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HIGH AND LOW PERTURBATIONS OF THE CRITICAL CHOQUARD EQUATION ON THE HEISENBERG GROUP

Shujie Bai
College of Mathematics, Changchun Normal University
Changchun, 130032, P.R. China

Dušan D. Repovš
Faculty of Education, Faculty of Mathematics and Physics
University of Ljubljana, Ljubljana, 1000, Slovenia
Institute of Mathematics, Physics and Mechanics, Ljubljana, 1000, Slovenia

Yueqiang Song [1]
College of Mathematics, Changchun Normal University
Changchun, 130032, P.R. China

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Abstract. In this paper, our aim is to study the following critical Choquard equation on the Heisenberg group:

$$\begin{cases} -\Delta_H u = \mu|u|^{q-2}u + \displaystyle\int_\Omega \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u|^{Q_\lambda^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{H}^N$ is a smooth bounded domain, Δ_H is the Kohn-Laplacian on the Heisenberg group $\mathbb{H}^N$, $1 < q < 2$ or $2 < q < Q_\lambda^*$, $\mu > 0$, $0 < \lambda < Q = 2N + 2$, and $Q_\lambda^* = \frac{2Q-\lambda}{Q-2}$ is the critical exponent. Using the concentration compactness principle and the critical point theory, we prove that the above problem has the least two positive solutions for $1 < q < 2$ in the case of low perturbations (small values of μ), and has a nontrivial solution for $2 < q < Q_\lambda^*$ in the case of high perturbations (large values of μ). Moreover, for $1 < q < 2$, we also show that there is a positive ground state solution, and for $2 < q < Q_\lambda^*$, there are at least n pairs of nontrivial weak solutions.

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[1]Corresponding author: songyq16@mails.jlu.edu.cn

1. Introduction and main results

In this paper, our aim is to study the existence of solutions for the following critical Choquard equation on the Heisenberg group:

$$\begin{cases} -\Delta_H u = \mu |u|^{q-2}u + \displaystyle\int_\Omega \frac{|u(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta |u|^{Q^*_\lambda - 2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{H}^N$ is a smooth bounded domain, Δ_H is the Kohn-Laplacian on the Heisenberg group $\mathbb{H}^N$, $1 < q < 2$ or $2 < q < Q^*_\lambda$, $\mu > 0$, $0 < \lambda < Q = 2N + 2$, and $Q^*_\lambda = \frac{2Q-\lambda}{Q-2}$ is the critical exponent.

The study of this problem was mainly inspired by two aspects. On the one hand, in the Euclidean case, more and more mathematicians are beginning to pay attention to the Choquard equation. As is well known, Fröhlich [12] and Pekar [24] established the following Choquard equation

$$-\Delta u + u = \Big(\frac{1}{|x|} * |u|^2\Big)u \quad \text{in } \mathbb{R}^3,$$

for the first time in their pioneering work of the modeling of quantum polaron. Such problems are often referred to as the nonlinear Schrödinger-Newton equation. Many authors began to study these problems by using variational methods. For example, Lions [19] obtained the existence of an infinite number of radially symmetric solutions in $H^1(\mathbb{R}^N)$. Ackermann [1] proved the existence of an infinite number of geometrically different weak solutions for a general case was established. Moroz and Van Schaftingen [22, 23] obtained the properties of the ground state solutions, and also proved that these solutions decay asymptotically at infinity. Recently, more and more mathematicians have shown a strong interest in studying critical Choquard type equations. Brézis and Lieb [7] originally addressed the critical problem in his seminal paper, which dealt with the Laplacian equations. Liang et al. [17] proved the multiplicity results of the Choquard-Kirchhoff type equations with Hardy-Littlewood-Sobolev critical exponents. More results about Choquard equations are available in [18, 31, 35, 37].

On the other hand, the study of nonlinear partial differential equations on the Heisenberg group has brought about widespread attention of many researchers. One of the reasons to study such equations is due to their many significant applications. Over the last few decades, many scholars have paid close attention to Heisenberg group's geometric analysis because of its significant applications in quantum mechanics, partial differential equations and other fields. For example, Liang and Pucci [16] applied the Symmetric Mountain Pass Theorem to considering a class of the the critical Kirchhoff-Poisson systems on the Heisenberg group. Pucci and Temperini [29] proved the existence of entire nontrivial solutions for the (p, q) critical systems on the Heisenberg group by an application of variational methods. Pucci [26], applied the Mountain Pass Theorem and the Ekeland variational principle to prove the existence of nontrivial nonnegative solutions of the Schrödinger-Hardy system on the Heisenberg group. For more fascinating results, see [3, 6, 20, 21, 25, 26, 28, 30]. However, once we turn our attention to the critical Choquard equation on the Heisenberg group, we immediately notice that the literature is relatively sparse. Recently, Goel and Sreenadh [13] have studied the following critical Choquard equation on the Heisenberg group:

$$\begin{cases} -\Delta_H u = au + \Big(\displaystyle\int_\Omega \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}\Big)|u|^{Q_\lambda^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

They applied the boot-strap method, iteration techniques, the linking theorem, and the Mountain Pass Theorem to obtain the regularity of solutions and nonexistence of solutions for this kind of problems.

Sun et al. [33] studied the following critical Choquard-Kirchhoff problem on the Heisenberg group:

$$M(\|u\|^2)(-\Delta_H u + u) = \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u|^{Q_\lambda^*-2}u + \mu f(\xi, u),$$

where f is a Carathéodory function, M is the Kirchhoff function, $\mu > 0$ is a parameter, and $Q_\lambda^* = \frac{2Q-\lambda}{Q-2}$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. A new version of the concentration-compactness principle of the Choquard equation on the Heisenberg group was established. Moreover, they also applied the Mountain Pass Theorem to obtain the existence of nontrivial solutions for the above-mentioned problem under non-degenerate and degenerate conditions.

Inspired by the above achievements, with the help of the concentration compact principle and the critical point theory, we prove that problem (1.1) has at least two positive solutions for $1 < q < 2$ and μ small enough, and this equation has a nontrivial solution for $2 < q < Q_\lambda^*$ and μ large enough. Moreover, for $1 < q < 2$, we also show that there is a positive ground state solution for problem (1.1), and for $2 < q < Q_\lambda^*$, there are at least n pairs of nontrivial weak solutions.

Before presenting the main results of this paper, we list some notions about the Heisenberg group. Let $\mathbb{H}^N$ be the Heisenberg group. If $\xi = (x, y, t) \in \mathbb{H}^N$, then the definition of this group operation is

$$\tau_\xi(\xi') = \xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x)) \quad \text{for all} \quad \xi, \xi' \in \mathbb{H}^N.$$

$\xi^{-1} = -\xi$ is the inverse, and therefore $(\xi \circ \xi')^{-1} = (\xi')^{-1} \circ \xi^{-1}$.

The definition of a natural group of dilations on $\mathbb{H}^N$ is $\delta_s(\xi) = (sx, sy, s^2t)$ for all $s > 0$. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. It can be easily proved that the Jacobian determinant of dilatations $\delta_s : \mathbb{H}^N \to \mathbb{H}^N$ is constant and equal to s^Q for all $\xi = (x, y, t) \in \mathbb{H}^N$. The natural number $Q = 2N + 2$ is called the homogeneous dimension of $\mathbb{H}^N$ and the critical exponents is $Q^* := \frac{2Q}{Q-2}$. The Korányi norm is defined as follows

$$|\xi|_H = \left[(x^2 + y^2)^2 + t^2\right]^{\frac{1}{4}} \quad \text{for all} \quad \xi \in \mathbb{H}^N,$$

and is derived from an anisotropic dilation on the Heisenberg group. Hence, the homogeneous degree of the Korányi norm is equal to 1, in terms of dilations

$$\delta_s : (x, y, t) \mapsto (sx, sy, s^2t) \quad \text{for all} \quad s > 0.$$

The set

$$B_H(\xi_0, r) = \{\xi \in \mathbb{H}^N : d_H(\xi_0, \xi) < r\}$$

denotes the Korányi open ball of radius r centered at ξ_0. For the sake of simplicity, we denote $B_r = B_r(O)$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^N$.

