq \ell(y) + \alpha_2 > 0, \tag{4.2}$$

for every $y \in \overline{\mathcal{U}}$. Let $t \in (0, s)$. We use the continuity of p, q, ℓ, (4.1), and (4.2) to find a constant $\varepsilon = \varepsilon(p, q, \ell, \alpha_1, \alpha_2)$ and a finite family of disjoint Lipschitz sets $\mathcal{U}_j$, such that $\mathcal{U} = \bigcup_{j=1}^{N} \mathcal{U}_j$ and $\operatorname{diam}(\mathcal{U}_j) < \varepsilon$,

$$\frac{Np(m,z)}{N - tp(m,z)} \geq \ell(y) + \frac{\alpha_2}{2} > 0, \;\; q(y) \geq p(m,z) + \frac{\alpha_1}{2} > 0, \quad \forall (y,z,m) \in \mathcal{U}_j^3.$$

Put $p_j = \inf_{(y,z)\in\mathcal{U}_j\times\mathcal{U}_j}\{p(y,z) - \delta\}$. By the continuity of the involved exponents, we can choose $\delta = \delta(\alpha_2)$, with $p^- - 1 > \delta > 0$, such that

$$\frac{Np_j}{N - tp_j} \geq \ell(y) + \frac{\alpha_2}{3}, \quad \text{for every } y \in \mathcal{U}_j.$$

Therefore, we have the following

$$\frac{Np_j}{N - tp_j} \geq \ell(y) + \frac{\alpha_2}{3}, \quad \text{for every } y \in \mathcal{U}_j. \tag{4.3}$$

$$p_j + \frac{\alpha_1}{2} \leq q(y), \quad \text{for every } y \in \mathcal{U}_j. \tag{4.4}$$

By [24, Lemma 2.4], there exists a constant $C = C(N, t, \varepsilon, p_j, \mathcal{U}_j)$, such that (see [24] for more details):

$$\|w\|_{L^{p_j^*}(\mathcal{U}_j)} \leq C(\|w\|_{L^{p_j}(\mathcal{U}_j)} + [w]^{t,p_j}(\mathcal{U}_j)) \quad \text{for every } w \in W^{s,p_j}(\mathcal{U}_j). \tag{4.5}$$

Now, we shall prove the following three inequalities.

(a) There exists a constant $c_1 > 0$, such as: $c_1|w|_{L^{\ell(y)}(\mathcal{U})} \leq \sum_{j=0}^{N} \|w\|_{L^{p_j^*}(\mathcal{U}_j)}$.

(b) There exists a constant $c_2 > 0$, such as: $\sum_{j=0}^{N} \|w\|_{L^{p_j}(\mathcal{U}_j)} \leq c_2|w|_{L^{q(y)}(\mathcal{U})}$.

(c) There exists a constant $c_3 > 0$, such as: $\sum_{j=0}^{N}[w]^{t,p_j}(\mathcal{U}_j)) \leq c_3[w]^{s,p(y,z)}(\mathcal{U})$.

We shall first prove (a). We have that

$$w(y) = \sum_{j=0}^{N} |w(y)|\chi_{\mathcal{U}_j},$$

where $\chi_{\mathcal{U}_j}$ is a characteristic function. Hence, we have

$$|w|_{L^{\ell(y)}(\mathcal{U})} \leq \sum_{j=0}^{N} |w|_{L^{\ell(y)}(\mathcal{U}_j)}.$$

Combining the statement (4.3) with the Hölder inequality, we obtain

$$|w|_{L^{\ell(y)}(\mathcal{U}_j)} \leq C\|w\|_{L^{p_j^*}(\mathcal{U}_j)}|1|_{L^{a_j(y)}(\mathcal{U}_j)} \leq C(\mathcal{U}_j, a_j)\|w\|_{L^{p_j^*}(\mathcal{U}_j)},$$

$$\text{where } \frac{1}{\ell(y)} + \frac{1}{p_j^*} = \frac{1}{a_j(y)}, \text{ for every } y \in \mathcal{U}_j. \tag{4.6}$$

Similarly, by using the fact that $q(y) > p_j$ for every $y \in \mathcal{U}_j$, we get (b).

