



Mixed order elliptic problems driven by a singularity, a Choquard type term and a discontinuous power nonlinearity with critical variable exponents

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Abstract

We prove the existence of solutions for the following critical Choquard type problem with a variable-order fractional Laplacian and a variable singular exponent

$$\begin{aligned} a(-\Delta)^{s(\cdot)}u + b(-\Delta)u &= \lambda|u|^{-\gamma(x)-1}u + \left(\int_{\Omega} \frac{F(y, u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x, u) \\ &+ \eta H(u - \alpha)|u|^{r(x)-2}u, \text{ in } \Omega, \\ u &= 0, \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

where $a(-\Delta)^{s(\cdot)} + b(-\Delta)$ is a mixed operator with variable order $s(\cdot) : \mathbb{R}^{2N} \rightarrow (0, 1)$, $a, b \geq 0$ with $a + b > 0$, H is the Heaviside function (i.e., $H(t) = 0$ if $t \leq 0$, $H(t) = 1$ if $t > 0$), $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $\lambda > 0$, $0 < \gamma^- = \inf_{x \in \bar{\Omega}} \{\gamma(x)\} \leq \gamma(x) \leq \gamma^+ = \sup_{x \in \bar{\Omega}} \{\gamma(x)\} < 1$, μ is a continuous variable

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parameter, and F is the primitive function of a suitable f . The variable exponent $r(x)$ can be equal to the critical exponent $2_s^*(x) = \frac{2N}{N-2\bar{s}(x)}$ with $\bar{s}(x) = s(x, x)$ for some $x \in \bar{\Omega}$, and η is a positive parameter. We also show that as $\alpha \rightarrow 0^+$, the corresponding solution converges to a solution for the above problem with $\alpha = 0$.

Keywords Choquard type · Variable-order fractional operator · Mixed operator · Variable singular exponent · Discontinuous power nonlinearity

Mathematics Subject Classification 35R11 · 35J75 · 35J60 · 46E35.

1 Introduction

In this paper, we shall consider the following mixed order elliptic problem

$$\begin{aligned}
 a(-\Delta)^{s(\cdot)}u + b(-\Delta)u &= \lambda|u|^{-\gamma(x)-1}u + \left(\int_{\Omega} \frac{F(y, u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x, u) \\
 &+ \eta H(u - \alpha)|u|^{r(x)-2}u, \text{ in } \Omega, \\
 u &= 0, \text{ in } \mathbb{R}^N \setminus \Omega.
 \end{aligned}
 \tag{1.1}$$

Here, $(-\Delta)^{s(\cdot)}$ is the fractional Laplacian of a variable order $s(\cdot)$ which is defined as follows.

$$(-\Delta)^{s(\cdot)}u(x) = C_{N,s(\cdot)} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_{\epsilon}(x)} \frac{(u(x) - u(y))}{|x-y|^{N+2s(x,y)}} dy, \quad x \in \mathbb{R}^N$$

and $-\Delta = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the well-known Laplace operator.

We suppose that $s(\cdot) : \mathbb{R}^{2N} \rightarrow (0, 1)$ is a uniformly continuous function and $\mu(\cdot) : \mathbb{R}^{2N} \rightarrow (0, N)$ is a continuous function that satisfy the following hypotheses

- (H₁) : $0 < s^- \leq s^+ < 1, 0 < \mu^- < \mu^+ < N$ and $2s^+ < N$;
- (H₂) : $s(\cdot), \mu(\cdot)$ are symmetric, i.e.
 $s(x, y) = s(y, x), \mu(x, y) = \mu(y, x), (x, y) \in \mathbb{R}^{2N}$.

We assume that $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$ is a continuous odd function with respect to the second variable that satisfies the following hypotheses

- (F₁) : there exist $c_1 > 0$ and $\tau \in C_+(\bar{\Omega}) \cap \mathcal{M}$ with $2 < \tau^- < 2_s^{*-}$ and $2 < \tau^+ < 2_s^{*+}$ such that $0 \leq f(x, t) \leq c_1|t|^{\tau(x)-1}$, for any $(x, t) \in \Omega \times \mathbb{R}$, where $\mathcal{M} = \{\tau \in C_+(\bar{\Omega}) : 2 \leq \tau(x)\beta^- \leq \tau(x)\beta^+ < 2_s^*(x) \text{ for all } x \in \mathbb{R}^N\}$ with $\beta \in C_+(\mathbb{R}^{2N})$ such that $\frac{2}{\beta(x, y)} + \frac{\mu(x, y)}{N} = 2$

for any $(x, y) \in \mathbb{R}^{2N}$, and

$$1 < 2_s^{*-} = \inf_{x \in \Omega} \{2_s^*(x)\} \leq 2_s^*(x) \leq 2_s^{*+} = \sup_{x \in \Omega} \{2_s^*(x)\};$$

(F_2) : there exists $\theta > 2$ such that

$$0 < \theta F(x, t) \leq 2tf(x, t), \text{ for any } t \in \mathbb{R} \setminus \{0\} \text{ and for any } x \in \Omega.$$

Remark 1 We shall sometimes denote the *Choquard* type of term

$$\int_{\Omega} \frac{F(y, u(y))}{|x - y|^{\mu(x,y)}} dy$$

by $F * \frac{1}{|x|^{\mu(\cdot)}}$.

In recent years, the problem of combining local and nonlocal Laplacian operators has attracted a huge amount of attention due to theoretical and practical applications, such as in plasma physics (see [8]) and population dynamics (see [20]). Another key reason as to why the study of such equations holds importance is due to the fact that many of the free boundary problems studied in mathematical physics are formulated in this form. Some of them are the surface problem, the obstacle problem, and the Elenbaas equation (see [12–14]). Interested readers may also refer to [3].

Biagi et al. [6] considered the corresponding Sobolev inequality, detecting the optimal constant for a critical local and nonlocal problem, furthermore, they proved an existence (and nonexistence) result of the corresponding subcritical perturbation problem. Hu and Yang [30] investigated a mixed order problem and proved a bifurcation and multiplicity result. Arora and Rădulescu [4] studied an elliptic problem involving both local-nonlocal operators and proved the existence (or nonexistence) of weak solution to it. Cassani et al. [10] analyzed spectral properties, established the validity of the maximum principle, showed related properties of weak solutions of mixed local-nonlocal elliptic equations. In particular, this makes the problem more interesting when the order of nonlocal operator can be variable, and to the best of our knowledge the first fractional variable-order problem can be traced back to the work of Lorenzo and Hartley [32]. Since the fractional problem of variable order can better describe the temperature change of the object in the process of nonlinear diffusion, we shall focus on a class of problems with the fractional variable-order Laplace operator (see [37, 38, 40]).

The starting point for the study of singular problems was the pioneering work of Crandall [17]. We also refer to the monograph by Ghergu and Rădulescu [27]. At a time when scholars were concerned about nonlinearities with constant exponent, few works also involved variable singular exponent case. For instance, Garain et al. [26] proved the existence of one solution and two solutions for a quasilinear variable singular exponent fractional elliptic problems when the parameters are in a suitable range. Ghosh et al. [28] proved the existence of infinite number of small energy solutions by using the symmetric mountain pass lemma for the elliptic PDEs of variable exponent with a singular nonlinearity. Interested readers can also refer to Chammem et al. [11].