The following vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

generate the real Lie algebra of left invariant vector fields for $j = 1, \cdots, n$, which forms a basis satisfying the Heisenberg regular commutation relation on $\mathbb{H}^N$. This means that

$$[X_j, Y_j] = -4\delta_{jk}T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

The so-called horizontal vector field is just a vector field with the span of $[X_j, Y_j]_{j=1}^n$. The Heisenberg gradient on $\mathbb{H}^N$ is

$$\nabla_H = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n),$$

and the Kohn Laplacian on $\mathbb{H}^N$ is given by

$$\begin{aligned}\Delta_H &= \sum_{j=1}^{N} X_j^2 + Y_j^2 \\ &= \sum_{j=1}^{N} \Big[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial x_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2}\Big].\end{aligned}$$

The Haar measure is invariant under the left translations of the Heisenberg group and is Q-homogeneous in terms of dilations. More precisely, it is consistent with the $(2n+1)$-dimensional Lebesgue measure. Hence, as is shown in Leonardi and Masnou [15], the topological dimension $2N+1$ of $\mathbb{H}^N$ is strictly less than its Hausdorff dimension $Q = 2N+2$. Next, $|\Omega|$ denotes the $(2N+1)$ dimensional Lebesgue measure of any measurable set $\Omega \subseteq \mathbb{H}^N$. Therefore,

$$|\delta_s(\Omega)| = s^Q|\Omega|, \quad d(\delta_s \xi) = s^Q d\xi$$

and

$$|B_H(\xi_0, r)| = \alpha_Q r^Q, \quad \text{where} \quad \alpha_Q = |B_H(0,1)|.$$

Now, we are ready to present our main results.

Theorem 1.1. *Let $\Omega \subset \mathbb{H}^N$ be a smooth bounded domain and $1 < q < 2$. Then there exists $\mu_* > 0$ such that if $\mu \in (0, \mu_*)$, then problem* (1.1) *has at least two positive solutions. Moreover, problem* (1.1) *has a positive ground state solution.*

Theorem 1.2. *Let $\Omega \subset \mathbb{H}^N$ be a smooth bounded domain and $2 < q < Q_\lambda^*$. Then there exists $\mu^* > 0$ such that if $\mu > \mu^*$, then problem* (1.1) *has a nontrivial solution.*

Theorem 1.3. *Let $\Omega \subset \mathbb{H}^N$ be a smooth bounded domain and $2 < q < Q_\lambda^*$. Then there exists $\mu^{**} > 0$ such that if $\mu > \mu^{**}$, then problem* (1.1) *has at least n pairs of nontrivial weak solutions.*

The paper is organized as follows. In Section 2, we collect some notations and known facts, and introduce some properties of the Folland-Stein space $\mathring{S}_1^2(\Omega)$. Moreover, a key estimate, i.e., Lemma 2.2, is introduced. In Section 3, we make use of the variational methods to prove some basic lemmas. Then we demonstrate Theorem 1.1. To be more specific, in the first subsection, Ekeland variational principle is used to prove the existence of the first positive solution, and in the second subsection, Mountain Pass Lemma is used to prove the existence of the second positive solution. Furthermore, in the third subsection, we prove that problem (1.1) has a positive ground state solution. In Section 4, we use the general Mountain Pass Theorem to accomplish the proof of Theorem 1.2. Finally, in Section 5, we prove Theorem 1.3 by using Krasnoselskii's genus theory.

2. PRELIMINARIES

In this section, we have collected some known facts which will be useful in the sequel. Set $Q = 2N + 2$ and $Q^* = \frac{2Q}{Q-2}$. Let $\|u\|_p^p = \int_\Omega |u|^p d\xi$ for all $u \in L^p(\Omega)$, represent the usual L^p-norm. Following Folland and Stein [10], we define the space $\mathring{S}_1^2(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $S_1^2(\mathbb{H}^N)$. Then $\mathring{S}_1^2(\Omega)$ is a Hilbert space with respect to the norm

$$\|u\|^2_{\mathring{S}_1^2(\Omega)} = \int_\Omega |\nabla_H u|^2 d\xi.$$

For the sake of brevity, we shall denote $\|u\| = \|u\|^2_{\mathring{S}_1^2(\Omega)}$. By [10], we know that the Folland-Stein space is a Hilbert space and the embedding $\mathring{S}_1^2(\Omega) \hookrightarrow L^p(\Omega)$ for all $p \in [1, Q^*)$ is compact. However, it is only continuous if $p = Q^*$.

By Jerison and Lee [14], we have the following Best Sobolev constant

$$S = \inf_{u \in \mathring{S}_1^2(\Omega)} \frac{\int_\Omega |\nabla_H u|^2 d\xi}{\left(\int_\Omega |u|^{Q^*} d\xi\right)^{\frac{2}{Q^*}}}. \tag{2.1}$$

Proposition 2.1. (see Goel and Sreenadh [13]) *Let $r, s > 1$ and $0 < \lambda < Q$ with $\frac{1}{r} + \frac{\lambda}{Q} + \frac{1}{s} = 2$, $g \in L^r(\Omega)$, and $d \in L^s(\Omega)$. There is a sharp constant $C(t, r, \lambda, Q)$, independent of g, d, such that*

$$\int_\Omega \int_\Omega \frac{g(\xi) d(\eta)}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \le C(t, r, \lambda, Q)|g|_r |d|_s. \tag{2.2}$$

If $r = s = \frac{2Q}{2Q-\lambda}$, then

$$C(t, r, \lambda, Q) = C(\lambda, Q) = \left(\frac{\pi^{N+1}}{2^{N-1}N!}\right)^{\lambda/Q} \frac{N!\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/2)},$$

where Γ is the standard Gamma function.

From Goel and Sreenadh [13], we get

$$\int_\Omega \int_\Omega \frac{|u(\xi)|^{Q_\lambda^*} |u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \le C(\lambda, Q)|u|^{2Q_\lambda^*}_{Q^*}$$

and the best constant S_{HG} is defined by

$$S_{HG} = \inf_{u \in \mathring{S}_1^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_H u|^2 d\xi}{\left(\int_\Omega \int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \right)^{\frac{1}{Q_\lambda^*}}}. \tag{2.3}$$

Lemma 2.1. (see Goel and Sreenadh [13]) *We obtain the best constant S_{HG} if and only if*

$$u(\xi) = u(x,y,t) = CZ(\delta_\theta(a^{-1}\xi)),$$

where $C > 0$ is a fixed constant, $\theta \in (0,\infty)$ are parameters, $a \in \mathbb{H}^N$ and Z is defined in [13, (1.6)]. *Furthermore,*

$$S_{HG} = S(C(Q,\lambda))^{\frac{-1}{Q_\lambda^*}},$$

where S is the best constant defined in [13, (1.5)].

On the other hand, from the proof of [13, Lemma 2.1], we know that a unique minimizer of S_{HG} is the function

$$P(\eta) = S^{\frac{(Q-\lambda)(2-Q)}{4(Q-\lambda+2)}} C(Q,\lambda)^{\frac{2-Q}{2(Q-\lambda+2)}} Z(\eta)$$

and it satisfies the following:

$$-\Delta_H u = \Big(\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta \Big) |u|^{Q_\lambda^*-2} u \ \text{ in } \ \mathbb{H}^N$$

and

$$\int_{\mathbb{H}^N} |\nabla_H P|^2 d\xi = \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|P(\xi)|^{Q_\lambda^*}|P(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = S_{HG}^{\frac{2Q-\lambda}{Q-\lambda+2}}.$$

Furthermore, for $\gamma > 0$, the function P_γ is defined as follows

$$P_\gamma = \frac{\gamma^{\frac{Q-2}{2}} S^{\frac{(Q-\lambda)(2-Q)}{4(Q-2+\lambda)}} C(\lambda,Q)^{\frac{2-Q}{2(Q-\lambda+2)}} C}{(\gamma^4 t^2 + (1+\gamma^2|x|^2+\gamma^2|y|^2)^2)^{(Q-2)/4}},$$

and satisfies

$$\int_\Omega |\nabla_H P_\gamma|^2 d\xi = \int_\Omega \int_\Omega \frac{|P_\gamma(\xi)|^{Q_\lambda^*}|P_\gamma(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = S_{HG}^{\frac{2Q-\lambda}{Q-\lambda+2}}$$

and

$$\int_\Omega |P_\gamma|^{Q^*} d\xi = S^{\frac{Q}{Q-\lambda+2}} C(\lambda,Q)^{\frac{-Q}{Q-\lambda+2}}.$$

More generally, we can suppose that $0 \in \Omega$ and that there is $r > 0$ such that $B(0,4r) \subset \Omega \subset B(0,kr)$ for some $k > 0$. Choose $\nu \in C_c^\infty(\Omega)$ such that $0 \le \nu \le 1$, $|\nabla_H \nu|$ is bounded, and

$$\nu(\eta) = \begin{cases} 1, & \text{if} \quad \eta \in B(0,r), \\ 0, & \text{if} \quad \eta \in \Omega \backslash B(0,2r). \end{cases} \tag{2.4}$$

Then for the following function

$$\upsilon_\gamma = \nu P_\gamma \in \mathring{S}_1^2(\Omega), \tag{2.5}$$

we have asymptotic estimates as follows.