Now, we show (c). Put

$$G(y,z) = \frac{|w(y) - w(z)|}{d_g(y,z)^s}.$$

We use the Hölder inequality and the definition of p_j, to get

$$\begin{aligned}
[w]^{t,p_j}(\mathcal{U}_j) &= \left(\int_{\mathcal{U}_j \times \mathcal{U}_j} \frac{|w(y) - w(z)|^{p_j}}{d_g(y,z)^{N+tp_j}} \mathrm{d}v_g(y)\mathrm{d}v_g(z) \right)^{\frac{1}{p_j}} \\
&= \left(\int_{\mathcal{U}_j \times \mathcal{U}_j} \left(\frac{|w(y) - w(z)|}{d_g(y,z)^s} \right)^{p_j} \frac{\mathrm{d}v_g(y)\mathrm{d}v_g(z)}{d_g(y,z)^{N+(t-s)p_j}} \right)^{\frac{1}{p_j}} \\
&\leq C|G|_{L^{p(y,z)}(\mu_g, \mathcal{U}_j \times \mathcal{U}_j}|1|_{L^{\mathfrak{B}_j(y,z)}(\mu_g, \mathcal{U}_j \times \mathcal{U}_j)} \\
&= C(\mathcal{U}_j, \mathfrak{B}_j)|G|_{L^{p(y,z)}(\mu_g, \mathcal{U}_j \times \mathcal{U}_j)},
\end{aligned}$$

where

$$\frac{1}{p_j} = \frac{1}{p(y,z)} + \frac{1}{\mathfrak{B}_j(y,z)} \text{ and } d\mu_g(y,z) = \frac{\mathrm{d}v_g(y)\mathrm{d}v_g(z)}{d_g(y,z)^{N+(t-s)p_j}}.$$

Next, we prove that $|G|_{L^{p(y,z)}} \leq C[w]^{s,p(y,z)}(\mathcal{U}_j)$. Let $\lambda > 0$ be such that

$$\int_{\mathcal{U}_j \times \mathcal{U}_j} \frac{|w(y) - w(z)|^{p(y,z)}}{\lambda^{p(y,z)} d_g(y,z)^{N+sp(y,z)}} \mathrm{d}v_g(y)\mathrm{d}v_g(z) < 1.$$

Put $k = \sup\{1, \sup_{(y,z)\in\mathcal{U}\times\mathcal{U}} d_g(y,z)^{s-t}\}$ and $\breve{\lambda} = \lambda k$. Then we have

$$\int_{\mathcal{U}_j^2} \frac{|w(y) - w(z)|^{p(y,z)}}{((d_g(y,z)^s \breve{\lambda})^{p(y,z)}} \frac{\mathrm{d}v_g(y)\mathrm{d}v_g(z)}{d_g(y,z)^{N+(t-s)p_j}} \leq \int_{\mathcal{U}_j^2} \frac{|w(y) - w(z)|^{p(y,z)}}{d_g(y,z)^{N+sp(y,z)} \lambda^{p(y,z)}} < 1, \tag{4.7}$$

therefore

$$|G|_{L^{p(y,z)}(\mu_g(\mathcal{U}_j \times \mathcal{U}_j)} \leq \lambda k.$$

It now follows from (1.5), inequalities $(a), (b), (c)$, and [24, Lemma 2.4] that

$$\begin{aligned}
|w|_{L^{\ell(y)}(\mathcal{U})} &\leq c \sum_{j=0}^{N} \|u\|_{L^{p_j^*}(\mathcal{U})} \\
&\leq c \sum_{j=0}^{N} \left(\|w\|_{L^{p_j}(\mathcal{U}_j)} + [w]^{t,p_j}(\mathcal{U}_j) \right) \\
&\leq c \left(|w|_{L^{q(y)}(\mathcal{U})} + [w]^{s,p(y,z)}(\mathcal{U}) \right) \\
&\leq c\alpha_0 \left(|w|_{L^{q(y)}(\mathcal{U})} + [w]^{K,p(y,z)}(\mathcal{U}) \right) \\
&= c\alpha_0 \|w\|_{W_K^{q(y),p(y,z)}(\mathcal{U})}.
\end{aligned}$$