The Choquard type problem is a hot topic which many mathematical scholars have begun paying close attention to recently. It becomes more interesting when combined

with the variable exponents. Alves et al. [1] established a kind of Hardy-Littlewood-Sobolev inequality with variable exponents, which usually is used to estimate the convolution term in a problem. With the help of Nehari manifold methods, Li and Tang [31] obtained the existence of positive ground state solution for a class of stationary Choquard equations with variable exponent growth. By using variational methods, Zuo et al. [42] and Biswas et al. [7] proved the existence and multiplicity of solutions for a $(p(\cdot)&q(\cdot))$ Choquard problem with nonstandard growth.

In practical problems, some functional models are not always continuous, and it is necessary to consider non-smooth functions on a bounded domain, Figueiredo et al. [25] studied a critical $p&q$ elliptic problem with discontinuous nonlinearities and obtained the existence of positive solutions. Yuan-Yu [41] also considered a class of discontinuous problems to which they obtained the existence of definite sign and sign-changing solutions. Moreover, Xiang-Zhang [39] dealt with a class of critical p -Kirchhoff type problems with a discontinuous nonlinearity in the whole space \mathbb{R}^N and proved the existence and multiplicity of solutions. Albuquerque et al. [18] discussed a coupled elliptic system with a discontinuous nonlinearity and showed the existence and behavior of positive solutions. For more information on non-smooth analysis, one can also see Alves-Yuan [2], dos Santos-Tavares [23], dos Santos-Figueiredo [22], Saoudi et al. [36].

Inspired by all of the above, we are going to consider a critical problem involving the mixed operator, variable singularity exponents, the convolution term and the discontinuity term simultaneously. To this end, we need to overcome the difficulty posed by each part. Here are the main results of this article

Theorem 1 *Let $(H_1) - (H_2)$ and $(F_1) - (F_2)$ hold for functions s, μ and f . Then there exist $\alpha', \lambda' > 0$ such that for every $\alpha \in (0, \alpha'), \lambda \in (0, \lambda')$, problem (1.1) admits at least one nontrivial weak solution, say u_α . Furthermore, $|\{x \in \Omega : u_\alpha(x) > \alpha\}| > 0$, where $|\cdot|$, when considered for a set, denotes the Lebesgue measure.*

Theorem 2 *Let u_α be a weak solution to problem (1.1). Then for any sequence $(\alpha_i) \subset (0, \alpha_0)$ with $\alpha_i \rightarrow 0^+$, there exists a subsequence such that $u_{\alpha_i} \rightarrow u_0$ in Z , where u_0 is a nontrivial weak solution to the following problem*

$$\begin{aligned} a(-\Delta)^{s(\cdot)}u + b(-\Delta)u &= \lambda|u|^{-\gamma(x)-1}u + \left(\int_{\Omega} \frac{F(y, u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x, u) \\ &+ \eta H(u)|u|^{r(x)-2}u, \text{ in } \Omega, \\ u &= 0, \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

which we shall denote by (P_0) .

The paper is organized as follows. In Section 2, we introduce some basic workspace and properties and crucial technical lemmas. In Section 3, we construct cut-off functions, auxiliary results, and verify all conditions of the non-smooth critical point theory. In Section 4, we prove the main results. In Section 5, we explain some more details.

2 Preliminaries

Let X be a space which is defined as follows

$$X = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} : u \text{ is measurable, } u|_{\Omega} \in L^2(\Omega), \frac{u(x) - u(y)}{|x - y|^{\frac{N+2s(x,y)}{2}}} \in L^2(Q) \right\}$$

and is equipped with the Gagliardo norm

$$\|u\|_X = \|u\|_2 + \left(\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dy dx \right)^{\frac{1}{2}},$$

where

$$\Omega \subset \mathbb{R}^N, \quad Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)).$$

Here, $\|u\|_2$ refers to the L^2 -norm of u .

We shall mostly use the subspace X_0 of X which is defined as

$$X_0 = \left\{ u \in X : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

with the norm

$$\|u\|_{s(\cdot),2} = \left(\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dy dx \right)^{\frac{1}{2}}.$$

Then $(X_0, \|\cdot\|_{s(\cdot),2})$ is a Hilbert space with respect to the norm $\|\cdot\|_{s(\cdot),2}$ induced by the inner product

$$\langle u, v \rangle = \iint_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s(x,y)}} dx dy$$

for each $u, v \in X_0$. The space X_0 is also separable and reflexive, see Azroul et al. [5, Lemma 2.3].

In order to study (1.1), we define a new workspace

$$Z = \left\{ u \in L^2(\Omega) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega, a\|u\|_{s(\cdot),2}^2 + b\|u\|_{1,2}^2 < \infty \right\}$$

with the norm:

$$\|u\| = (a\|u\|_{s(\cdot),2}^2 + b\|u\|_{1,2}^2)^{\frac{1}{2}}, \tag{2.1}$$

where

$$\|u\|_{1,2}^2 = \int_{\Omega} |\nabla u|^2 dx.$$

A standard approach from functional analysis guarantees the reflexivity of Banach space Z .

Remark 2 We make some useful observations.

1. The spaces $X, X_0 \subset W^{s(x),2}(\Omega)$, where

$$W^{s(x),2}(\Omega) = \left\{ u \in L^2(\Omega) : \frac{u(x) - u(y)}{|x - y|^{\frac{N+2s(x,y)}{2}}} \in L^2(\Omega) \right\}$$

is the usual fractional order Sobolev space, endowed with the Gagliardo norm

$$\|u\|_{W^{s(\cdot),2}(\Omega)} = \|u\|_2 + \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dy dx \right)^{\frac{1}{2}}.$$

2. Due to the “boundary condition” that $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the space X_0 is equivalent to $W_0^{s(x),2}(\mathbb{R}^N)$.
3. The first order Sobolev space will be defined as

$$W_0^{1,2}(\Omega) = \left\{ u \in L^2(\Omega) : u|_{\partial\Omega} = 0, \int_{\Omega} |\nabla u|^2 dx < \infty \right\}.$$

4. Any uniform constant appearing in an inequality will be denoted by the symbol C with or without a prefix/suffix.

Recall that the Lebesgue space of variable exponent is defined as follows (see Papa-georgiou et al. [35]).

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\} \quad (2.2)$$

and it is equipped with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx < 1 \right\}.$$

Furthermore, we define the best Sobolev constants as follows:

$$S_1 = \inf_{u \in X_0 \setminus \{0\}} \frac{\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dy dx}{\|u\|_{r(\cdot)}^2}, \quad (2.3)$$

$$S_2 = \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\|u\|_{2^*}^2}, \quad (2.4)$$

where $2^* = \frac{2N}{N-2}$. An important inequality which will be used consistently throughout the paper is the Hölder inequality for the Lebesgue space of variable exponent, see Fan-Zhao [24, Theorem 1.3].