Lemma 2.2. (see Goel and Sreenadh [13]) *Let $0 < \lambda < Q$. Then the following holds:*

(*i*)

$$\int_\Omega |\upsilon_\gamma|^2 d\xi \ge C \begin{cases} \gamma^{-2} + O(\gamma^{-Q+2}), & Q > 4, \\ \gamma^{-2}\log\gamma + O(\gamma^{-2}), & Q = 4. \end{cases}$$

(*ii*)

$$\int_\Omega |\upsilon_\gamma|^{Q^*} d\xi = S^{\frac{Q}{Q-\lambda+2}} C(\lambda,Q)^{\frac{-Q}{Q-\lambda+2}} + O(\gamma^{-Q}).$$

(*iii*)

$$\int_\Omega\int_\Omega \frac{|\upsilon_\gamma(\xi)|^{Q_\lambda^*}|\upsilon_\gamma(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \le S_{HG}^{\frac{2Q-\lambda}{Q-\lambda+2}} + O(\gamma^{-Q}).$$

(*iv*)

$$\int_\Omega\int_\Omega \frac{|\upsilon_\gamma(\xi)|^{Q_\lambda^*}|\upsilon_\gamma(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \ge S_{HG}^{\frac{2Q-\lambda}{Q-\lambda+2}} - O(\gamma^{-\frac{2Q-\lambda}{2}}).$$

(*v*)

$$\int_\Omega |\nabla_H \upsilon_\gamma|^2 d\xi \le S_{HG}^{\frac{2Q-\lambda}{Q-\lambda+2}} + O(\gamma^{-\min\{\frac{2Q-\lambda}{2}, Q-2\}}).$$

3. Low perturbations of problem (1.1)

We say that $u \in \mathring{S}_1^2(\Omega)$ is a solution of problem (1.1) if

$$\int_\Omega \nabla_H u \nabla_H v d\xi - \mu\int_\Omega |u|^{q-2}uvd\xi - \int_\Omega\int_\Omega \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}|u|^{Q_\lambda^*-2}uvd\eta d\xi = 0$$

for any $v \in \mathring{S}_1^2(\Omega)$. Furthermore, if $u > 0$, then we call $u \in \mathring{S}_1^2(\Omega)$ a positive solution to problem (1.1). In order to prove our results, it is necessary to define the energy functional $I_\mu : \mathring{S}_1^2(\Omega) \to \mathbb{R}$ related to problem (1.1):

$$I_\mu(u) = \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\mu}{q}\int_\Omega |u|^q d\xi - \frac{1}{2Q_\lambda^*}\int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi. \tag{3.1}$$

Then I_μ is C^1 on $\mathring{S}_1^2(\Omega)$ and its critical points are solutions of problem (1.1). Indeed, let $I'_\mu(u)$ denote the derivative of I_μ at u, that is, for any $u \in \mathring{S}_1^2(\Omega)$,

$$\begin{aligned}\langle I'_\mu(u), v\rangle = &\int_\Omega \nabla_H u \nabla_H v d\xi - \mu\int_\Omega |u|^{q-2}uv d\xi \\ &- \int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*-2}u(\eta)v(\eta)}{|\eta^{-1}\xi|^\lambda} d\eta d\xi.\end{aligned}$$

Then $I'_\mu(u)$ continuously maps $\mathring{S}_1^2(\Omega)$ in the dual space of $\mathring{S}_1^2(\Omega)$, which can be shown by standard calculations. Therefore, we conclude that u is a solution of problem (1.1) if and only if I_μ is C^1 on $\mathring{S}_1^2(\Omega)$ and $I'_\mu(u) = 0$.

3.1. **The existence of a positive solution of problem** (1.1).

Lemma 3.1. *Let $1 < q < 2$. Then for all*

$$c < \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}} - D\mu^{\frac{2}{2-q}}, \tag{3.2}$$

where

$$D = \Big(\frac{2Q_\lambda^* - q}{2Q_\lambda^* q}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}} S^{-\frac{q}{2}}\Big)^{\frac{2}{2-q}} \Big(\frac{qQ_\lambda^*}{Q_\lambda^*-1}\Big)^{\frac{q}{2-q}},$$

I_μ satisfies the $(PS)_c$ condition.

Proof. Suppose that $\{u_n\} \subset \mathring{S}_1^2(\Omega)$ satisfies

$$I_\mu(u_n) \to c, \quad I'_\mu(u_n) \to 0 \quad as \quad n \to \infty, \tag{3.3}$$

where c is taken from (3.2). It follows from the Young inequality that

$$\begin{aligned}1 + c + o(\|u_n\|) &\geq I_\mu(u_n) - \frac{1}{2Q_\lambda^*} I'_\mu(u_n)u_n \qquad (3.4)\\ &= \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big)\|u_n\|^2 - \mu\Big(\frac{1}{q} - \frac{1}{2Q_\lambda^*}\Big)\int_\Omega |u_n|^q d\xi \\ &\geq \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big)\|u_n\|^2 - \mu\Big(\frac{1}{q} - \frac{1}{2Q_\lambda^*}\Big) S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}}\|u_n\|^q.\end{aligned}$$

This means that $\{u_n\}$ is bounded in $\mathring{S}_1^2(\Omega)$ since $1 < q < 2$. More generally, let us assume that $u_n \rightharpoonup u$ weakly in $\mathring{S}_1^2(\Omega)$ and $u_n \to u$ strongly in $L^p(\Omega)$ with $1 \le p < Q^*$. Applying the concentration compactness principle on the Heisenberg group (see Sun et al. [33], Theorem 3.1), one has

$$|u_n|^{Q^*} \rightharpoonup \zeta \ge |u|^{Q^*} + \sum_{j\in J} \zeta_j \delta_{\xi_j},$$

$$\|\nabla_H u_n|^2 \rightharpoonup d\mu \ge |\nabla_H u|^2 + \sum_{j\in J} \mu_j \delta_{\xi_j},$$

$$\Big(\int_\Omega \frac{|u_n(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta\Big)|u_n(\xi)|^{Q^*_\lambda} \rightharpoonup \Big(\int_\Omega \frac{|u(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta\Big)|u(\xi)|^{Q^*_\lambda} + \sum_{j\in J} \nu_j \delta_{\xi_j},$$

where J is at most countable index set, $\xi_j \in \Omega$ and δ_{ξ_j} is the Dirac mass at ξ_j. Furthermore, we have

$$\zeta_j, \mu_j, \nu_j > 0, \quad S_{HG}\nu_j^{\frac{1}{Q^*_\lambda}} \le \mu_j, \quad \nu_j^{\frac{Q}{2Q-\lambda}} \le C(Q,\lambda)^{\frac{Q}{2Q-\lambda}}\zeta_j. \tag{3.5}$$

Now, we claim that $J = \emptyset$. In fact, let us assume that the hypothesis $\mu_j \neq 0$ holds for some $j \in J$. Then for $\varepsilon > 0$ small enough, by Lemma 3.2 of Capogna et al. [8], we can take the cut-off function $\psi_{\varepsilon,j} \in C_0^\infty(B_H(\xi_j, \varepsilon))$ such that $0 \le \psi_{\varepsilon,j} \le 1$ and

$$\begin{cases} \psi_{\varepsilon,j} = 1 & \text{in} \quad B_H(\xi_j, \frac{\varepsilon}{2}), \\ \psi_{\varepsilon,j} = 0 & \text{in} \quad \Omega\backslash B_H(\xi_j, \varepsilon), \\ |\nabla_H \psi_{\varepsilon,j}| \le \frac{4}{\varepsilon}. \end{cases} \tag{3.6}$$

Now, by the boundedness of $\{\psi_{\varepsilon,j}u_n\}$ and (3.3), we have

$$\begin{aligned} \lim_{n\to\infty} I'_\mu(u_n)[\psi_{\varepsilon,j}u_n] = \lim_{n\to\infty} \Big(\int_\Omega \nabla_H u_n \nabla_H \psi_{\varepsilon,j} u_n d\xi - \mu \int_\Omega u_n^{q-1}\psi_{\varepsilon,j}u_n d\xi \\ - \int_\Omega\int_\Omega \frac{|u_n|^{Q^*_\lambda}|u_n(\eta)|^{Q^*_\lambda - 1}\psi_{\varepsilon,j}u_n(\eta)}{|\eta^{-1}\xi|^\lambda} d\eta d\xi\Big) = 0, \end{aligned}$$

which gives

$$\begin{aligned} &\int_\Omega |\nabla_H u_n|^2 \psi_{\varepsilon,j} d\xi + \int_\Omega \nabla_H u_n \nabla_H \psi_{\varepsilon,j} u_n d\xi \\ &= \mu \int_\Omega u_n^{q-1}\psi_{\varepsilon,j}u_n d\xi + \int_\Omega\int_\Omega \frac{|u_n(\xi)|^{Q^*_\lambda}|u_n(\eta)|^{Q^*_\lambda - 1}\psi_{\varepsilon,j}u_n(\eta)}{|\eta^{-1}\xi|^\lambda} d\eta d\xi + o(1), \end{aligned} \tag{3.7}$$

where $o(1) \to 0$ as $n \to \infty$. From (3.6), we obtain

$$\int_\Omega u_n^{q-1}\psi_{\varepsilon,j}u_n d\xi \le |B_H(\xi_j,\varepsilon)|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}}\Big(\int_{B_H(\xi_j,\varepsilon)}|u_n|^{Q_\lambda^*}d\xi\Big)^{\frac{q}{Q_\lambda^*}}$$
$$\le |\alpha_Q\varepsilon^Q|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}}S^{-\frac{q}{2}}\|u_n\|^q.$$