We show that this embedding is compact. Let $\{w_n\}$ be a bounded sequence in $W_K^{q(y),p(y,z)}(\mathcal{U})$, we need to prove that there exists $w \in L^{\ell(y)}(\mathcal{M})$, such that for every $\ell(y) \in (1, p_s^*)$:

$$w_n \to w \text{ in } L^{\ell(y)}(\mathcal{M}) \quad \text{as } n \to +\infty.$$

Since $\mathcal{M}$ is a compact Riemannian N-manifold, we can cover $\mathcal{M}$ by a finite number of charts $(\mathcal{U}_j, \varphi_j)_{j=1,\dots,m}$ satisfying

$$\frac{1}{Q}\delta_{ij} \leq g_{ij}^s \leq Q\delta_{ij},$$

where g_{ij}^s are bilinear forms and $Q > 1$. Let η_j be a smooth partition of unity subordinate to the chart $(\mathcal{U}_j, \varphi_j)_{j=1,\dots,m}$. Let $w_n \in W_K^{q(y),p(y,z)}(\mathcal{M})$. Then

$$\eta_j w_n \in W_K^{q(y),p(y,z)}(\mathcal{M}) \text{ and } (\eta_j w_n) \circ \varphi_j^{-1} \in W_K^{q(y),p(y,z)}(B_0(1)),$$

where $B_0(1)$ is an open unit ball of $\mathbb{R}^N$. By [27, Theorem 1.1], there exists $w_j \in L^{\ell(y)}(\varphi_j(\mathcal{U}_j))$, such that

$$(\eta_j w_n) \circ \varphi_j^{-1} \to w_j \text{ strongly in } L^{\ell(y)}(\varphi_j(\mathcal{U}_j)) \text{ as } n \to +\infty.$$

Hence

$$\eta_j w_n \to w_j \circ \varphi_j = a_j \text{ strongly in } L^{\ell(y)}(\varphi_j(\mathcal{U}_j)) \quad \text{as } n \to +\infty.$$

Finally, we put

$$w = \sum_{j=1}^{m} a_j = \sum_{j=1}^{m} w_j \circ \varphi_j \in L^{\ell(y)}(\mathcal{M}).$$

□

5. Proof of Theorem 1.2

In this section, we shall prove our existence result stated in Theorem 1.2.

Definition 5.1. We say that $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$ is a weak solution of problem (1.1) if for every $h \in (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ we have,

$$\int_{\mathcal{M}\times\mathcal{M}} |w(y) - w(z)|^{p(y,z)-2}(w(y) - w(z))(h(y) - h(z))K(y, z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)$$
$$= \lambda \int_{\mathcal{M}} \beta(y)|w(y)|^{r(y)-2}w(y)h(y)\mathrm{d}v_g(y) + \int_{\mathcal{M}} f(y, w(y))h(y)\mathrm{d}v_g(y).$$

We consider the functional $\zeta : W_K^{q(y),p(y,z)}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\zeta(w) = \int_{\mathcal{M}\times\mathcal{M}} \frac{1}{p(y, z)}|w(y) - w(z)|^{p(y,z)}K(y, z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)$$
$$- \lambda \int_{\mathcal{M}} \frac{1}{r(y)}\beta(y)|w(y)|^{r(y)}\mathrm{d}v_g(y) - \int_{\mathcal{M}} F(y, w(y))\mathrm{d}v_g(y).$$