Finally, the modular function is defined to be the mapping $\rho : L^{q(x)}(\Omega) \rightarrow \mathbb{R}$ which is defined as

$$\rho_{q(x)}(u) := \int_{\Omega} |u|^{q(x)} dx.$$

For some important properties of this modular function, we refer the reader to [35].

3 Auxiliary results

We define

$$J_{\alpha}(t) = \int_0^t j_{\alpha}(s) ds = \frac{1}{r(x)} H(t - \alpha)(t^{r(x)} - \alpha^{r(x)})$$

to be the primitive of

$$j_{\alpha}(t) = H(t - \alpha)|t|^{r(x)-2}t, g(x, t) = \left(F * \frac{1}{|x|^{\mu(\cdot)}}\right) f(x, t), F(x, t) = \int_0^t f(x, s) ds.$$

We call $u \in Z$ a *weak solution* to problem (1.1) if for every $\varphi \in Z$, we have

$$\begin{aligned} & a \iint_Q \frac{(u(x) - u(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy + b \int_{\Omega} \nabla u \cdot \nabla \varphi dx \\ & = \lambda \int_{\Omega} |u|^{-\gamma(x)-1} u \varphi dx + \int_{\Omega} g(x, u) \varphi dx + \eta \int_{\Omega} j_{\alpha}(u) \varphi dx. \end{aligned} \tag{3.1}$$

Clearly, the weak solution for equation (1.1) is a *critical point* of the following functional

$$\begin{aligned} I(u) &= \frac{a}{2} \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dx dy + \frac{b}{2} \int_{\Omega} |\nabla u|^2 dx - \lambda \int_{\Omega} \frac{|u|^{1-\gamma(x)}}{1 - \gamma(x)} dx \\ &\quad - \frac{1}{2} \int_{\Omega} \left(F * \frac{1}{|x|^{\mu(\cdot)}}\right) F(x, u) dx - \eta \int_{\Omega} J_{\alpha}(u) dx \\ &= \frac{1}{2} \|u\|^2 - \lambda \int_{\Omega} \frac{|u|^{1-\gamma(x)}}{1 - \gamma(x)} dx - \frac{1}{2} \int_{\Omega} \left(F * \frac{1}{|x|^{\mu(\cdot)}}\right) F(x, u) dx \\ &\quad - \eta \int_{\Omega} J_{\alpha}(u) dx. \end{aligned} \tag{3.2}$$

However, the functional I is not even differentiable, owing to the existence of the singular term and discontinuous nonlinear term. Therefore, we replace the original functional I with the cut-off functional \bar{I} .

3.1 Cut-off functional

To overcome the nondifferentiability of the functional I due to the singular term, we define the following cut-off function

$$\phi(x, t) = \begin{cases} |t|^{-\gamma(x)-1}t, & \text{if } |t| > \underline{u}_\lambda(x) \\ \underline{u}_\lambda(x)^{-\gamma(x)}, & \text{if } |t| \leq \underline{u}_\lambda(x). \end{cases}$$

Here, \underline{u}_λ refers to the solution of the singular problem as in Theorem 5 (see Section 5). We get the well-defined *cut-off* problem:

$$\begin{aligned} a(-\Delta)^{s(\cdot)}u - b\Delta u &= \lambda\phi(x, t) + \left(F * \frac{1}{|x|^{\mu(\cdot)}} \right) f(x, u) \\ &\quad + \eta H(u - \alpha)|u|^{r(x)-2}u \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned} \quad (3.3)$$

We say that $\tilde{u} \in Z$ is a *weak solution* for equation (3.3) if we have

$$\begin{aligned} a \iint_Q \frac{(\tilde{u}(x) - \tilde{u}(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy + b \int_\Omega \nabla \tilde{u} \cdot \nabla \varphi dx \\ = \lambda \int_\Omega \phi(x, \tilde{u}) \varphi dx + \int_\Omega \left(\int_\Omega \frac{F(y, \tilde{u}(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, \tilde{u}) \varphi dx + \eta \int_\Omega j_\alpha(\tilde{u}) \varphi dx \end{aligned}$$

for any $\varphi \in Z$. Consequently, we shall define the energy functional

$$\begin{aligned} \bar{I}(u) &= \frac{a}{2} \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dx dy + \frac{b}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \Phi(x, u) dx \\ &\quad - \frac{1}{2} \int_\Omega \left(\int_\Omega \frac{F(y, u(y))}{|x - y|^{\mu(x,y)}} dy \right) F(x, u) dx - \eta \int_\Omega J_\alpha(u) dx, \end{aligned} \quad (3.4)$$

where Φ is the primitive of the function ϕ , i.e. $\Phi(x, t) = \int_0^t \phi(x, s) ds$.

Remark 3 If u_0 is a global minimizer of I , then clearly $I(u_0) = I(|u_0|)$. Trivially, we have $I(u_0) \leq I(|u_0|)$. Furthermore, using

$$\begin{aligned} \iint_Q \frac{||u(x)| - |u(y)||^2}{|x - y|^{N+2s(x,y)}} dx dy &\leq \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dx dy, \\ - \int_\Omega H(|u| - \alpha)|u|^{r(x)} dx &\leq - \int_\Omega H(u - \alpha)|u|^{r(x)} dx, \end{aligned}$$

and the property of F that it is even, we have $I(u_0) \geq I(|u_0|)$. Hence, we seek for a minimizer of the functional I which is nonnegative. Furthermore, the solution is positive a.e. in Ω due to singular terms.

Remark 4 We note that the weak solution of new problem (3.3) with $u > u_\lambda$ is also a weak solution to original problem (1.1). Refer to the proof of Theorem 5 in Section 5.

3.2 Definitions from convex analysis

Let us now see a few definitions that are very important in the analysis of functionals that are nondifferentiable but *locally Lipschitz continuous* (see [23, Section 2]).

Definition 1 (see [23, Section 2]) Let J be a locally Lipschitz continuous functional. The *directional derivative* of J at u in the direction of $z \in V$ is defined by

$$\tilde{J}(u; z) = \limsup_{\substack{h \rightarrow 0 \\ \xi \downarrow 0}} \frac{J(u + h + \xi z) - J(u + h)}{\xi}.$$

From dos Santos-Tavares [23] we know that $\tilde{J}(u; \cdot)$ is convex and continuous. The subdifferential of $\tilde{J}(u; \cdot)$ at $w \in V$ is defined to be the following set:

$$\partial \tilde{J}(u; w) = \{v \in V^* : \tilde{J}(u; z) \geq \tilde{J}(u; w) + \langle v, z - w \rangle, z \in V\}.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pair between V and V^* (the dual space of V). The *generalized gradient* (see [23]) of J at u is defined as

$$\partial J(u) = \{v \in V^* : \langle v, z \rangle \leq \tilde{J}(u; z), z \in V\},$$

which is convex and weak*-compact by [14]. Clearly, $\partial J(u)$ is nonempty and is the subdifferential of $\tilde{J}(u; 0)$ as $\tilde{J}(u; 0) = 0$. Then by [14], the function

$$\Lambda_J(u) = \min\{\|v\|_{V^*} : v \in \partial J(u)\} \tag{3.5}$$

exists and is lower semi-continuous.