Thus, by the boundedness of $\{u_n\}$, we obtain that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n^{q-1}\psi_{\varepsilon,j}u_n d\xi = 0. \tag{3.8}$$

Furthermore, by (3.7) and the Hölder inequality, we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_\Omega u_n\nabla_H u_n\nabla_H\psi_{\varepsilon,j}d\xi\Big|$$
$$\le C\lim_{\varepsilon\to 0}\Big(\int_{B_H(\xi_j,\varepsilon)}|u_n|^{Q_\lambda^*}d\xi\Big)^{\frac{1}{Q_\lambda^*}}\Big(\int_{B_H(\xi_j,\varepsilon)}|\nabla_H\psi_{\varepsilon,j}|^{Q_\lambda^*}d\xi\Big)^{\frac{1}{Q_\lambda^*}} = 0. \tag{3.9}$$

Hence,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\nabla_H u_n|^2\psi_{\varepsilon,j}(\xi)d\xi \tag{3.10}$$
$$\ge\lim_{\varepsilon\to 0}\Big(\mu_j+\int_{B_H(\xi_j,\varepsilon)}|\nabla_H u|^2\psi_{\varepsilon,j}(\xi)d\xi\Big) = \mu_j$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega\int_\Omega\frac{|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*}\psi_{\varepsilon,j}(\xi)}{|\eta^{-1}\xi|^\lambda}d\eta d\xi \tag{3.11}$$
$$=\lim_{\varepsilon\to 0}\Big(\nu_j+\int_{B_H(\xi_j,\varepsilon)}\int_\Omega\frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}\psi_{\varepsilon,j}(\xi)}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\Big) = \nu_j.$$

So, from (3.7)-(3.11), we conclude that $\nu_j \ge \mu_j$. Hence, it follows from (3.5) that $\mu_j \ge S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}$.

Furthermore, according to (3.3) and the Young inequality, we obtain

$$c = \lim_{n\to\infty}\Big\{I_\mu(u_n) - \frac{1}{2Q_\lambda^*}I'_\mu(u_n)u_n\Big\}$$
$$=\lim_{n\to\infty}\Big\{(\frac{1}{2}-\frac{1}{2Q_\lambda^*})\|u_n\|^2 - \mu(\frac{1}{q}-\frac{1}{2Q_\lambda^*})\int_\Omega|u_n|^q d\xi\Big\} \tag{3.12}$$
$$\ge(\frac{1}{2}-\frac{1}{2Q_\lambda^*})\mu_j + (\frac{1}{2}-\frac{1}{2Q_\lambda^*})\|u\|^2 - \mu(\frac{1}{q}-\frac{1}{2Q_\lambda^*})S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}}\|u\|^q.$$

Let

$$f(t) = (\frac{1}{2} - \frac{1}{2Q_\lambda^*})t^2 - \mu(\frac{1}{q} - \frac{1}{2Q_\lambda^*})S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}} t^q.$$

Then by a simple calculation, we see that

$$t_0 = (\mu(\frac{1}{q} - \frac{1}{2Q_\lambda^*})S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}})^{\frac{Q_\lambda^*}{(Q_\lambda^*-1)(2-q)}}$$

is the minimum value point of $f(x)$, and the minimum value of $f(x)$ is

$$\begin{aligned} f(t_0) &= (\mu(\frac{1}{q} - \frac{1}{2Q_\lambda^*})S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}})^{\frac{2}{2-q}}(2q)^{\frac{q}{2-q}}(\frac{q}{2}-1) \\ &< (\mu(\frac{1}{q} - \frac{1}{2Q_\lambda^*})S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}})^{\frac{2}{2-q}}(\frac{Q_\lambda^*}{Q_\lambda^*-1}q)^{\frac{q}{2-q}} = D\mu^{\frac{2}{2-q}}. \end{aligned}$$

Thus,

$$c > (\frac{1}{2} - \frac{1}{2Q_\lambda^*})S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}} - D\mu^{\frac{2}{2-q}},$$

which contradicts (3.3). Thus, $J = \emptyset$, and one has

$$\int_\Omega \frac{|u_n(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u_n(\xi)|^{Q_\lambda^*} \to \int_\Omega \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u(\xi)|^{Q_\lambda^*} \quad \text{as} \quad n \to \infty. \tag{3.13}$$

From (3.3) and (3.13), we get

$$\begin{aligned} &\int_\Omega \nabla_H u_n \nabla_H \varphi d\xi - \mu \int_\Omega |u_n|^{q-1}\varphi d\xi \\ &- \int_\Omega \int_\Omega \frac{|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*-1}\varphi}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = o(1). \end{aligned} \tag{3.14}$$

Choose $\varphi = u$ in (3.14). Then

$$\|u\|^2 - \mu \int_\Omega |u|^q d\xi - \int_\Omega \int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = 0. \tag{3.15}$$

By (3.3) and (3.13), we also have

$$\lim_{n\to\infty} \|u_n\|^2 - \mu \int_\Omega |u|^q d\xi - \int_\Omega \int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = 0. \tag{3.16}$$

Combining (3.15) and (3.16), we obtain that $\lim_{n\to\infty} \|u_n\| = \|u\|$. Thus, uniform convexity follows from $\mathring{S}_1^2(\Omega)$, so we can conclude that $u_n \to u$ in $\mathring{S}_1^2(\Omega)$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let* $1 < q < 2$. *Then there exist* $\Lambda_0, \rho_0 > 0$ *such that if* $\mu \in (0, \Lambda_0)$, *then*

$$\inf_{u \in B\rho_0} I_\mu(u) < 0$$

and

$$I_\mu(u) > \frac{1}{2} g(\rho_0)\rho_0^q > 0 \quad \text{for all} \quad u \in S_{\rho_0},$$

where $g(s) = \frac{1}{2}s^{2-q} - a_0 s^{2Q_\lambda^* - q}$.

Proof. First, by the Young inequality, we obtain that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{q}\int_\Omega |u|^q d\xi - \frac{1}{2Q_\lambda^*}\int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \qquad (3.17) \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\mu}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}}\|u\|^q - \frac{C(\lambda, Q)}{2Q_\lambda^*}\Big(\int_\Omega |u|^{Q^*} d\xi\Big)^{\frac{2Q-\lambda}{Q}} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\mu}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}}\|u\|^q - \frac{1}{2Q_\lambda^* S_{HG}^{Q_\lambda^*}}\|u\|^{2Q_\lambda^*} \\ &= \|u\|^q\Big\{\frac{1}{2}\|u\|^{2-q} - \frac{\mu}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}} - \frac{1}{2Q_\lambda^* S_{HG}^{Q_\lambda^*}}\|u\|^{2Q_\lambda^*-q}\Big\}. \end{aligned}$$

Let $a_0 = \frac{1}{2Q_\lambda^* S_{HG}^{Q_\lambda^*}} > 0$. Then it follows from (3.17) that

$$I_\mu(u) \geq \|u\|^q\Big\{\frac{1}{2}\|u\|^{2-q} - \frac{\mu}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}} - a_0\|u\|^{2Q_\lambda^*-q}\Big\}. \qquad (3.18)$$

Take

$$g(s) = \frac{1}{2}s^{2-q} - a_0 s^{2Q_\lambda^*-q}.$$

Then the maximum value point of $g(s)$ is

$$\rho_0 = \Big(\frac{2-q}{2a_0(2Q_\lambda^* - q)}\Big)^{\frac{1}{2Q_\lambda^*-2}},$$

and the maximum value of $g(s)$ is

$$g(\rho_0) = \frac{\rho_0^{2-q}}{2}\Big(1 - \frac{2-q}{2Q_\lambda^* - q}\Big) > 0.$$

Hence, if

$$\Lambda_0 = \frac{1}{2}qS^{\frac{q}{2}}|\Omega|^{\frac{q-Q_\lambda^*}{Q_\lambda^*}} g(\rho_0),$$

then for all $\mu \in (0, \Lambda_0)$, we have from (3.18) that

$$I_\mu(u) \geq \frac{1}{2} g(\rho_0)\rho_0^q > 0 \quad \text{for all} \quad u \in S_{\rho_0}.$$

Furthermore, for any $u \in \mathring{S}_1^2(\Omega)\backslash\{0\}$, one has

$$\lim_{s\to 0^+} \frac{I_\mu(su)}{s^q} = -\frac{\mu}{q}\int_\Omega |u|^q d\xi < 0,$$

which implies that $u \in B_{\rho_0}$ makes $I_\mu(u) < 0$. From this, we can conclude that $\inf_{u\in B_{\rho_0}} I_\mu(u) < 0$, and the proof of Lemma 3.2 is complete. □

Lemma 3.3. *Let* $1 < q < 2$ *and assume that* $\mu \in (0, \Lambda_0)$. *Then problem* (1.1) *has a positive solution* $u_1 \in \mathring{S}_1^2(\Omega)$ *such that* $I_\mu(u_1) < 0$.