Then it follows from [9,27] that $\zeta \in C^1(W_K^{q(y),p(y,z)}(\mathcal{U}),\mathbb{R})$ and

$$\begin{aligned}
&\langle \zeta'(w), h\rangle \\
&\quad = \int_{\mathcal{M}^2} |w(y)-w(z)|^{p(y,z)-2}(w(y)-w(z))(h(y)-h(z))K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z) \\
&\qquad - \lambda \int_{\mathcal{M}} \beta(y)|w(y)|^{r(y)-2}w(y)h(y)\mathrm{d}v_g(y) - \int_{\mathcal{M}} f(y,w(y))h(y)\mathrm{d}v_g(y) \\
&\quad = \langle L(w),h\rangle - \langle S_1(w),h\rangle - \langle S_2(w),h\rangle.
\end{aligned}$$

Proof. Let $w \in W_K^{q(y),p(y,z)}(\mathcal{U})$. Then w is a weak solution of problem (1.1) if and only if

$$Lw + Sw = 0, \tag{5.1}$$

where L, S are the operators defined in Lemmas 6.1 and 6.2. Since S is bounded, continuous, and quasi-monotone (see Lemma 6.1) and L is strictly monotone, thanks to the Minty–Browder Theorem [36, Theorem 26 A], we have that $L^{-1}=G$ is bounded continuous of type (S_+).

Equation (5.1) is equivalent to

$$w = Gh \quad \text{and} \quad h + S\circ Gh = 0. \tag{5.2}$$

To solve (5.2), we shall use the Berkovits topological degree introduced in Sect. 2. To this end, we first show that the set

$$D = \left\{ h \in (W_K^{q(y),p(y,z)}(\mathcal{U}))^* : h + tS\circ Gh = 0 \quad \text{for some } t\in[0,1]\right\},$$

is bounded. Let $h \in D$ and take $w = Gh$. Using the growth condition $(\mathcal{B}_1)$, the Hölder inequality, the Young inequality, and continuous embedding $W_K^{q(y),p(y,z)}(\mathcal{U}) \hookrightarrow L^{q(y)}(\mathcal{U})$, we get

$$\begin{aligned}
&\|Gh\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} \le \int_{\mathcal{M}\times\mathcal{M}} |w(y)-w(z)|^{p(y,z)}\, K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z) \\
&\quad = \langle Lw,h\rangle = \langle h, Gh\rangle \\
&\quad \le |t|\,\langle S\circ Gh, Gh\rangle \\
&\quad \le \lambda \int_{\mathcal{M}} \beta(y)|w(y)|^{r(y)}\mathrm{d}v_g(y) + \int_{\mathcal{M}} f(y,w(y))w(y)\mathrm{d}v_g(y) \\
&\quad \le \lambda\|\beta\|_\infty C_1 \|w\|^{r^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + C_2\left(\int_{\mathcal{M}} |f(y,w)|^{q'(y)}\mathrm{d}v_g(y)\right)^{\frac{1}{q'(y)}} \\
&\qquad + C_3\left(\int_{\mathcal{M}} |w(y)|^{q(y)}\mathrm{d}v_g(y)\right)^{\frac{1}{q(y)}} \\
&\quad \le \lambda\|\beta\|_\infty C_1 \|w\|^{r^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + C_2\|\beta\|_\infty \int_{\mathcal{M}} \left((1+|w(y)|^{(q(y)-1)q'(y)})\right)^{\frac{1}{q'(y)}} \\
&\qquad + C_3\left(\int_{\mathcal{M}} |w(y)|^{q(y)}\mathrm{d}v_g(y)\right)^{\frac{1}{q(y)}} \\
&\quad \le \lambda\|\beta\|_\infty C_1 \|w\|^{r^+}_{W_K^{q(y),p(y,z)}(\mathcal{U})} + 2^{q^+} C_4\|\beta\|_\infty \|w\|_{W_K^{q(y),p(y,z)}(\mathcal{U})} \\
&\qquad + C_4\|w\|_{W_K^{q(y),p(y,z)}(\mathcal{U})}.
\end{aligned}$$

Since S is bounded, it follows that D is bounded in $(W_K^{q(y),p(y,z)}(\mathcal{U}))^*$. As a result, there exists a positive constant $\eta > 0$, such that

$$\|h\|_{W_K^{q(y),p(y,z)}(\mathcal{U})^*} < \eta, \quad \text{for every } h \in D.$$