A function \bar{u} is said to be a *critical point* of J if $0 \in \partial J(\bar{u})$, whereas $c \in \mathbb{R}$ is said to be a *critical value* of J if $J(\bar{u}) = c$ for a critical point $\bar{u} \in V$. If J is a C^1 functional then $\partial J(u) = \{J'(u)\}$.

Remark 5 Define

$$\bar{I}_\eta(u) = \int_\Omega J_\alpha(u) dx,$$

and

$$E_0(u) = \frac{\|u\|^2}{2} - \int_\Omega \Phi(x, u) dx - \int_\Omega \left(F * \frac{1}{|x|^{\mu(\cdot)}} \right) F(x, u) dx.$$

Then

$$\partial \bar{I}(w) \subset \{E'_0(w)\} - \partial \bar{I}_\eta(w) \text{ for all } w \in Z. \tag{3.6}$$

Remark 6 We shall be interested in solutions that are greater than $\alpha > 0$. We shall show that $\{u > \alpha\}$ is of positive Lebesgue measure for a small range of α .

Next, consider the energy functional

$$\begin{aligned} \mathcal{I}(u) = & \frac{1}{2} \|u\|^2 - \lambda \int_{\Omega} \frac{(u^+)^{1-\gamma(x)}}{1-\gamma(x)} dx - \frac{1}{2} \int_{\Omega} \left(F * \frac{1}{|x|^{\mu(\cdot)}} \right) F(x, u^+) dx \\ & - \eta \int_{\Omega} J_{\alpha}(u^+) dx. \end{aligned}$$

A critical point of \mathcal{I} is clearly a weak solution of (1.1). Therefore we have

$$\begin{aligned} 0 = & a \iint_Q \frac{(u(x) - u(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy + b \int_{\Omega} \nabla u \cdot \nabla \varphi dx \\ & - \lambda \int_{\Omega} (u^+)^{-\gamma(x)} \varphi dx - \frac{1}{2} \int_{\Omega} \left(F * \frac{1}{|x|^{\mu(\cdot)}} \right) f(x, u^+) \varphi dx - \eta \int_{\Omega} j_{\alpha}(u^+) \varphi dx. \end{aligned}$$

On testing with $\varphi = u^-$, we get that the Lebesgue measure of the set $\{u < 0\}$ is zero. Thus without loss of generality we may assume that $u > 0$ a.e. in Ω .

We recall some properties of the functionals \bar{I}_{η} in the following lemma, the proof of which follows from the argument in dos Santos-Figueiredo [21, Lemma 3.1].

Lemma 1 $\bar{I}_{\eta} : L^{r(x)}(\Omega) \rightarrow \mathbb{R}$ is a locally Lipschitz functional such that $\partial \bar{I}_{\eta}(u) \subset [f_{-\alpha}(u(x)), \bar{f}_{\alpha}(u(x))]$ a.e. in Ω and

$$[f_{-\alpha}(u), \bar{f}_{\alpha}(u)] = \begin{cases} \{0\}, & \text{if } u < \alpha \\ [0, u^{r(x)-1}], & \text{if } u = \alpha \\ \{u^{r(x)-1}\}, & \text{if } u > \alpha. \end{cases} \quad (3.7)$$

3.3 The mountain pass theorem

An essential result which will be required henceforth is the Mountain Pass Theorem for a locally Lipschitz nondifferentiable functional due to Grossinho-Tersian [29] and Rădulescu [34].

The following theorem gives an energy level, below which the $(PS)_c$ condition is satisfied by the functional \bar{I} .

Theorem 3 The functional \bar{I} satisfies the $(PS)_c$ -condition for

$$c < \left(\frac{1}{2} - \frac{1}{\kappa} \right) S_1^{\frac{\mathcal{V}}{\mathcal{V}-2}} - \left(\frac{1}{2} - \frac{1}{\kappa} \right)^{-\frac{1-\delta}{1+\delta}} \left(\frac{\lambda}{1-\gamma^+} C(\Omega, N) \right)^{\frac{2}{1+\delta}} = c_*$$

where $c_* > 0$ for any $\lambda \in (0, \lambda')$, λ' being sufficiently small, where \mathcal{V} is a constant and $2 < \kappa < r^- < 2_s^{*-}$.

Proof We first choose those sequences that will be sufficient to prove this lemma. We shall discard it if $\{u_i\} \subset X_0$ is an eventually zero sequence. Assume that $\{u_i\} \subset X_0$ is a sequence with infinitely many terms equal to 0. Then we can choose a subsequence of $\{u_i\}$ that is nonzero. Let this sequence $\{u_i\}$ be such that

$$\bar{I}(u_i) \rightarrow c \text{ and } \Lambda_{\bar{I}}(u_i) \rightarrow 0 \tag{3.8}$$

as $i \rightarrow \infty$. Let $\{v_i\} \subset Z'$ be such that $\Lambda_{\bar{I}}(u_i) = \|v_i\|_{Z'}$, where Z' is the dual space of Z . By Remark 5, there exists $\eta_i \in \partial \bar{I}_{\eta}(u_i)$ satisfying the following

$$\begin{aligned} \langle v_i, \varphi \rangle = & a \iint_Q \frac{(u(x) - u_i(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy + b \int_{\Omega} \nabla u_i \cdot \nabla \varphi dx \\ & - \lambda \int_{\Omega} \phi(x, u_i) \varphi dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_i(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, u_i) \varphi dx - \langle \eta_i, \varphi \rangle. \end{aligned}$$

For $2 < \kappa < r^- < 2_s^{*-}$ and $\kappa < \theta$, consider

$$\begin{aligned} & \bar{I}(u_i) - \frac{1}{\kappa} \langle u_i, u_i \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\kappa} \right) \|u_i\|^2 + \frac{\lambda}{\kappa} \int_{\Omega} \phi(x, u_i) u_i dx \\ & \quad - \lambda \int_{\Omega} \Phi(x, u_i) dx - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_i(y))}{|x - y|^{\mu(x,y)}} dy \right) F(x, u_i) dx \\ & \quad + \frac{1}{\kappa} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_i(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, u_i) u_i dx - \eta \int_{\Omega} J_{\alpha}(u_i) dx + \frac{\eta}{\kappa} \langle \eta_i, u_i \rangle \\ & \geq \left(\frac{1}{2} - \frac{1}{\kappa} \right) \|u_i\|^2 + \lambda \left(\frac{1}{\kappa} - \frac{1}{1 - \gamma^-} \right) \int_{\Omega} \psi(x, u_i) dx \\ & \quad + \left(\frac{\theta}{2\kappa} - \frac{1}{2} \right) \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_i(y))}{|x - y|^{\mu(x,y)}} dy \right) F(x, u_i) dx \\ & \quad + \eta \left(\frac{1}{\kappa} - \frac{1}{r^-} \right) \int_{\Omega \cap \{u_i > \alpha\}} |u_i|^{r(x)} dx + \frac{\eta}{\kappa} \langle \eta_i, u_i \rangle + \frac{\eta}{r^+} \alpha^{\gamma} |\{u_i = \alpha\}| \\ & \geq \left(\frac{1}{2} - \frac{1}{\kappa} \right) \|u_i\|^2 - \frac{\lambda}{1 - \gamma^+} C(\Omega, N) \|u_i\|^{1-\delta} \\ & \quad + \eta \left(\frac{1}{\kappa} - \frac{1}{r^-} \right) \int_{\Omega \cap \{u_i > \alpha\}} |u_i|^{r(x)} dx. \end{aligned}$$