Proof. Let ρ_0 be as in Lemma 3.2 and set

$$w = \inf_{u\in B_{\rho_0}} I_\mu(u) < 0 < \inf_{u\in S_{\rho_0}} I_\mu(u). \tag{3.19}$$

Note that $I_\mu(|u|) = I_\mu(u)$. According to the Ekeland variational principle (see [9]), we know that

$$I_\mu(u_n) \leq \inf_{u\in B_{\rho_0}} I_\mu(u) + \frac{1}{n}, \quad I_\mu(v) \geq I_\mu(u_n) - \frac{1}{n}\|v - u_n\|$$

for all $v \in B_{\rho_0}$ and some nonnegative minimizing sequence $\{u_n\} \subset B_{\rho_0}$. From this and (3.19), we get $I'_\mu(u) \to 0$ and $I_\mu(u_n) \to w$. Because of $u_n \geq 0$ and $\|u_n\| \leq \rho_0$, there is $u_1 \in B_{\rho_0}$ and $u_1 \geq 0$ satisfying $u_n \rightharpoonup u_1$ in $\mathring{S}_1^2(\Omega)$ as $n \to \infty$. It follows from Lemma 3.1 that $u_n \to u_1$ in $\mathring{S}_1^2(\Omega)$ and

$$w = \lim_{n\to\infty} I_\mu(u_n) = I_\mu(u_1) < 0.$$

Hence, we have $u_1 \geq 0$ and $u_1 \not\equiv 0$. Moreover, u_1 is a solution of problem (1.1), that is

$$-\Delta_H u_1 = \mu|u_1|^{q-1} + \int_\Omega \frac{|u_1(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u_1|^{Q_\lambda^*-1}.$$

The maximum principle (see Bony [5]) implies that $u_1 > 0$ in Ω. Thus, u_1 is a positive solution of problem (1.1). This completes the proof of Lemma 3.3. □

3.2. **The existence of the second positive solution of problem** (1.1)**.**

Lemma 3.4. *Let* $1 < q < 2$ *and* $\mu \in (0, \Lambda_0)$*. Then* $I_\mu(u) > 0$ *for all* $u \in S_{\rho_0}$*. Moreover, there is* $e \in \mathring{S}_1^2(\Omega)\backslash B_{\rho_0}$ *satisfying* $I_\mu(e) < 0$*, where* Λ_0, ρ_0 *are as in Lemma* 3.2.

Proof. It is evident that Lemma 3.2 proves the first assertion. Therefore, we only need to prove the rest of Lemma 3.4. Let $u \in \mathring{S}_1^2(\Omega)\backslash\{0\}$. Then one has

$$\begin{aligned} I_\mu(su) &= \frac{s^2}{2}\|u\|^2 - \frac{\mu s^q}{q}\int_\Omega |u|^q d\xi - \frac{s^{2Q_\lambda^*}}{2Q_\lambda^*}\int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \\ &\le \frac{s^2}{2}\|u\|^2 - \frac{C(\lambda,Q)s^{2Q_\lambda^*}}{2Q_\lambda^*}\Big(\int_\Omega |u|^{Q^*} d\xi\Big)^{\frac{2Q-\lambda}{Q}} \\ &\to -\infty \quad as \quad s \to +\infty. \end{aligned} \tag{3.20}$$

Thus there exists $e \in \mathring{S}_1^2(\Omega)\backslash B_{\rho_0}$ satisfying $I_\mu(e) < 0$. This completes the proof of Lemma 3.4. □

Lemma 3.5. *Let* $1 < q < 2$ *and assume that* υ_γ *is defined by* (2.4)*. Then there exists* $\Lambda_1 > 0$ *such that* $\mu \in (0, \Lambda_1)$*, and*

$$\sup_{s\ge 0} I_\mu(u_1 + s\upsilon_\gamma) < \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}} - D\mu^{\frac{2}{2-q}}, \tag{3.21}$$

where u_1 *is the positive solution from Lemma* 3.4 *and* D *is from* (3.2)*.*

Proof. Because u_1 is the positive solution from Lemma 3.4, there exist positive constants t and T satisfying $t \le u_1(\xi) \le T$ for any $\xi \in \sup\nu$, where ν is as in (2.4). Moreover, one has $I'_\mu(u_1)u_1 = 0$ and $I_\mu(u_1) < 0$.

Next, it is easy to prove that for any $a, b > 0$, we have

$$(a+b)^\sigma \ge a^\sigma + \sigma a^{\sigma-1} b, \quad 1 < \sigma < 2 \tag{3.22}$$

and

$$(a_1+b_1)^{Q_\lambda^*}(a_2+b_2)^{Q_\lambda^*} \ge a_1^\sigma a_2^\sigma + b_1^\sigma b_2^\sigma + 2\sigma a_1^\sigma a_2^{\sigma-1} b_2 + 2\sigma a_1^\sigma a_2 b_2^{\sigma-1},\ 2 \le \sigma. \tag{3.23}$$

Hence, for any $s \ge 0$, we have

$$\begin{aligned} I_\mu(u_1 + s\upsilon_\gamma) &= I_\mu(u_1) + \frac{s^2}{2}\|\upsilon_\gamma|^2 + sI'_\mu(u_1)\upsilon_\gamma \\ &- \frac{\mu}{q}\int_\Omega \Big[(u_1 + s\upsilon_\gamma)^q - u_1^q - qsu_1^{q-1}\upsilon_\gamma\Big] d\xi \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2Q_\lambda^*}\int_\Omega\int_\Omega\Big[\frac{|u_1(\xi)+s\upsilon_\gamma(\xi)|^{Q_\lambda^*}|u_1(\eta)+s\upsilon_\gamma(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}\\
&- \frac{|u_1(\xi)|^{Q_\lambda^*}|u_1(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} - 2Q_\lambda^* s\frac{|u_1(\xi)|^{Q_\lambda^*}|u_1(\eta)|^{Q_\lambda^*-1}|\upsilon_\gamma(\eta)|}{|\eta^{-1}\xi|^\lambda}\Big]d\eta d\xi\\
&\le \frac{s^2}{2}\|\upsilon_\gamma\|^2 - \frac{s^{2Q_\lambda^*}}{2Q_\lambda^*}\int_\Omega\int_\Omega\frac{|\upsilon_\gamma(\xi)|^{Q_\lambda^*}|\upsilon_\gamma(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\
&- s^{Q_\lambda^*-1}\int_\Omega\int_\Omega\frac{|u_1(\xi)|^{Q_\lambda^*}|u_1(\eta)||\upsilon_\gamma(\eta)|^{Q_\lambda^*-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\
&\le \frac{s^2}{2}\|\upsilon_\gamma\|^2 - \frac{C(\lambda,Q)s^{2Q_\lambda^*}}{2Q_\lambda^*}(\int_\Omega|\upsilon_\gamma|^{Q^*}d\xi)^{\frac{2Q-\lambda}{Q}}\\
&- ts^{Q_\lambda^*-1}\int_\Omega\int_\Omega\frac{|u_1(\xi)|^{Q_\lambda^*}|\upsilon_\gamma(\eta)|^{Q_\lambda^*-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi. \qquad (3.24)
\end{aligned}
$$

Let

$$
\begin{aligned}
\phi(s) = \frac{s^2}{2}\|\upsilon_\gamma\|^2 - \frac{C(\lambda,Q)s^{2Q_\lambda^*}}{2Q_\lambda^*}\Big(\int_\Omega|\upsilon_\gamma|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{Q}}\\
- ts^{Q_\lambda^*-1}\int_\Omega\int_\Omega\frac{|u_1(\xi)|^{Q_\lambda^*}|\upsilon_\gamma(\eta)|^{Q_\lambda^*-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi.
\end{aligned}
$$

