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Research article

On p-Laplacian Kirchhoff-Schrödinger-Poisson type systems with critical growth on the Heisenberg group

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Abstract: In this article, we investigate the Kirchhoff-Schrödinger-Poisson type systems on the Heisenberg group of the following form:

$$\begin{cases} -(a + b\int_\Omega |\nabla_H u|^p d\xi)\Delta_{H,p}u - \mu\phi|u|^{p-2}u = \lambda|u|^{q-2}u + |u|^{Q^*-2}u & \text{in } \Omega, \\ -\Delta_H\phi = |u|^p & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where a, b are positive real numbers, $\Omega \subset \mathbb{H}^N$ is a bounded region with smooth boundary, $1 < p < Q$, $Q = 2N + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^N$, $Q^* = \frac{pQ}{Q-p}$, $q \in (2p, Q^*)$ and $\Delta_{H,p}u = \mathrm{div}(|\nabla_H u|^{p-2}\nabla_H u)$ is the p-horizontal Laplacian. Under some appropriate conditions for the parameters μ and λ, we establish existence and multiplicity results for the system above. To some extent, we generalize the results of An and Liu (Israel J. Math., 2020) and Liu et al. (Adv. Nonlinear Anal., 2022).

Keywords: Kirchhoff-Schrödinger-Poisson system; Heisenberg group; p-Laplacian operator; critical growth; concentration-compactness principle

1. Introduction

In this article, we investigate the following Kirchhoff-Schrödinger-Poisson type systems on the Heisenberg group:

$$\begin{cases} -(a+b\int_\Omega |\nabla_H u|^p d\xi)\Delta_{H,p}u - \mu\phi|u|^{p-2}u = \lambda|u|^{q-2}u + |u|^{Q^*-2}u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^p & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where a, b are positive real numbers, $\Omega \subset \mathbb{H}^N$ is a bounded region with smooth boundary, $1 < p < Q$, $Q = 2N + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^N$, $Q^* = \frac{pQ}{Q-p}$, $q \in (2p, Q^*)$, $\Delta_{H,p}u = \mathrm{div}(|\nabla_H u|^{p-2}\nabla_H u)$ is known as the p-horizontal Laplacian, and μ and λ are some positive real parameters.

In recent years, geometrical analysis of the Heisenberg group has found significant applications in quantum mechanics, partial differential equations and other fields, which has attracted the attention of many scholars who tried to establish the existence and multiplicity of solutions of partial differential equations on the Heisenberg group. For instance, in the subcase of problem (1.1), when $p = 2$ and $b = \mu = 0$, the existence of solutions for some nonlinear elliptic problems in bounded domains has been established. Tyagi [1] studied a class of singular boundary value problem on the Heisenberg group:

$$\begin{cases} -\Delta_H u = \mu \frac{g(\xi)u}{(|z|^4+t^2)^{\frac{1}{2}}} + \lambda f(\xi, t), \quad \xi \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

and under appropriate conditions, obtained some existence results using Bonanno's three critical point theorem. Goel and Sreenadh [2] dealt with a class of Choquard type equation on the Heisenberg group, and they established regularity of solutions and nonexistence of solutions invoking the mountain pass theorem, the linking theorem and iteration techniques and boot-strap method.

In the case $b \neq 0$ and $\mu = 0$, problem (1.1) becomes the Kirchhoff problem, which has also been widely studied. For example, Sun et al. [3] dealt with the following Choquard-Kirchhoff problem with critical growth:

$$M(\|u\|^2)(-\Delta_H u + u) = \int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\lambda^*}}{|\eta^{-1}\xi|^\lambda} d\eta |u|^{Q_\lambda^*-2}u + \mu f(\xi, u),$$

where f is a Carathéodory function, M is the Kirchhoff function, Δ_H is the Kohn Laplacian on the Heisenberg group $\mathbb{H}^N$, $\mu > 0$ is a parameter and $Q_\lambda^* = \frac{2Q-\lambda}{Q-2}$ is the critical exponent. In their paper, a new version of the concentration-compactness principle on the Heisenberg group was established for the first time. Moreover, the existence of nontrivial solutions was obtained even under nondegenerate and degenerate conditions. Zhou et al. [4] proved the existence of solutions of Kirchhoff type nonlocal integral-differential operators with homogeneous Dirichlet boundary conditions on the Heisenberg group using the variational method and the mountain pass theorem. Deng and Xian [5] obtained the existence of solutions for Kirchhoff type systems involving the Q-Laplacian operator on the Heisenberg group with the help of the Trudinger-Moser inequality and the mountain pass theorem. For more related results, see [6–13].