Here,

$$\psi(x, u_i) = \begin{cases} |u_i|^{1-\gamma(x)}, & \text{if } u_i > \underline{u}_{\lambda} \\ u_i \underline{u}_{\lambda}^{-\gamma(x)}, & \text{if } u_i \leq \underline{u}_{\lambda} \end{cases}$$

and

$$\gamma = \begin{cases} r^-, & \|w\|_{r(\cdot)} < 1 \\ r^+, & \|w\|_{r(\cdot)} > 1. \end{cases} \tag{3.9}$$

Define

$$\delta = \begin{cases} \gamma^+, & \|w^{1-\gamma(x)}\|_{\frac{1}{1-\gamma(x)}} < 1 \\ \gamma^-, & \|w^{1-\gamma(x)}\|_{\frac{1}{1-\gamma(x)}} > 1. \end{cases} \tag{3.10}$$

Furthermore,

$$\bar{I}(u_i) - \frac{1}{\kappa} \langle v_i, u_i \rangle \leq |\bar{I}(u_i)| + \frac{1}{\kappa} \|v_i\|_{Z'} \|u_i\| \leq c + 1 + \|u_i\| + o_i(1). \tag{3.11}$$

Finally, we have

$$\begin{aligned} c + 1 + \|u_i\| + o_i(1) &\geq \bar{I}(u_i) - \frac{1}{\kappa} \langle v_i, u_i \rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right) \|u_i\|^2 - \frac{\lambda}{1 - \gamma^+} C(\Omega, N) \|u_i\|^{1-\delta} \\ &\quad + \eta \left(\frac{1}{\kappa} - \frac{1}{r^-}\right) \int_{\Omega} H(u_i - \alpha) |u_i|^{r(x)} dx. \end{aligned} \tag{3.12}$$

This implies the boundedness of the sequence $\{u_i\}$ in Z . Thus up to a subsequence, still denoted by the notation of the sequence, the following holds

$$\begin{aligned} u_i &\rightharpoonup u \text{ in } Z, \quad u_i(x) \rightarrow u(x) \text{ a.e. in } \Omega, \quad \eta_i \overset{*}{\rightharpoonup} D \text{ in } L^{r(x)'}(\Omega) \text{ as } i \rightarrow \infty, \tag{3.13} \\ \text{and } \|u_i - u\|^2 &\rightarrow L \text{ as } i \rightarrow \infty \tag{3.14} \end{aligned}$$

where $r(x)' = \frac{r(x)}{r(x)-1}$ is the conjugate of $r(x)$. If $L = 0$, then $u_i \rightarrow u$ and we have nothing to prove. Therefore, let $L > 0$. From (3.13) we have

$$\int_{\Omega} |u_i|^{r(x)-1} \chi_{\{u_i > \alpha - \frac{1}{n}\}} \varphi dx \rightarrow \int_{\Omega} |u|^{r(x)-1} \chi_{\{u > \alpha\}} \varphi dx \text{ for all } \varphi \in L^{r(x)}(\Omega),$$

Here, χ denotes the indicator function. Invoking the Brezis-Lieb Lemma [9] we have

$$\begin{aligned} \|u_i\|^2 &= \|u_i - u\|^2 + \|u\|^2 + o_i(1), \tag{3.15} \\ \int_{\Omega} |u_i \chi_{\{u_i > \alpha - \frac{1}{n}\}}|^{r(x)} dx &= \int_{\Omega} |u_i \chi_{\{u_i > \alpha - \frac{1}{n}\}} - u \chi_{\{u > \alpha\}}|^{r(x)} dx \\ &\quad + \int_{\Omega} |u \chi_{\{u > \alpha\}}|^{r(x)} dx + o_i(1). \tag{3.16} \end{aligned}$$

The embedding result $W^{s(\cdot),2}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, proved in [43, Theorem 3.1], guarantees that $u_i \rightarrow u$ in $L^{q(x)}(\Omega)$, where $N > 2s(x, y)$ for $(x, y) \in \bar{\Omega} \times \bar{\Omega}$ and there exists $\varepsilon = \varepsilon(x) > 0$ such that

$$\sup_{y \in \Omega_{x,\varepsilon}} \{q(y)\} \leq \frac{2N}{N - 2 \inf_{(y,z) \in \Omega_{x,\varepsilon} \times \Omega_{x,\varepsilon}} \{s(y, z)\}}, \tag{3.17}$$

where $\Omega_{z,\varepsilon} = B_\varepsilon(z) \cap \Omega$, for $z \in \Omega$. Also, the following always holds

$$\int_\Omega \phi(x, u_i) dx \leq \int_\Omega u_i^{-\gamma(x)} dx.$$

Thus it follows that

$$\left| \int_\Omega \phi(x, u_i)(u_i - u) dx \right| \leq o_i(1). \tag{3.18}$$

Consider now

$$\begin{aligned} o_i(1) &= \langle u_i, u_i - u \rangle \\ &= a \iint_Q \frac{(u_i(x) - u_i(y))}{|x - y|^{N+2s(x,y)}} ((u_i - u)(x) - (u_i - u)(y)) dx dy \\ &\quad + b \int_\Omega \nabla u_i \cdot \nabla (u_i - u) dx - \lambda \int_\Omega \phi(x, u_i)(u_i - u) dx \\ &\quad - \int_\Omega \left(\int_\Omega \frac{F(y, u_i(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, u_i)(u_i - u) dx - \langle \eta_i, u_i - u \rangle \\ &= \|u_i\|^2 - \iint_Q \frac{(u_i(x) - u_i(y))}{|x - y|^{N+2s(x,y)}} (u(x) - u(y)) dx dy \\ &\quad - \lambda \int_\Omega \phi(x, u_i)(u_i - u) dx \\ &\quad - \int_\Omega \left(\int_\Omega \frac{F(y, u_i(y))}{|x - y|^{\mu(x,y)}} dy \right) f(x, u_i)(u_i - u) dx - \langle \eta_i, u_i - u \rangle \\ &= \|u_i - u\|^2 - \int_\Omega |u_i|^{r(x)-1} (u_i - u) \chi_{\{u_i > \alpha - \frac{1}{n}\}} dx + o_i(1) \\ &\quad \text{(invoking (3.13), (3.15))} \\ &= \|u_i - u\|^2 - \int_\Omega |u_i|^{r(x)} \chi_{\{u_i > \alpha - \frac{1}{n}\}} dx + \int_\Omega |u|^{r(x)} \chi_{\{u > \alpha\}} dx \\ &\quad + o_i(1) \\ &= \|u_i - u\|^2 - \int_\Omega |u_i \chi_{\{u_i > \alpha - \frac{1}{n}\}} - u \chi_{\{u > \alpha\}}|^{r(x)} dx + o_i(1) \\ &\geq \|u_i - u\|^2 - \|u_i \chi_{\{u_i > \alpha - \frac{1}{n}\}} - u \chi_{\{u > \alpha\}}\|_{r(\cdot)}^V + o_i(1). \end{aligned}$$