The definition of $\phi(s)$ enables us to obtain $\phi(0) = 0$ and $\phi(s) \to -\infty$ as $s \to +\infty$. Thus, we can find $s_\gamma > 0$ and positive constants s_1, s_2 independent of γ, μ, satisfying

$$\phi(s_\gamma) = \sup_{s\ge 0}\phi(s), \quad \phi'(s_\gamma) = 0 \qquad (3.25)$$

and

$$0 < s_1 \le s_\gamma \le s_2 < \infty. \qquad (3.26)$$

Therefore, one has

$$s_\gamma\|\upsilon_\gamma\|^2 - C(\lambda,Q)s_\gamma^{2Q_\lambda^*-1}\Big(\int_\Omega|\upsilon_\gamma|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{Q}} \qquad (3.27)$$

$$- t(Q_\lambda^*-1)s_\gamma^{Q_\lambda^*-2}\int_\Omega\int_\Omega\frac{|u_1(\xi)|^{Q_\lambda^*}|\upsilon_\gamma(\eta)|^{Q_\lambda^*-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi = 0 \qquad (3.28)$$

and

$$\|\upsilon_\gamma\|^2 - C(\lambda,Q)(2Q_\lambda^*-1)s_\gamma^{2Q_\lambda^*-2}\Big(\int_\Omega|\upsilon_\gamma|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{Q}} \qquad (3.29)$$

$$- t(Q^*_\lambda - 1)(Q^*_\lambda - 2)s_\gamma^{Q^*_\lambda - 3}\int_\Omega\int_\Omega \frac{|u_1(\xi)|^{Q^*_\lambda}|\upsilon_\gamma(\eta)|^{Q^*_\lambda-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi < 0. \quad (3.30)$$

From (3.29), we obtain that there exists $s_1 > 0$ (independent of γ, μ) satisfying $0 < s_1 \le s_\gamma$.

Next, from (3.27) one has

$$\begin{aligned}&\frac{\|\upsilon_\gamma\|^2}{s_\gamma^{2Q^*_\lambda-2}} - C(\lambda, Q)\Big(\int_\Omega |\upsilon_\gamma|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{Q}}\\ &- \frac{t(Q^*_\lambda - 1)}{s_\gamma^{Q^*_\lambda+1}}\int_\Omega\int_\Omega \frac{|u_1(\xi)|^{Q^*_\lambda}|\upsilon_\gamma(\eta)|^{Q^*_\lambda-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi = 0.\end{aligned} \quad (3.31)$$

This implies that s_γ has an upper bound for $\gamma > 0$ small enough. If not, making $s_\gamma \to \infty$ in (3.31), one gets $\int_\Omega |\upsilon_\gamma|^{Q^*}d\xi = 0$, which contradicts Lemma 2.2 for γ small enough. It follows from (3.24), (3.25), (3.26), Lemmas 2.1, and 2.2 that

$$\begin{aligned}&\sup_{s\ge0} I_\mu(u_1 + s\upsilon_\gamma) \le \sup_{s\ge0}\Phi(s) = \Phi(s_\gamma)\\ &\le \sup_{s\ge0}\Big\{\frac{s^2}{2}\|\upsilon_\gamma\|^2 - \frac{C(\lambda,Q)s^{2Q^*_\lambda}}{2Q^*_\lambda}\Big(\int_\Omega|\upsilon_\gamma|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{Q}}\Big\}\\ &\quad - ts^{Q^*_\lambda-1}\int_\Omega\int_\Omega \frac{|u_1(\xi)|^{Q^*_\lambda}|\upsilon_\gamma(\eta)|^{Q^*_\lambda-1}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\\ &\le \sup_{s\ge0}\Big\{\frac{s^2}{2}\|\upsilon_\gamma\|^2 - \frac{C(\lambda,Q)s^{2Q^*_\lambda}}{2Q^*_\lambda}\Big(\int_\Omega|\upsilon_\gamma|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{Q}}\Big\}\\ &\quad - ts^{Q^*_\lambda-1}C(\lambda,Q)\Big(\int_\Omega|u_1(\xi)|^{Q^*}d\xi\Big)^{\frac{2Q-\lambda}{2Q}}\Big(\int_\Omega|\upsilon_\gamma(\eta)|^2d\eta\Big)^{\frac{Q-\lambda+2}{2(Q-2)}}\\ &\le \sup_{s\ge0}\Big\{\frac{s^2}{2}S_{HG}^{\frac{2Q-\lambda}{Q-\lambda+2}} - \frac{C(\lambda,Q)s^{2Q^*_\lambda}}{2Q^*_\lambda}\Big(S^{\frac{Q}{Q-\lambda+2}}C(\lambda,Q)^{\frac{-Q}{Q-\lambda+2}}\Big)^{\frac{2Q-\lambda}{Q}}\Big\}\\ &\quad - C_1(\gamma^{-2})^{\frac{Q-\lambda+2}{2(Q-2)}}\\ &= \sup_{s\ge0}\Big\{\frac{s^2}{2}S_{HG}^{\frac{Q^*_\lambda}{Q^*_\lambda-1}} - \frac{C(\lambda,Q)s^{2Q^*_\lambda}}{2Q^*_\lambda}S_{HG}^{\frac{Q^*_\lambda}{Q^*_\lambda-1}}\Big\} - C_1\gamma^{-\frac{Q-\lambda+2}{Q-2}}\\ &< \Big(\frac{1}{2} - \frac{1}{2Q^*_\lambda}\Big)S_{HG}^{\frac{Q^*_\lambda}{Q^*_\lambda-1}} - C_1\gamma^{-\frac{Q-\lambda+2}{Q-2}},\end{aligned} \quad (3.32)$$

where $C_1 > 0$ is independent of γ and μ.

Let $\gamma^{-\frac{Q-\lambda+2}{Q-2}} = \mu^{\frac{q}{2-q}}$ and $\Lambda_1 = \frac{C_1}{D}$. Then for any $\mu \in (0, \Lambda_1)$, one has

$$C_1\gamma^{-\frac{Q-\lambda+2}{Q-2}} > D\mu^{\frac{2}{2-q}}. \tag{3.33}$$

By (3.32) and (3.33), equation (3.21) holds if $\mu \in (0, \Lambda_1)$. This completes the proof of Lemma 3.5. □

From the above discussion, we get the following result.

Lemma 3.6. *Let* $1 < q < 2$. *Then there exists* $\mu_* > 0$ *such that for all* $\mu \in (0, \Lambda_*)$, *problem* (1.1) *has a positive solution* $u_2 \in \mathring{S}_1^2(\Omega)$ *satisfying* $I_\mu(u_2) > 0$.

3.3. **Existence of a positive ground state solution of problem** (1.1)**.** In this subsection, we will show that problem (1.1) has a positive ground state solution. Indeed, let

$$\mathcal{N} = \{u \in \mathring{S}_1^2(\Omega) : u \neq 0, \langle I'_\mu(u), u\rangle = 0\}.$$

Since any nontrivial solution of problem (1.1) belongs to $\mathcal{N}$, we can set $\tau = \inf_{u\in\mathcal{N}} I_\mu(u)$. Clearly, if $u \in \mathcal{N}$, one also has $|u| \in \mathcal{N}$ and $I_\mu(|u|) = I_\mu(u)$, and therefore we can consider a nonnegative minimizing sequence $\{u_n\} \subset \mathcal{N}$ and such that

$$I_\mu(u_n) \to \tau \quad \text{as } n \to \infty. \tag{3.34}$$

By Lemma 3.3, $\tau < 0$ and $\{u_n\}$ is bounded in $\mathring{S}_1^2(\Omega)$. More generally, suppose that $u_n \rightharpoonup u_\lambda$ weakly in $\mathring{S}_1^2(\Omega)$ and $u_n \to u_\lambda$ strongly in $L^p(\Omega)$ with $1 < p < Q^*$. Thus, $u_\lambda \neq 0$. In fact, if $u_\lambda = 0$ and $l = \lim_{n\to\infty} \|u_n\|$, then since $u_n \in \mathcal{M}$, one has

$$\begin{aligned}\|u_n\|^2 &= I'_\mu(u_n)[u_n] + \mu\int_\Omega |u_n|^q d\xi + \int_\Omega\int_\Omega \frac{|u_n(\xi)|^{Q^*_\lambda}|u_n(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \\ &\leq C(\lambda, Q)\Big(\int_\Omega |u_n|^{Q^*} d\xi\Big)^{\frac{2Q-\lambda}{Q}} + o(1) \leq S_{HG}^{-Q^*_\lambda}\|u_n\|^{2Q^*_\lambda} + o(1). \end{aligned} \tag{3.35}$$

From this, one can infer that either $l = 0$ or $l \geq S_{HG}^{\frac{Q^*_\lambda}{2Q^*_\lambda - 2}}$. Besides, from (3.34), one has

$$\begin{aligned}\tau &= \lim_{n\to\infty}\Big\{\frac{1}{2}\|u_n\|^2 - \frac{\mu}{q}\int_\Omega |u_n|^q d\xi - \frac{1}{2Q^*_\lambda}\int_\Omega\int_\Omega \frac{|u_n(\xi)|^{Q^*_\lambda}|u_n(\eta)|^{Q^*_\lambda}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi\Big\} \\ &= \Big(\frac{1}{2} - \frac{1}{2Q^*_\lambda}\Big)\lim_{n\to\infty}\|u_n\|^2 = \Big(\frac{1}{2} - \frac{1}{2Q^*_\lambda}\Big)l^2. \end{aligned} \tag{3.36}$$

If $l = 0$, then from (3.36), we get $\tau = 0$, which is a contradiction. Thus

$$\tau \geq \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}.$$

It follows from Lemma 3.5 that

$$\Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}} \leq \tau < \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}} - D\mu^{\frac{2}{2-q}},$$

which is also a contradiction. Therefore, we must have $u_\lambda \neq 0$ in $\mathring{S}_1^2(\Omega)$.