Furthermore, $h + tS \circ Gh \neq 0$ for every $(h, t) \in \partial B_\eta(0) \times [0, 1]$. Using Lemma 2.13, and $i + S \circ G \in \mathcal{F}_{\mathfrak{B}}(\overline{B_\eta(0)})$ and $i = L \circ G \in \mathcal{F}_{\mathfrak{B}}(\overline{B_\eta(0)})$ are present. Next, $i + S \circ G$ is also bounded because the operators i, S and G are all bounded. We come to the conclusion that

$$i + S \circ G \in \mathcal{F}_{\mathfrak{B},1}(\overline{B_\eta(0)}) \text{ and } i \in \mathcal{F}_{\mathfrak{B},1}(\overline{B_\eta(0)}).$$

We consider the map $H : [0, 1] \times \overline{B_\eta(0)} \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ given by

$$H(t, w) = w + tS \circ Lw.$$

By the statements (1)–(2) in Theorem 2.17, we can deduce

$$d(i + S \circ G, B_\eta(0), 0) = d(i, B_\eta(0), 0) = 1,$$

by applying the homotopy invariance and normalization properties of the degree d from Theorem 2.17. Therefore there exists $w \in B_\eta(0)$, such that $h + S \circ Gh = 0$. We can now deduce that $w = Gh$ is a weak solution to problem (1.1) in $W_K^{q(y),p(y,z)}(\mathcal{U})$. This completes the proof. $\square$

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Declarations

Conflict of Interest The authors declare to have no conflict of interest.

Appendix

Lemma 6.1. *Suppose that $(\mathcal{M},g)$ is a compact N-dimensional Riemannian manifold, $\mathcal{U}$ is a smooth open subset of $\mathcal{M}$, $K:\mathcal{U}\times\mathcal{U}\to(0,+\infty)$ is a symmetric function satisfying Lévy-integrability and coercivity conditions. Assume that assumption $(\mathcal{B}_1)$ holds. Then, the operator $L: W_K^{q(y),p(y,z)}(\mathcal{U})\to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ is continuous, bounded and strictly monotone, and*

(i) *L is an operator of type (S_+),*
(ii) *$L: W_K^{q(y),p(y,z)}(\mathcal{U})\to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ is a homeomorphism.*

Proof. It is obvious that L is bounded. We show that L is continuous. Assume that $w_n\to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$ and we show that $L(w_n)\to L(w)$ in $(W_K^{q(y),p(y,z)}(\mathcal{U}))^*$. Indeed

$$\begin{aligned}\langle Lw_n-Lw,\varphi\rangle &= \int_{\mathcal{M}^2}\Big[(|w_n(y)-w_n(z)|^{p(y,z)-2}(w_n(y)-w_n(z))\\ &\quad -|w(y)-w(z)|^{p(y,z)-2}(w(y)-w(z))\Big]\\ &\quad\times K(y,z)(\varphi(y)-\varphi(z))\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &=\int_{\mathcal{M}^2}\Big[|w_n(y)-w_n(z)|^{p(y,z)-2}(w_n(y)-w_n(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}}\\ &\quad -|w(y)-w(z)|^{p(y,z)-2}(w(y)-w(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}}\Big]\\ &\quad\times(\varphi(y)-\varphi(z))K(y,z)^{\frac{1}{p(y,z)}}\mathrm{d}v_g(y)\mathrm{d}v_g(z).\end{aligned}$$