Therefore

$$S_1 \leq \frac{\|u_i - u\|^2}{\|u_i \chi_{\{u_i > \alpha - \frac{1}{n}\}} - u \chi_{\{u > \alpha\}}\|_{r(\cdot)}^2} \leq L^{\frac{\nu-2}{\nu}} + o_i(1). \tag{3.19}$$

Thus $L > S_1^{\frac{\nu}{\nu-2}} + o_i(1)$. It now follows from (3.12) that

$$\begin{aligned} c &\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right) \|u_i\|^2 - \frac{\lambda}{1 - \gamma^+} C(\Omega, N) \|u_i\|^{1-\delta} \\ &\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right) (L + \|u\|^2) - \frac{\lambda}{1 - \gamma^+} C(\Omega, N) \|u_i\|^{1-\delta} \\ &\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right) (L + \|u\|^2) - \left(\frac{1}{2} - \frac{1}{\kappa}\right) \|u\|^2 \\ &\quad - \left(\frac{1}{2} - \frac{1}{\kappa}\right)^{-\frac{1-\delta}{1+\delta}} \left(\frac{\lambda}{1 - \gamma^+} C(\Omega, N)\right)^{\frac{2}{1+\delta}} \\ &\geq \left(\frac{1}{2} - \frac{1}{\kappa}\right) S_1^{\frac{\nu}{\nu-2}} - \left(\frac{1}{2} - \frac{1}{\kappa}\right)^{-\frac{1-\delta}{1+\delta}} \left(\frac{\lambda}{1 - \gamma^+} C(\Omega, N)\right)^{\frac{2}{1+\delta}} + o_i(1) \\ &= c^* + o_i(1) \end{aligned}$$

with $c^* > 0$ for a small range of λ , say $\lambda \in (0, \lambda')$. This leads to a contradiction to the assumption that $c < c^*$. Therefore $L = 0$ and hence $u_i \rightarrow u$ in Z as $i \rightarrow \infty$. \square

The next lemma will guarantee the verification of the hypotheses of the Mountain Pass Theorem for nondifferentiable functionals.

Lemma 2 *Let $(H_1) - (H_2)$ and $(F_1) - (F_2)$ hold. Then there exists $r_1 > 0, \lambda_0 = \lambda_0(r_1), r_2 > 0, m_0 > 0, \sigma \in Z$ such that for every $\alpha > 0$ and $\lambda \in (0, \lambda_0)$ the following holds*

1. $\sup_{m \in [0, m_0]} \bar{I}(m\sigma) < c^*$;
2. $\bar{I}(v) \geq r_2$ for every $v \in \partial B_{r_1}(0) \cap Z$, where r_1, r_2 are independent of α , and $\partial B_{r_1}(0) = \{v \in Z : \|v\| = r_1\}$;
3. $\bar{I}(m_0\sigma) < 0$ for $\|m_0\sigma\| > r_1$.

Proof Let $\sigma \in Z$ be fixed with $\sigma > 0$ in Ω and $\|\sigma\| = 1$. Now for sufficiently small $m_0 > 0$ and $\lambda \in (0, \lambda'')$ for sufficiently small λ'' , we have

$$\bar{I}(m\sigma) \leq \frac{1}{2} m_0^2 - \frac{\lambda}{1 - \gamma^-} \int_{\Omega} m_0^{1-\gamma(x)} \sigma^{1-\gamma(x)} dx < 0 \text{ and } \frac{1}{2} m_0^2 < c^* \tag{3.20}$$

for every $m \in [0, m_0]$, thus proving properties 1 and 3. Furthermore, by the embedding result in [43, Theorem 3.1],

$$\bar{I}(u) \geq \frac{1}{2} \|u\|^2 - \lambda C_1 \|u\|^{1-\delta} - C_2 \|u\|^{2A} - \eta C_3 \|u\|^\nu \tag{3.21}$$

where

$$\mathcal{A} = \begin{cases} \tau^-, & \|v\|_{\beta+\tau(\cdot)} < 1 \\ \tau^+, & \|v\|_{\beta+\tau(\cdot)} > 1. \end{cases} \tag{3.22}$$

Here, we have used the Hardy-Littlewood-Sobolev inequality for variable exponent given in [1, Proposition 2.4] and the conditions in (F_1) , to estimate the Choquard term. Define

$$\tilde{m}(t) = t^{1+\delta} - 2C_2t^{2\mathcal{A}-1+\delta} - 2\eta C_3t^{\nu-1+\delta} - 2\lambda C_1.$$

This function achieves a maximum, say, at $t = r_1 > 0$, i.e. $\max_{t \in [0,1]} \{\tilde{m}(t)\} = \tilde{m}(r_1) > 0$.

We let

$$\lambda''' = \frac{1}{2C_1} \tilde{m}(r_1) \text{ and } \lambda_0 = \min\{\lambda', \lambda'', \lambda'''\},$$

where λ' is obtained from Theorem 3. Then for any $u \in Z$ such that $\|u\| = r_1$ and for any $\lambda \in (0, \lambda_0)$, we have

$$\bar{I}(u) \geq \frac{1}{2}r_1^{1-\delta} \tilde{m}(r_1) = r_2.$$

Thus for $\lambda \in (0, \lambda_0)$, property 2 holds. □

4 Proof of the main results

This section is devoted to the proofs of the main results stated in Theorems 1 and 2.

4.1 Proof of Theorem 1

Let $m_0, \sigma, r_1, r_2, \lambda'$ be as given in Lemma 2. We further denote

$$c_\alpha = \inf_{\zeta \in \Gamma} \max_{t \in [0,1]} \bar{I}(\zeta(t)) \text{ and } \Gamma = \{\zeta \in C([0, 1]; Z) : \zeta(0) = 0, \zeta(1) = m_0\sigma\}.$$