On the other hand, $u_n \to u_\lambda$ in $\mathring{S}_1^2(\Omega)$ is derived from Lemma 3.1. In other words, u_λ is a positive solution of problem (1.1) and $I_\mu(u_\lambda) \geq \tau$.

Next, we show that $I_\mu(u_\lambda) \leq \tau$. Indeed, by the Fatou Lemma and (3.33), we get

$$\begin{aligned}\tau &= \lim_{n\to\infty} \Big\{ I_\mu(u_n) - \frac{1}{2Q_\lambda^*} I'_\mu(u_n)u_n \Big\} \\ &= \lim_{n\to\infty} \Big\{ \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big)\|u_n\|^2 - \mu\Big(\frac{1}{q} - \frac{1}{2Q_\lambda^*}\Big)\int_\Omega u_n^q d\xi \Big\} \\ &\geq \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big)\|u_\lambda\|^2 - \mu\Big(\frac{1}{q} - \frac{1}{2Q_\lambda^*}\Big)\int_\Omega u_\lambda^q d\xi. \end{aligned} \tag{3.37}$$

Furthermore, because u_λ is a positive solution of problem (1.1), we have

$$\begin{aligned} I_\mu(u_\lambda) &= I_\mu(u_\lambda) - \frac{1}{2Q_\lambda^*} I'_\mu(u_\lambda)u_\lambda \\ &= \Big(\frac{1}{2} - \frac{1}{2Q_\lambda^*}\Big)\|u_\lambda\|^2 - \mu\Big(\frac{1}{q} - \frac{1}{2Q_\lambda^*}\Big)\int_\Omega u_\lambda^q d\xi. \end{aligned}$$

From (3.37), we have $I_\lambda(u_\lambda) \leq \tau$ and $I_\lambda(u_\lambda) = \tau$ and $u_\lambda \neq 0$. This means that u_λ is a positive ground state solution of problem (1.1). Consequently, invoking Lemmas 3.3 and 3.6 completes the proof of Theorem 1.1.

4. High perturbations of problem (1.1)

This section focuses on the proof of Theorem 1.2. To this end, we shall apply the general Mountain Pass Theorem.

Theorem 4.1. (see Rabinowitz [32]) *Let E be a real Banach space and $I \in C^1(E, R)$ satisfying (PS) condition. Suppose that $I(0) = 0$ and that*

(i) there are constants $\rho, \alpha > 0$ satisfying $I(u)|_{\partial B_\rho} \geq \alpha$;

(ii) there exists $e \in E \backslash \overline{B_\rho}$ satisfying $I(e) \leq 0$.

Then I has a critical value $c \geq \alpha$. Moreover,

$$c = \inf_{h\in\Gamma} \max_{0\leq t\leq 1} I(h(t)) \geq \alpha,$$

where

$$\Gamma = \{h \in C([0,1], E) : h(0) = 1, h(1) = e\}.$$

Next, we prove that the geometric properties (i) and (ii) of Theorem 4.1 are satisfied by I_μ.

Lemma 4.1. *Let $2 < q < Q_\lambda^*$. Then the properties (i) and (ii) of Theorem 4.1 are satisfied by the energy functional I_μ.*

Proof. From (3.17), we have

$$I_\mu(u) \geq \frac{1}{2}\|u\|^2 - \frac{\mu}{q} S^{-\frac{q}{2}} |\Omega|^{\frac{Q_\lambda^*-q}{Q_\lambda^*}} \|u\|^q - \frac{1}{2Q_\lambda^* S_{HG}^{Q_\lambda^*}} \|u\|^{2Q_\lambda^*}. \tag{4.1}$$

Now, we can take $\rho, \alpha > 0$ satisfying $I_\mu(u) \geq \alpha$ for all $u \in \partial B_\rho$, since $2 < q$, $2 < 2Q_\lambda^*$. Thus, property (i) of Theorem 4.1 holds.

Now, we show that property (ii) of Theorem 4.1 also holds. From (3.20), one has

$$\begin{aligned} I_\mu(su) &= \frac{s^2}{2}\|u\|^2 - \frac{\mu s^q}{q}\int_\Omega |u|^q d\xi - \frac{s^{2Q_\lambda^*}}{2Q_\lambda^*}\int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \\ &\leq \frac{s^2}{2}\|u\|^2 - \frac{C(\lambda,Q)s^{2Q_\lambda^*}}{2Q_\lambda^*}\Big(\int_\Omega |u|^{Q^*} d\xi\Big)^{\frac{2Q-\lambda}{Q}}. \end{aligned} \tag{4.2}$$

Since $2 < 2Q_\lambda^*$, we can deduce that $I_\mu(s_0 u) < 0$ and $s_0\|u\| > \rho$ for s_0 large enough. Let $e = s_0 u$. Hence, e is the desired function and the proof of property (ii) of Theorem 4.1 is complete. □

Lemma 4.2. *Let $2 < q < Q_\lambda^*$. Then for all*

$$c_\mu < \Big(\frac{1}{2} - \frac{1}{q}\Big) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}, \tag{4.3}$$

I_μ satisfies the $(PS)_{c_\mu}$ condition.

Proof. We assume that $\{u_n\} \subset \mathring{S}_1^2(\Omega)$ satisfies

$$I_\mu(u_n) \to c_\mu, \quad I'_\mu(u_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{4.4}$$

where c_μ is as in (4.4). It follows that

$$1 + c_\mu + o(\|u_n\|) \geq I_\mu(u_n) - \frac{1}{q} I'_\mu(u_n)u_n = \Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|^2$$

$$+\Big(\frac{1}{q}-\frac{1}{2Q_\lambda^*}\Big)\int_\Omega\int_\Omega\frac{|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi$$
$$\geq\Big(\frac{1}{2}-\frac{1}{q}\Big)\|u_n\|^2. \tag{4.5}$$

From this, we can derive that $\{u_n\}$ is bounded in $\mathring{S}_1^2(\Omega)$.

The rest of the proof is similar to the proof of Lemma 3.1. We have $\mu_j\geq S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}$ and by using the concentration compactness principle, we have

$$\begin{aligned}c_\mu&=\lim_{n\to\infty}\Big\{I_\mu(u_n)-\frac{1}{q}I'_\mu(u_n)u_n\Big\}\\&=\lim_{n\to\infty}\Big\{(\frac{1}{2}-\frac{1}{q})\|u_n\|^2+\Big(\frac{1}{q}-\frac{1}{2Q_\lambda^*}\Big)\int_\Omega\int_\Omega\frac{|u_n(\xi)|^{Q_\lambda^*}|u_n(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi\Big\}\\&\geq\lim_{n\to\infty}\Big\{(\frac{1}{2}-\frac{1}{q})\|u_n\|^2\Big\}\geq(\frac{1}{2}-\frac{1}{q})\mu_j+(\frac{1}{2}-\frac{1}{q})\|u\|^2>\Big(\frac{1}{2}-\frac{1}{q}\Big)S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}.\end{aligned}$$

From (4.3), we have $J=\emptyset$. As in the discussion of Lemma 3.1, one knows that $u_n\to u$ in $\mathring{S}_1^2(\Omega)$. This completes the proof of Lemma 4.2. □

Proof of Theorem 1.2. We claim that

$$0<c_\mu=\inf_{h\in\Gamma}\max_{0\leq t\leq 1}I_\mu(h(t))<\Big(\frac{1}{2}-\frac{1}{q}\Big)S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}. \tag{4.6}$$

Assume that (4.6) holds. Then Lemmas 4.2 and 4.1 and Theorem 4.1 assure the existence of nontrivial critical points of I_μ.