When $b = 0$, $p = 2$ and $\mu \neq 0$, problem (1.1) becomes the Schrödinger-Poisson system. This is a very interesting subject and has recently witnessed very profound results. For example, An and

Liu [14] dealt with the following forms of Schrödinger-Poisson type system on the Heisenberg group:

$$\begin{cases} -\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u + |u|^2 u & \text{in } \Omega, \\ -\Delta_H \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in \mathbb{R}$ and $\lambda > 0$ are some real parameters and $1 < q < 2$. By applying the concentration compactness and the critical point theory, they found at least two positive solutions and a positive ground state solution.

Liang and Pucci [15] studied the following critical Kirchhoff-Poisson system on the Heisenberg group:

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \phi|u|^{q-2}u = h(\xi, u) + \lambda|u|^2 u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$ and $1 < q < 2$. By applying the symmetric mountain pass lemma, they obtained the multiplicity of solutions with λ sufficiently small.

The Kirchhoff-Poisson system on the Heisenberg group with logarithmic and critical nonlinearity was considered by Pucci and Ye [13]:

$$\begin{cases} -M\left(\int_\Omega |\nabla_H u|^2 d\xi\right)\Delta_H u + \phi u = |u|^2 u + \lambda|u|^{q-2}u \ln|u|^2 & \text{in } \Omega, \\ -\Delta_H \phi = |u|^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Under suitable assumptions on the Kirchhoff function M covering the degenerate case, they showed that for a sufficiently large $\lambda > 0$, there exists a nontrivial solution to the above problem.

When $p \neq 2$ and $\mu \neq 0$, as far as we know, for Kirchhoff-Schrodinger-Poisson systems (1.1) with critical nonlinearities on the Heisenberg group, existence and multiplicity results are not yet available. In the Euclidean case, Du et al. [16] first studied the existence results for the Kirchhoff-Poisson systems with p-Laplacian under the subcritical case using the mountain pass theorem. Later, Du et al. [17] studied quasilinear Schrödinger-Poisson systems. For the critical case, Du et al. [18] also obtained the existence of ground state solutions with the variational approach.

Inspired by the above achievements, we aim to establish some results on the existence and multiplicity of nontrivial solutions of the Kirchhoff-Schrödinger-Poisson systems (1.1). The major difficulties in dealing with problem (1.1) are the presence of a nonlocal term and critical nonlinearities making the study of this problem very challenging.

Before presenting the main results of this article, we first present some concepts of the Heisenberg group. The Heisenberg group is represented by $\mathbb{H}^N$. If $\xi = (x, y, t) \in \mathbb{H}^N$, then the definition of this group operation is

$$\tau_\xi(\xi') = \xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x)), \text{ for every } \xi, \xi' \in \mathbb{H}^N,$$

$\xi^{-1} = -\xi$ is the inverse, and therefore $(\xi')^{-1} \circ \xi^{-1} = (\xi \circ \xi')^{-1}$.

The definition of a natural group of dilations on $\mathbb{H}^N$ is $\delta_s(\xi) = (sx, sy, s^2t)$, for every $s > 0$. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. It can be easily proved that the Jacobian determinant of dilatations $\delta_s : \mathbb{H}^N \to \mathbb{H}^N$ is constant and equal to s^Q, for every $\xi = (x, y, t) \in \mathbb{H}^N$. The critical exponent is $Q^* := \frac{pQ}{Q-p}$, where the natural number $Q = 2N + 2$ is called the homogeneous dimension of $\mathbb{H}^N$. We define the Korányi norm as follows

$$|\xi|_H = \left[(x^2 + y^2)^2 + t^2\right]^{\frac{1}{4}}, \text{ for every } \xi \in \mathbb{H}^N,$$

and we derive this norm from the Heisenberg group's anisotropic dilation. Hence, the homogeneous degree of the Korányi norm is equal to 1, in terms of dilations

$$\delta_s : (x, y, t) \mapsto (sx, sy, s^2t), \text{ for every } s > 0.$$

The set

$$B_H(\xi_0, r) = \{\xi \in \mathbb{H}^N : d_H(\xi_0, \xi) < r\},$$

denotes the Korányi open ball of radius r centered at ξ_0. For the sake of simplicity, we shall denote $B_r = B_r(O)$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^N$.

The following vector fields

$$T = \frac{\partial}{\partial t},\ X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\ Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},$$

generate the real Lie algebra of left invariant vector fields for $j = 1, \cdots, n$, which forms a basis satisfying the Heisenberg regular commutation relation on $\mathbb{H}^N$. This means that

$$[X_j, Y_j] = -4\delta_{jk}T,\ [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

The so-called horizontal vector field is just a vector field with the span of $[X_j, Y_j]_{j=1}^n$.