Since \bar{I} fulfills the conditions of the theorem (as proved in Lemma 2), the existence of a $(PS)_{c_\alpha}$ sequence $\{w_i\} \subset Z$, such that $|w_i| > \underline{u}_\lambda$ a.e. in Ω , is guaranteed that obeys $\bar{I}(w_i) = c_\alpha + o_i(1)$ and $\Lambda_{\bar{I}}(w_i) = o_i(1)$. The properties given in Lemma 2 imply

$$r_2 \leq c_\alpha < c^* = \left(\frac{1}{2} - \frac{1}{\kappa}\right) S_1^{\frac{\nu}{\nu-2}} - \left(\frac{1}{2} - \frac{1}{\kappa}\right)^{-\frac{1-\delta}{1+\delta}} \left(\frac{\lambda}{1-\gamma^+} C(\Omega, N)\right)^{\frac{2}{1+\delta}} \tag{4.1}$$

for all $\alpha > 0$. Thus, by [43, Theorem 3.1] there exists $w_\alpha \in Z$ such that, up to a subsequence which is still denoted by the notation of the sequence, $w_i \rightarrow w_\alpha$ in Z

as $n \rightarrow \infty$. This establishes $\bar{I}(w_\alpha) = c_\alpha$ and $0 \in \partial \bar{I}(w_\alpha)$. Further, by (3.6) and Lemma 1, we have

$$\begin{aligned}
 a \iint_Q \frac{(w_\alpha(x) - w_\alpha(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy + b \int_\Omega \nabla w_\alpha \cdot \nabla \varphi dx \\
 = \lambda \int_\Omega \Phi(x, w_\alpha) \varphi dx + \int_\Omega \left(\frac{F(y, u)}{|x - y|^{\mu(x,y)}} \right) dy \int_\Omega f(x, u) \varphi dx + \eta \int_\Omega j_\alpha \varphi dx,
 \end{aligned}$$

for every $\varphi \in Z$, where $f_\alpha(x) \in [f_\alpha(w_\alpha(x)), \bar{f}_\alpha(w_\alpha(x))]$ a.e. in Ω . Clearly $|w_\alpha| \geq u_\lambda > 0$ a.e. in Ω . Therefore, w_α is a nontrivial weak solution of (3.3).

We now claim is that the Lebesgue measure of the set $\{x \in \Omega : w_\alpha(x) > \alpha\}$ is positive. Suppose to the contrary, that $|\{x \in \Omega : w_\alpha(x) > \alpha\}| = 0$. Then we have $w_\alpha \leq \alpha$ a.e. in Ω . Therefore, from the weak formulation of the problem (3.3) and on testing with $\varphi = w_\alpha$ we get

$$\begin{aligned}
 2r_2 \leq \|w_\alpha\|^2 &= \lambda \int_\Omega \phi(x, w_\alpha) w_\alpha dx + \int_\Omega \int_\Omega \left(\frac{F(y, w_\alpha)}{|x - y|^{\mu(x,y)}} dy \right) f(x, w_\alpha) w_\alpha dx \\
 &\quad + \int_\Omega j_\alpha w_\alpha dx \\
 &\leq \lambda \alpha^{1-\delta} |\Omega| + \frac{C}{\tau^-} |\Omega|^{\frac{1}{\tau^+}} \alpha^2 + \alpha^\nu |\Omega|.
 \end{aligned} \tag{4.2}$$

This is impossible since r_2 is independent of α . Thus there exists a small α , say α' , such that for all $\alpha \in (0, \alpha')$ we have $|\{x \in \Omega : w_\alpha(x) > \alpha\}| > 0$. \square

Since by Theorem 1 we have proved the existence of a nontrivial solution to (1.1) it will now be interesting to investigate the nature of this obtained solution under the limit $\alpha \rightarrow 0^+$.

Theorem 4 *Let r_2 be as in Lemma 2 and c_α, c_0 be as in the proof of Theorem 1. Then $\lim_{\alpha \rightarrow 0^+} c_\alpha = c_0 \geq r_2$.*

Proof Let $\bar{I}_0 : Z \rightarrow \mathbb{R}$ be the associated functional to the problem (1.1) for $\alpha = 0$ which is defined as follows.

$$\begin{aligned}
 \bar{I}_0(u) &= \frac{1}{2} \|u\|^2 - \lambda \int_\Omega \frac{|u|^{1-\gamma(x)}}{1 - \gamma(x)} dx - \int_\Omega \int_\Omega \left(\frac{F(y, u)}{|x - y|^{\mu(x,y)}} dy \right) F(x, u) dx \\
 &\quad - \int_\Omega H(u) \frac{|u|^{r(x)}}{r(x)} dx \text{ for all } u \in Z.
 \end{aligned} \tag{4.3}$$

Define

$$c_0 = \inf_{\xi \in \Gamma} \max_{t \in [0,1]} \bar{I}_0(\xi(t)) \tag{4.4}$$

where

$$\Gamma = \{\xi \in C([0, 1]; Z) : \xi(0) = 0, \xi(1) = m_0 \sigma\}.$$

The notations m_0, σ are as in Theorem 2. Since

$$J_\alpha(t) = \int_0^t H(t - \alpha)t^{r(x)-1} dt = \chi_{\{t>\alpha\}} \frac{1}{r(x)} (|t|^{r(x)} - \alpha^{r(x)}),$$

this implies that

$$\begin{aligned} & \left| \int_\Omega \frac{1}{r(x)} |u|^{r(x)} dx - \int_\Omega J_\alpha(u) dx \right| \\ &= \left| \int_\Omega \frac{|u|^{r(x)}}{r(x)} \chi_{\{u \leq \alpha\}} dx + \int_\Omega \frac{\alpha^{r(x)}}{r(x)} \chi_{\{u > \alpha\}} dx \right| \leq \frac{2\alpha^\nu}{r^-}. \end{aligned} \tag{4.5}$$

Clearly, $\bar{I}_0(u) \leq \bar{I}(u)$ for all $u \in Z$. Thus $c_0 \leq c_\alpha$ for any $\alpha > 0$. From (4.5) we establish that

$$\bar{I}(u) = \bar{I}_0(u) + o_\alpha(1) \text{ for all } u \in Z. \tag{4.6}$$

This further gives

$$\bar{I}(\xi(t)) = \bar{I}_0(\xi(t)) + o_\alpha(1) \text{ for all } \xi \in \Gamma, t \in [0, 1] \tag{4.7}$$

and thus $c_\alpha = c_0 + o_\alpha(1)$. So from (4.7) and using the properties proved in Lemma 2 we conclude from the Mountain Pass Theorem that $\lim_{\alpha \rightarrow 0^+} c_\alpha = c_0$. \square

4.2 Proof of Theorem 2

Let $w_\alpha \in Z$ be a weak solution of (1.1). So,

$$a \iint_Q \frac{(w_\alpha(x) - w_\alpha(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy + b \int_\Omega \nabla w_\alpha \cdot \nabla \varphi dx \tag{4.8}$$

$$\begin{aligned} &= \lambda \int_\Omega |w_\alpha|^{-\gamma(x)-1} w_\alpha \varphi dx + \int_\Omega \int_\Omega \left(\frac{F(y, w_\alpha)}{|x - y|^{\mu(x,y)}} dy \right) f(x, w_\alpha) \varphi dx \\ &+ \eta \int_\Omega j_\alpha \varphi dx \end{aligned} \tag{4.9}$$

for all $\varphi \in Z$. Consider a $(PS)_c$ sequence, say (w_i) , such that $w_i \rightarrow w_\alpha$ for a subsequence in Z as $i \rightarrow \infty$. Therefore the following holds.

$$c_\alpha + 1 + \|w_i\| + o(1) \geq \left(\frac{1}{2} - \frac{1}{\kappa} \right) \|w_\alpha\|^2 - \lambda \left(\frac{1}{1 - \gamma^+} - \frac{1}{\kappa} \right) C(\Omega, N) \|w_i\|^{1-\delta}.$$

Thus,

$$c_\alpha + 1 + \|w_\alpha\| \geq \left(\frac{1}{2} - \frac{1}{\kappa} \right) \|w_\alpha\|^2 - \lambda \left(\frac{1}{1 - \gamma^+} - \frac{1}{\kappa} \right) C(\Omega, N) \|w_\alpha\|^{1-\delta}$$

for all $\alpha > 0$. This implies that w_α is uniformly bounded in Z .