In order to prove (4.6), we choose $v_0\in\mathring{S}_1^2(\Omega)$ such that

$$\|v_0\|=1\quad\text{and}\quad\lim_{s\to+\infty}I_\mu(sv_0)=-\infty.$$

Then

$$\sup_{s\geq 0}I_\mu(sv_0)=I_\mu(s_\mu v_0)\quad\text{for some } s_\mu>0.$$

So, s_μ satisfies

$$\|s_\mu v_0\|^2=\mu\int_\Omega|s_\mu v_0|^q d\xi+\int_\Omega\int_\Omega\frac{|s_\mu v_0(\xi)|^{Q_\lambda^*}|s_\mu v_0(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda}d\eta d\xi. \tag{4.7}$$

Now, we demonstrate that $\{s_\mu\}_{\mu>0}$ is bounded. Indeed, suppose that the following hypothesis $s_\mu\geq 1$ is satisfied for all $\mu>0$. Furthermore, we can deduce from (4.7) that

$$s_\mu^q \geq \|s_\mu v_0\|^2 = \mu \int_\Omega |s_\mu v_0|^q d\xi + \int_\Omega \int_\Omega \frac{|s_\mu v_0(\xi)|^{Q_\lambda^*} |s_\mu v_0(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi \quad (4.8)$$

$$\geq s_\mu^{2Q_\lambda^*} \int_\Omega \int_\Omega \frac{|v_0(\xi)|^{Q_\lambda^*} |v_0(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi. \quad (4.9)$$

It follows from (4.8) that $\{s_\mu\}_{\mu>0}$ is bounded, since $2 < 2Q_\lambda^*$.

Next, we demonstrate that $s_\mu \to 0$ as $\mu \to \infty$. Suppose to the contrary, there are $s_\mu > 0$ and a sequence $(\mu_n)_n$ with $\mu_n \to \infty$ as $n \to \infty$, satisfying $s_{\mu_n} \to s_\mu$ as $n \to \infty$. Invoking the Lebesgue dominated convergence theorem, we have

$$\int_\Omega |s_{\mu_n} v_0|^q d\xi \to \int_\Omega |s_\mu v_0|^q d\xi \quad \text{as} \quad n \to \infty.$$

It now follows that

$$\mu_n \int_\Omega |s_\mu v_0|^q d\xi \to \infty \quad \text{as} \quad n \to \infty.$$

Thus, invoking (4.7), we can show that this cannot happen. Therefore, $s_\mu \to 0$ as $\mu \to \infty$. Furthermore, we can apply (4.7) to show that

$$\lim_{\mu\to\infty} \mu \int_\Omega |s_\mu v_0|^q d\xi = 0$$

and

$$\lim_{\mu\to\infty} \int_\Omega \int_\Omega \frac{|s_\mu v_0(\xi)|^{Q_\lambda^*} |s_\mu v_0(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi = 0.$$

Therefore, $s_\mu \to 0$ as $\mu \to \infty$ and by the definition of I_μ, we get that

$$\lim_{\mu\to\infty} (\sup_{s\geq 0} I_\mu(sv_0)) = \lim_{\mu\to\infty} I_\mu(s_\mu v_0) = 0.$$

So, there is $\mu^* > 0$ such that if $\mu > \mu^*$, we have

$$\sup_{s\geq 0} I_\mu(sv_0) < (\frac{1}{2} - \frac{1}{q}) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}.$$

Letting $e = s_1 v_0$ with s_1 large enough to have $I_\mu(e) < 0$, we get

$$0 < c_\mu \leq \max_{0\leq t\leq 1} I_\mu(h(t)) \quad \text{where} \quad h(t) = ts_1 v_0.$$

Therefore,

$$0 < c_\mu \leq \sup_{s\geq 0} I_\mu(sv_0) < (\frac{1}{2} - \frac{1}{q}) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}$$

for μ large enough. The proof of Theorem 1.2 is now complete. □

5. Proof of Theorem 1.3

In order to prove Theorem 1.3, we shall use the Krasnoselskii genus theory [32]. Let X be a Banach space and let Λ denote the family of all closed subsets $A \subset X\backslash\{0\}$ which are symmetric with respect to the origin, that is, $u \in A$ implies that also $-u \in A$. If $z_1, \cdots, z_k$ are in Z, then the span of all liner combinations of $z_1, \cdots, z_k$ is denoted by $\text{span}\{z_1, \cdots, z_k\}$ and is called the subset of Z generated by $z_1, \cdots, z_k$.

Theorem 5.1. (see Rabinowitz [32]) *Let E be an infinite-dimensional Banach space and let $I \in C^1(X)$ be even, with $I(0) = 0$. Assume that $E = X \oplus Y$, where X is a finite-dimensional space, and I satisfies the following properties:*

(*i*) *There is $\theta > 0$ such that I satisfies the $(PS)_c$ condition for all $c \in (0, \theta)$.*

(*ii*) *There are $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in \partial B_\rho \bigcap Y$.*

(*iii*) *For any finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > \rho$ such that $I(u) \leq 0$ on $\tilde{E} \setminus B_R$.*

Moreover, suppose that X is k-dimensional and $X = \text{span}\{z_1, \cdots, z_k\}$. For $n \geq k$, inductively select $z_{n+1} \notin X_n = \text{span}\{z_1, \cdots, z_n\}$. Let $R_n = R(X_n)$ and $\Upsilon_n = B_{R_n} \bigcap X_n$. Define

$$W_n = \{\varphi \in C(\Upsilon_n, E) : \varphi|\partial_{B_{R_n} \bigcap X_n} = id \text{ and } \varphi \text{ is odd}\}$$

and

$$\Gamma_i = \{\varphi(\overline{\Upsilon_n \setminus V}) : \varphi \in W_n, n \geq i, V \in \Lambda, \gamma(V) \leq n - i\},$$

where $\gamma(V)$ is the Krasnoselskii genus of V. For $i \in N$, let

$$c_i = \inf_{X \in \Gamma_i} \max_{u \in X} I(u).$$

Hence, $0 \leq c_i \leq c_{i+1}$ and $c_i < \theta$ for $i > k$. Then, we get that c_i is a critical value of I. Moreover, if $c_i = c_{i+1} = \cdots = c_{i+p} = c < \theta$ for $i > k$, then $\gamma(K_c) \geq p + 1$, where

$$K_c = \{u \in E : I(u) = c \quad \text{and} \quad I'(u) = 0\}.$$

Proof of Theorem 1.3. It is well known that $\mathring{S}_1^2(\Omega)$ is a reflexive Banach space and $I_\mu \in C^1(\mathring{S}_1^2(\Omega))$. The energy functional I_μ satisfies $I_\mu(0) = 0$. We have three steps to prove Theorem 1.3.

Step 1. The proof is similar to the proof of (i) and (ii) in Theorem 4.1. We can see that (*ii*) and (*iii*) of Theorem 5.1 are satisfied.

Step 2. We show that there is a sequence $(\Psi_n)_n \subset R^+$, with $\Psi_n \leq \Psi_{n+1}$, satisfying

$$c_n^\mu = \inf_{X\in\Gamma_n} \max_{u\in X} I_\mu(u) < \Psi_n.$$

For this purpose, applying an argument given in Wei and Wu [36], according to the definition of c_n^μ, one has

$$c_n^\mu = \inf_{X\in\Gamma_n} \max_{u\in X} I_\mu(u) \leq \inf_{X\in\Gamma_n} \max_{u\in X}\{\|u\|^2 - \frac{1}{2Q_\lambda^*}\int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi\}.$$

Set

$$\Psi_n = \inf_{X\in\Gamma_n} \max_{u\in X}\{\|u\|^2 - \frac{1}{2Q_\lambda^*}\int_\Omega\int_\Omega \frac{|u(\xi)|^{Q_\lambda^*}|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta d\xi\}.$$

Then $\Psi_n < \infty$ and $\Psi_n \leq \Psi_{n+1}$, by the definition of Γ_n.

Step 3. We show that problem (1.1) has at least n pairs of weak solutions. As in (4.6), a similar discussion yields that there exists $\mu^{**} > 0$ satisfying

$$c_n^\mu \leq \Psi_n < (\frac{1}{2} - \frac{1}{q}) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}} \quad \text{for all } \mu > \mu^{**}.$$

Thus, one has

$$0 < c_1^\mu \leq c_2^\mu \leq \cdots \leq c_n^\mu < \Psi_n < (\frac{1}{2} - \frac{1}{q}) S_{HG}^{\frac{Q_\lambda^*}{Q_\lambda^*-1}}.$$

An application of Rabinowitz [32, Proposition 9.30] ensures that the levels $c_1^\mu \leq c_2^\mu \leq \cdots \leq c_n^\mu$ are critical values of I_μ.

If $c_i^\mu = c_{i+1}^\mu$ where $i = 1, 2, \cdots, k-1$, then by Ambrosetti and Rabinowitz [2, Remark 2.12 and Theorem 4.2], the set $K_{c_i^\mu}$ consists of infinite number of different points and problem (1.1) has infinite numbers of weak solutions. Hence, problem (1.1) has at least n pairs of solutions. The proof of Theorem 1.3 is completed. □

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