The Heisenberg gradient on $\mathbb{H}^N$ is

$$\nabla_H = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n),$$

and the Kohn Laplacian on $\mathbb{H}^N$ is given by

$$\Delta_H = \sum_{j=1}^{N} X_j^2 + Y_j^2 = \sum_{j=1}^{N}[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j\frac{\partial^2}{\partial x_j \partial t} - 4x_j\frac{\partial^2}{\partial x_j \partial t} + 4(x_j^2 + y_j^2)\frac{\partial^2}{\partial t^2}].$$

The Haar measure is invariant under the left translations of the Heisenberg group and is Q-homogeneous in terms of dilations. More precisely, it is consistent with the $(2n + 1)$-dimensional Lebesgue measure. Hence, as shown Leonardi and Masnou [19], the topological dimension $2N + 1$ of $\mathbb{H}^N$ is strictly less than its Hausdorff dimension $Q = 2N+2$. Next, $|\Omega|$ denotes the (2N + 1)-dimensional Lebesgue measure of any measurable set $\Omega \subseteq \mathbb{H}^N$. Hence,

$$|\delta_s(\Omega)| = s^Q|\Omega|,\ d(\delta_s\xi) = s^Q d\xi \text{ and } |B_H(\xi_0, r)| = \alpha_Q r^Q, \text{ where } \alpha_Q = |B_H(0, 1)|.$$

Now, we can state the main result of the paper.

Theorem 1.1. *Let $q \in (2p, Q^*)$. Then there exist positive constants μ_1 and λ_1 such that for every $\mu \in (0, \mu_1)$ and $\lambda \in (\lambda_1, +\infty)$, the following assertions hold:*

(I) *Problem* (1.1) *has a nontrivial weak solution;*
(II) *Problem* (1.1) *has infinitely many nontrivial weak solutions if parameter a is large enough.*

We can give the following example for problem (1.1) with $p = 3$ and $\Omega \subset \mathbb{H}^1$:

$$\begin{cases} -(a + \int_\Omega |\nabla_H u|^2 d\xi)\Delta_H u + \mu\phi u = \lambda |u|^6 u + |u|^{10} u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^3 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, $N = 1$, $p = 3$ and $q = 8$, then $Q = 2N + 2 = 4$, $Q^* = 12$. If positive parameters μ small enough and λ large enough, by Theorem 1.1, we know that problem (1.1) has a nontrivial weak solution. Moreover, if in addition the parameter a is large enough, problem (1.1) has infinitely many nontrivial weak solutions. It should be noted that the methods in An and Liu [14] and Liang and Pucci [15] do not seem to apply to problem (1.1).

Remark 1.1. *Compared with previous results, this paper has the following key new features:*

1) *The presence of the nonlocal term $\phi|u|^{p-2}u$;*
2) *The lack of compactness caused by critical index;*
3) *The presence of the p-Laplacian makes this problem more complex and interesting.*

It is worth stressing that the nonlocal term and the critical exponent lead to the lack of compactness condition, and we use the concentration-compactness principle to overcome this difficulty. Moreover, we shall use some more refined estimates to overcome the presence of the p-Laplacian.

We need to emphasize here that despite the similarity of some properties between the classical Laplacian Δ and Kohn Laplacian Δ_H, similarities can be misleading (see Garofalo and Lanconelli [20]), so there are still many properties that deserve further study. Moreover, for the case $p \neq 2$, it is difficult to prove the boundedness of Palais-Smale sequences. In order to overcome these difficulties, we use some more accurate estimates of relevant expression. Additionally, we use the concentration-compactness principle on the Heisenberg group to prove the compactness condition.

The paper is organized as follows. In Section 2, we introduce some notations and known facts. Moreover, we introduce some key estimates. In addition, we define the corresponding energy functional I_λ and its derivative at u, that is, $I'_\lambda(u)$. In Section 3, we prove Theorem 1.1.

2. Preliminaries and $(PS)_c$ condition

2.1. Preliminaries

First of all, we collected some known facts, useful in the sequel. For additional background material, readers are advised to refer to Papageorgiou et al. [21].

Let

$$\|u\|_s^s = \int_\Omega |u|^s d\xi, \text{ for every } u \in L^s(\Omega),$$

represent the usual L^s-norm.