Let $\{\alpha_i\}$ be a sequence such that $\alpha_i \rightarrow 0^+$. Therefore, there exists up to a subsequence, w_0 such that $w_{\alpha_i} \rightharpoonup w_0$ in Z . From the embedding result in [43, Theorem 3.1] we have $w_{\alpha_i} \rightarrow w_0$ in $L^{q(x)}(\Omega)$. Hence, $w_{\alpha_i}(x) \rightarrow w_0(x)$ a.e. in Ω . Also, $j_{\alpha_i}(w_{\alpha_i}) \rightarrow |w_0|^{r(x)-1}w_0$. Thus

$$\iint_Q \frac{(w_0(x) - w_0(y))}{|x - y|^{N+2s(x,y)}} (\varphi(x) - \varphi(y)) dx dy = \lambda \int_\Omega |w_0|^{-\gamma(x)-1} w_0 \varphi dx + \int_\Omega \int_\Omega \left(\frac{F(y, w_0)}{|x - y|^{\mu(x,y)}} dy \right) f(x, w_0) \varphi dx + \eta \int_\Omega |w_0|^{r(x)-2} w_0 \varphi dx \text{ for all } \varphi \in Z.$$

In the above equation, Remark 7 in Section 5 has been used to pass the limit in the singular term. Furthermore, using (4.1), we obtain

$$r_2 \leq \bar{I}(w_{\alpha_i}) = c_{\alpha_i} < c^* = \left(\frac{1}{2} - \frac{1}{\kappa} \right) S_1^{\frac{\gamma}{\nu-2}} - \left(\frac{1}{2} - \frac{1}{\kappa} \right)^{-\frac{1-\delta}{1+\delta}} \left(\frac{\lambda}{1-\gamma^+} C(\Omega, N) \right)^{\frac{2}{1+\delta}} \tag{4.10}$$

for all $n \in \mathbb{N}$. From the inequality (4.5) we have

$$c_{\alpha_i} = \bar{I}(u_{\alpha_i}) = \bar{I}_0(u_{\alpha_i}) + o_{\alpha_i}(1). \tag{4.11}$$

Thus by (4.6)-(4.10) we see that

$$w_{\alpha_i} \rightarrow w_0, \text{ in } Z \text{ as } \alpha_i \rightarrow 0^+. \tag{4.12}$$

So $c_0 = \bar{I}_0(w_0) \geq r_2 > 0$. This completes the proof. □

5 Appendix

In this section we shall prove the existence of a solution to the *singular* problem.

Theorem 5 *For any $\lambda \in (0, \lambda_0)$ there exists a positive weak solution \underline{u}_λ to*

$$\begin{aligned} a(-\Delta)^{s(\cdot)} u + b(-\Delta)u &= \lambda |u|^{-\gamma(x)-1} u, \text{ in } \Omega \\ u &= 0, \text{ in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{5.1}$$

such that $\underline{u}_\lambda < u$ a.e. in Ω , where u is a solution to (1.1).

Proof We first prove the existence of \underline{u}_λ . We observe that the functional \bar{I}_{sing} associated to (5.1) can be defined as follows:

$$\bar{I}_{\text{sing}}(u) = \frac{a}{2} \iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dx dy + \frac{b}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega |u|^{1-\gamma(x)} dx \tag{5.2}$$

for $u \in Z$. The rest of the proof follows the argument in the proof of Choudhuri-Repovš [16, Lemma A.2].

□

Remark 7 We claim that

$$\lim_{\alpha_i \rightarrow 0^+} \int_{\Omega} \frac{\varphi}{v_{\alpha_i}^{\gamma(x)}} = \int_{\Omega} \frac{\varphi}{(v_0^+)^{\gamma(x)}} < \infty \text{ for all } \varphi \in Z, \tag{5.3}$$

where v_{α_i} is a positive weak solution to (1.1) with $\alpha = \alpha_i$ as obtained in Theorem 1 and u_0 is the uniform limit of the sequence (u_{α_i}) , up to a subsequence, in Ω obtained by [43, Theorem 3.1].

Proof of the claim We first denote

$$S_{\alpha_i} = \{x \in \Omega : v_{\alpha_i}(x) = 0\}.$$

As $v_{\alpha_i}^{-\gamma(x)} \varphi \in L^1(\Omega)$ for every $\varphi \in Z$, the Lebesgue measure of S_{α_i} (denoted in short as $|S_{\alpha_i}|$) is zero, i.e. $|S_{\alpha_i}| = 0$. Due to the sub-additive property of the Lebesgue measure implies, we have that $|\bigcup S_{\alpha_i}| = 0$.

Let $|D| < \epsilon$ such that for $x \in \Omega \setminus D$, $v_0(x) = 0$. We can construct the set D by the Egorov Theorem. Therefore if we are able prove that

$$|T_{\alpha_i}| = |\{x \in \Omega \setminus D : v_{\alpha_i}(x) \rightarrow 0 \text{ as } \alpha_i \rightarrow 0^+\}| = 0,$$

then (5.3) gets proven. We define

$$T_{\delta, \alpha_i} = \{x \in \Omega \setminus D : |v_{\alpha_i}(x)| < \delta\}.$$

Apparently, for a fixed α_i , by the uniform convergence, $|T_{\delta, \alpha_i}| \rightarrow 0$ as $\delta \rightarrow 0^+$. Furthermore, we have

$$\bigcup_{\delta, \alpha_i \in \mathbb{R}^+} T_{\delta, \alpha_i} = \bigcup_{\delta} \bigcap_{\delta \leq \alpha_i} T_{\delta, \alpha_i}.$$

Thus for a fixed α_i ,

$$\left| \bigcap_{\delta \leq \alpha_i} T_{\delta, \alpha_i} \right| = \lim_{\delta \rightarrow 0^+} |T_{\delta, \alpha_i}| = 0.$$

Hence,

$$\left| \bigcup_{\delta, \alpha_i \in \mathbb{R}^+} T_{\delta, \alpha_i} \right| = 0$$

and so $|T_{\alpha_i}| = 0$.

□

Conclusions

We have analyzed the elliptic problem driven by a singularity, a Choquard term and a discontinuous power nonlinearity and have proved the existence of a solution. In the process we have derived an embedding result and have proved the existence of a positive global minimizer of the associated *singular problem*. We have also proved that as $\alpha \rightarrow 0^+$, the corresponding limiting solutions also converge to a solution of the above problem when $\alpha = 0$. In the future, we also plan to extend the problem with a Kirchhoff type operator.

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Declarations

Conflicts of interest The authors declare to have no conflict of interest/competing interest.

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