Put

$$\begin{aligned}G_n(y,z)&=|w_n(y)-w_n(z)|^{p(y,z)-2}(w_n(y)-w_n(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}}\\ &\quad\in L^{p'(y,z)}(\mathcal{U}\times\mathcal{U}),\\ G(y,z)&=|w(y)-w(z)|^{p(y,z)-2}(w(y)-w(z))K(y,z)^{\frac{p(y,z)-1}{p(y,z)}}\\ &\quad\in L^{p'(y,z)}(\mathcal{U}\times\mathcal{U}),\end{aligned}$$

and

$$F(y,z)=(\varphi(y)-\varphi(z))K(y,z)^{\frac{1}{p(y,z)}}\in L^{p(y,z)}(\mathcal{U}\times\mathcal{U}),$$

where, $\frac{1}{p(y,z)}+\frac{1}{p'(y,z)}=1$. Thanks to the Hölder inequality, we have

$$\langle L(w_n)-L(w),\varphi\rangle\le 2|G_n(y,z)-G(y,z)|_{L^{p'(y,z)}(\mathcal{U}\times\mathcal{U})}|F|_{L^{p(y,z)}(\mathcal{U}\times\mathcal{U})}.$$

Thus

$$\|L(w_n)-L(w)\|_{(W_K^{q(y),p(y,z)}(\mathcal{U}))^*}\le 2|G_n(y,z)-G(y,z)|_{L^{p'(y,z)}(\mathcal{U}\times\mathcal{U})}.$$

Let

$$V_n(y,z)=(w_n(y)-w_n(z))K(y,z)^{\frac{1}{p(y,z)}}\in L^{p(y,z)}(\mathcal{U}\times\mathcal{U}),$$

and

$$V(y,z)=(w(y)-w(z))K(y,z)^{\frac{1}{p(y,z)}}\in L^{p(y,z)}(\mathcal{U}\times\mathcal{U}).$$

Since $w_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$, we have $V_n \to V$ in $L^{p(y,z)}(\mathcal{U} \times \mathcal{U})$. Therefore, there exists a subsequence of $\{V_n\}_{n\in\mathbb{N}}$ and $h(y,z) \in L^{p(y,z)}(\mathcal{U}\times\mathcal{U})$, such that $V_n \to V$ a.e in $\mathcal{U}\times\mathcal{U}$ and $|V_n| \leq h(y,z)$. Therefore we have $G_n \to G$ a.e in $\mathcal{U} \times \mathcal{U}$ and

$$|G_n(y,z)| = |V_n(y,z)|^{p(y,z)-1} \leq h(y,z)^{p(y,z)-1}.$$

We use the dominated convergence theorem to get

$$G_n \to G \quad \text{in } L^{p'(y,z)}(\mathcal{U} \times \mathcal{U}).$$

By Lemma 2.10, L is strictly monotone. Now, we show that L is mapping of type (S_+). Let $\{w_n\}_{n\in\mathbb{N}} \subset W_K^{q(y),p(y,z)}(\mathcal{U})$ be a sequence with $w_n \rightharpoonup w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$ and $\limsup_{n\to+\infty} \langle L(w_n) - \ L(w), w_n - w\rangle \leq 0$. Using (i), we get

$$0 = \lim_{n\to+\infty} \langle Lw_n - Lw, w_n - w\rangle \cdot \tag{6.1}$$

By Theorem 1.1, we have that $w_n(y) \to w(y)$ a.e. in $\mathcal{U}$. This, in combination with Fatou's lemma, gives us

$$\begin{aligned}\liminf_{n\to+\infty} &\int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &\geq \int_{\mathcal{M}^2} |w(y) - w(z)|^{p(y,z)} K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z).\end{aligned} \tag{6.2}$$

On the other hand, we have

$$\lim_{n\to+\infty} \langle L(w_n), w_n - w\rangle = \lim_{n\to+\infty} \langle L(w_n) - L(w), w_n - w\rangle = 0. \tag{6.3}$$

Using Young's inequality, we can see there is a positive constant $c > 0$, such that

$$\begin{aligned}&\langle L(w_n), w_n - w\rangle\\ &\quad= \int_{\mathcal{M}\times\mathcal{M}} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &\qquad - \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)-2}\\ &\qquad \times (w_n(y) - w_n(z))(w(y) - w(z))K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &\quad\geq \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &\qquad - \int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)-1} |w(y) - w(z)| K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &\quad\geq c\int_{\mathcal{M}^2} |w_n(y) - w_n(z)|^{p(y,z)} K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\\ &\qquad - c\int_{\mathcal{M}^2} |w(y) - w(z)|^{p(y,z)} K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z).\end{aligned}$$