Following Folland and Stein [22], we define the space $\mathring{S}^2_1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $S^2_1(\mathbb{H}^N)$. Then $\mathring{S}^2_1(\Omega)$ is a Hilbert space with the norm

$$||u||^2_{\mathring{S}^2_1(\Omega)} = \int_\Omega |\nabla_H u|^2 d\xi.$$

We define the Folland-Stein space $S^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with the norm

$$||u|| = (\int_\Omega |\nabla_H u|^p d\xi)^{\frac{1}{p}}.$$

Then the embedding

$$S^{1,p}(\Omega) \hookrightarrow L^s(\Omega), \text{ for every } s \in (1, Q^*),$$

is compact. However, if $s = Q^*$, the embedding is only continuous (see Vassiliev [23]).

Additionally, we say that $(u, \phi) \in S^{1,p}(\Omega) \times S^{1,p}(\Omega)$ is a solution of problem (1.1) if and only if

$$\begin{aligned} a\int_\Omega |\nabla_H u|^{p-2}\nabla_H u \nabla_H v d\xi + b||u||^p \int_\Omega |\nabla_H u|^{p-2}\nabla_H u \nabla_H v d\xi \\ -\mu \int_\Omega \phi |u|^{p-2} u v d\xi - \lambda \int_\Omega |u|^{q-2} u v d\xi - \int_\Omega |u|^{Q^*-2} u v d\xi = 0 \end{aligned}$$

and

$$\int_\Omega \nabla_H \phi \nabla_H \omega d\xi - \int_\Omega |u|^p \omega d\xi = 0,$$

for every $v, \omega \in S^{1,p}(\Omega) \times S^{1,p}(\Omega)$. Moreover, $(u, \phi) \in S^{1,p}(\Omega) \times S^{1,p}(\Omega)$ is a positive solution of problem (1.1) if u and ϕ are both positive. Therefore, in order to apply the critical point theory, we need to define the functional $J(u, \phi) : S^{1,p}(\Omega) \times S^{1,p}(\Omega) \to \mathbb{R}$ as follows

$$J(u, \phi) = \frac{a}{p}||u||^p + \frac{b}{2p}||u||^{2p} + \frac{\mu}{2p}\int_\Omega |\nabla_H \phi|^2 d\xi - \frac{\mu}{p}\int_\Omega \phi |u|^p d\xi - \frac{\lambda}{q}\int_\Omega |u|^q d\xi - \frac{1}{Q^*}\int_\Omega |u|^{Q^*} d\xi,$$

for every $(u, \phi) \in S^{1,p}(\Omega) \times S^{1,p}(\Omega)$. Then J is C^1 on $S^{1,p}(\Omega) \times S^{1,p}(\Omega)$ and its critical points are the solutions of problem (1.1). Indeed, the partial derivatives of J at (u, ϕ) are denoted by $J'_u(u, \phi)$, $J'_\phi(u, \phi)$, namely for every $v, \omega \in S^{1,p}(\Omega) \times S^{1,p}(\Omega)$,

$$\begin{aligned} J'_u(u, \phi)[v] =& a\int_\Omega |\nabla_H u|^{p-2}\nabla_H u \nabla_H v d\xi + b||u||^p \int_\Omega |\nabla_H u|^{p-2}\nabla_H u \nabla_H v d\xi \\ &-\mu \int_\Omega \phi |u|^{p-2} u v d\xi - \lambda \int_\Omega |u|^{q-2} u v d\xi - \int_\Omega |u|^{Q^*-2} u v d\xi = 0 \end{aligned}$$

and

$$J'_\phi(u, \phi) = \frac{\mu}{p}\int_\Omega \nabla_H \phi \nabla_H \omega d\xi - \frac{\mu}{p}\int_\Omega |u|^p \omega d\xi.$$

Standard computations show that J'_u (respectively J'_ϕ) continuously maps $S^{1,p}(\Omega) \times S^{1,p}(\Omega)$ into the dual of $S^{1,p}(\Omega)$. Moreover, the functional J is C^1 on $S^{1,p}(\Omega) \times S^{1,p}(\Omega)$ and

$$J'_u(u, \phi) = J'_\phi(u, \phi) = 0$$

if and only if (u, ϕ) is a solution of problem (1.1).