According to (6.1)–(6.3), we get

$$\lim_{n\to+\infty}\int_{\mathcal{M}^2}|w_n(y)-w_n(z)|^{p(y,z)}K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)$$
$$=\int_{\mathcal{M}^2}|w(y)-w(z)|^{p(y,z)}K(y,z)\mathrm{d}v_g(y)\mathrm{d}v_g(z)\cdot \tag{6.4}$$

As a consequence of the Brezis–Lieb Lemma [13], (6.1), and (6.4), L is of type $(S_+)\cdot$

We show that L is a homeomorphism. It is easy to see that L is coercive and injective. Thanks to the Minty–Browder Theorem [36, Theorem 26 A], L is surjective. Therefore, L is a bijection. There exists a map $G:(W_K^{q(y),p(y,z)}(\mathcal{U}))^*\to W_K^{q(y),p(y,z)}(\mathcal{U})$, such that $G\circ L=id_{W_K^{q(y),p(y,z)}(\mathcal{U})}$ and $L\circ G=id_{(W_K^{q(y),p(y,z)}(\mathcal{U}))^*}\cdot$

We show that G is continuous. Let $g_n,g\in W_K^{q(y),p(y,z)}(\mathcal{U})$ be such that $g_n\to g$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Let $t_n=G(g_n),w=G(g)$. Then $L(w_n)=g_n$ and $L(w)=g$. Since $\{t_n\}_{n\in\mathbb{N}}$ is bounded in $W_K^{q(y),p(y,z)}(\mathcal{U})$, we have $t_n\rightharpoonup w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. It follows that

$$\lim_{n\to+\infty}<L(t_n)-L(w),t_n-w>=\lim_{n\to+\infty}<g_n,t_n-w>=0.$$

Since L is of type (S_+), we get $t_n\to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. This completes the proof. □

Lemma 6.2. *If f satisfies $(\mathcal{B}_1)$, then the operator $S:W_K^{q(y),p(y,z)}(\mathcal{U})\to(W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ defined by*

$$\langle Sw,\varphi\rangle=-\lambda\int_{\mathcal{M}}\beta(y)|w(y)|^{r(y)-2}w\varphi\mathrm{d}v_g(y)-\int_{\mathcal{M}}f(y,w(y))\varphi\mathrm{d}v_g(y),$$

for every $\varphi\in\left(W_K^{q(y),p(y,z)}(\mathcal{U})\right)^*$ is compact.

Proof. Let

$$S_1:W_K^{q(y),p(y,z)}(\mathcal{U})\to L^{q'(y)}(\mathcal{U}) \qquad S_2:W_K^{q(y),p(y,z)}(\mathcal{U})\to L^{q'(y)}(\mathcal{U})$$
$$w\longmapsto S_1w=-\lambda\beta(y)|w(y)|^{r(y)-2}w(y) \qquad w\longmapsto S_2w=-f(y,w(y)).$$

We shall show that S_1 and S_2 are both bounded and continuous. For every $w\in W_K^{q(y),p(y,z)}(\mathcal{U})$,

$$\begin{aligned}|S_1w|_{q'(y)}&=\lambda\int_{\mathcal{M}}|\beta(y)|w(y)|^{r(y)-2}w(y)|^{q'(y)}\mathrm{d}v_g(y)\\&\le\lambda\|\beta\|_\infty\int_{\mathcal{M}}\||w(y)|^{r(y)-1}|^{q'(y)}\mathrm{d}v_g(y)\\&\le\lambda C\|\beta\|_\infty\int_{\mathcal{M}}\||w(y)|^{q(y)-1}|^{q'(y}\mathrm{d}v_g(y)\\&\le\lambda C\|\beta\|_\infty\int_{\mathcal{M}}|w(y)|^{q(y)}\mathrm{d}v_g(y).\end{aligned}$$

This implies that S_1 is bounded in $W_K^{q(y),p(y,z)}(\mathcal{U})$. By condition $(\mathcal{B}_1)$, there exists $\alpha > 0$, such that

$$|f(y, w(y))| \leq \alpha(1 + |w(y)|^{q(y)-1}).$$

Therefore

$$\begin{aligned} |S_2 w|_{q'(y)}^{q'(y)} &= \int_{\mathcal{M}} |f(y, w(y)|^{q'(y)} \mathrm{d}v_g(y) \\ &\leq \int_{\mathcal{M}} \alpha(1 + |w(y)|^{q(y)-1})^{q'(y)} \mathrm{d}v_g(y) \\ &\leq 2^{q'^+}(|\mathcal{M}| + \int_{\mathcal{M}} |w(y)|^{(q(y)-1)q'(y)} \mathrm{d}v_g(y) \\ &\leq \alpha c'(\mathcal{M}, q(y)) \int_{\mathcal{M}} |w(y)|^{(q(y)-1)q'(y)} \mathrm{d}v_g(y) \\ &\leq \alpha c'(\mathcal{U}, q(y)) |w(y)|_{L^{q'(y)}}, \end{aligned}$$

Hence, S_2 is bounded in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Next, we show that S_2 is continuous. Let $w_n \in W_K^{q(y),p(y,z)}(\mathcal{U})$, such that $w_n \to w$ in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Then $w_n \to w$ in $L^{q(y)}(\mathcal{U})$. Hence, there exists a subsequence, still denoted by w_n, and a measurable function g in $L^{q(y)}(\mathcal{U})$, such that $w_n(y) \to w$ and $|w_n(y)| \leq g(y)$, a.e in $\mathcal{U}$.

Since f is a Carathéodory function, we have

$$f(y, w_n) \to f(y, w(y)) \quad \text{a.e. in } \mathcal{U}. \tag{6.5}$$

According to condition $(\mathcal{B}_1)$, we have that

$$|f(y, w_n(y)| \leq \alpha(1 + g(y))^{q(y)-1} \in L^{q'(y)}(\mathcal{U}).$$

Using (6.5), we obtain

$$\int_{\mathcal{M}} |f(y, w_n(y)) - f(y, w(y))|^{q'(y)} \mathrm{d}v_g(y) \to 0 \quad \text{as } n \to \infty.$$

The dominated convergence theorem implies that $S_2 w_n \to S_2 w$ in $L^{q'(y)}(\mathcal{U})$, so S_2 is continuous in $W_K^{q(y),p(y,z)}(\mathcal{U})$. Because the canonical embedding $i : W_K^{q(y),p(y,z)}(\mathcal{U}) \hookrightarrow L^{q(y)}(\mathcal{U})$ is compact, its adjoint operator $i^* : L^{q'(y)}(\mathcal{U}) \to (W_K^{q(y),p(y,z)}(\mathcal{U}))^*$ is also compact. As a result, compositions $i^* \circ S_2$ and $S_2 \circ i^*$ are compact, so we come to the conclusion that the operator S is compact and this completes the proof. □

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Ahmed Aberqi
Laboratory LAMA, National School of Applied Sciences
Sidi Mohamed Ben Abdellah University
BP 1796
30000 Fez
Morocco
e-mail: ahmed.aberqi@usmba.ac.ma

Abdesslam Ouaziz
Laboratory LAMA, Faculty of Sciences Dhar Elmahraz
Sidi Mohamed Ben Abdellah University
BP 1796
30000 Fez
Morocco
e-mail: abdesslam.ouaziz@usmba.ac.ma

Dušan D. Repovš
Faculty of Education
University of Ljubljana
Kardeljeva ploščad 16
1000 Ljubljana
Slovenia

and

Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 21
1000 Ljubljana
Slovenia

and

Institute of Mathematics, Physics and Mechanics
Jadranska 19
1000 Ljubljana
Slovenia
e-mail: dusan.repovs@guest.arnes.si

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