Lemma 2.1. *Let $u \in S^{1,p}(\Omega)$. Then there is a unique nonnegative function $\phi_u \in \mathring{S}^2_1(\Omega)$ such that*

$$\begin{cases} -\Delta_H \phi = |u|^p & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Furthermore, $\phi_u \geq 0$ and $\phi_u > 0$ if $u \neq 0$. Also,
(i) $\phi_{tu} = t^p \phi_u$, for every $t > 0$;
(ii) $\|\phi_u\|_{\mathring{S}^2_1(\Omega)} \leq \hat{C}\|u\|^p$, where $\hat{C} > 0$;
(iii) Let $u_n \rightharpoonup u$ in $S^{1,p}(\Omega)$. Then, $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathring{S}^2_1(\Omega)$, and

$$\int_\Omega \phi_{u_n} |u_n|^{p-2} u_n v d\xi \to \int_\Omega \phi_u |u|^{p-2} u v d\xi, \text{ for every } v \in S^{1,p}(\Omega). \tag{2.2}$$

Proof. For any $u \in \mathring{S}^2_1(\Omega)$, we define $W : \mathring{S}^2_1(\Omega) \to \mathbb{R}$,

$$W(v) = \int_\Omega v|u|^p d\xi, \text{ for every } v \in \mathring{S}^2_1(\Omega).$$

Let $v_n \to v \in \mathring{S}^2_1(\Omega)$, as $n \to \infty$. It follows by the Hölder inequality that

$$\begin{aligned} |W(v_n) - W(v)| &\leq \int_\Omega (v_n - v)|u|^p d\xi \\ &\leq \Big(\int_\Omega |v_n - v|^{Q^*} d\xi\Big)^{\frac{1}{Q^*}} \Big(\int_\Omega |u|^{\frac{pQ^*}{Q^*-1}} d\xi\Big)^{\frac{Q^*-1}{Q^*}} \\ &\leq S^{-\frac{1}{p}} \|v_n - v\| \|u\|^p_{\frac{pQ^*}{Q^*-1}} \to 0, \text{ as } n \to \infty, \end{aligned}$$

where

$$S = \inf_{u \in S^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla_H u|^p d\xi}{(\int_\Omega |u|^{Q^*} d\xi)^{\frac{p}{Q^*}}} \tag{2.3}$$

is the best Sobolev constant. This implies that W is a continuous linear functional. Using the Lax-Milgram theorem, we see that there is a unique $\phi_u \in \mathring{S}^2_1(\Omega)$ satisfying

$$\int_\Omega \nabla_H \phi_u \nabla_H v d\xi = \int_\Omega v|u|^p d\xi, \text{ for every } v \in S^{1,p}(\Omega). \tag{2.4}$$

Thus, $\phi_u \in \mathring{S}^2_1(\Omega)$ is the unique solution of problem (2.1). Moreover, applying the maximum principle, one has $\phi_u \geq 0$ and $\phi_u > 0$ if $u \neq 0$. Indeed, for every $t > 0$, one has

$$-\Delta_H \phi_{tu} = t^p u^p = t^p(-\Delta_H \phi_u) = -\Delta_H(t^p \phi_u).$$

Hence $\phi_{tu} = t^p \phi_u$ due to the uniqueness of ϕ_u.

Furthermore, since $\phi_u \in S^{1,p}(\Omega)$, we can view it as a text function in problem (2.1). Then by (2.4), the Sobolev inequality and the Hölder inequality, we have (henceforth C_0, C_1, C_2 will denote positive constants)

$$\int_\Omega |\nabla_H \phi_u|^2 d\xi = \int_\Omega \phi_u |u|^p d\xi \leq |\phi_u|_{L^2(\Omega)} |u|^p_{L^{2p}(\Omega)} \leq C_1 \|\phi_u\|_{\mathring{S}^2_1(\Omega)} \|u\|^p.$$

Therefore, we get $\|\phi_u\|_{\mathring{S}^2_1(\Omega)} \leq C_1\|u\|^p$.

Since $u_n \rightharpoonup u$ in $S^{1,p}(\Omega)$, we can conclude that $u_n \to u$ a.e. in Ω and $\{|u_n|^p\}$ is bounded in $L^2(\Omega)$. Moreover, we have $|u_n|^p \rightharpoonup |u|^p$ in $L^2(\Omega)$. Then for every $v \in \mathring{S}^2_1(\Omega)$, it follows that

$$\int_\Omega v|u_n|^p d\xi \to \int_\Omega v|u|^p d\xi, \text{ as } n \to \infty.$$

Therefore, $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathring{S}^2_1(\Omega)$. By the Hölder inequality, the Sobolev inequality and (ii), one has

$$\begin{aligned}\int_\Omega |\phi_{u_n}|u_n|^{p-2}u_n|^{\frac{2p}{2p-1}} d\xi &\leq |\phi_{u_n}|^{\frac{2p}{2p-1}}_{L^2(\Omega)}\Big(\int_\Omega |u_n|^{2p}d\xi\Big)^{\frac{p-1}{2p-1}} \\ &\leq C_0\|\phi_{u_n}\|^{\frac{2p}{2p-1}}_{\mathring{S}^2_1(\Omega)}\Big(\int_\Omega |u_n|^{2p}d\xi\Big)^{\frac{p-1}{2p-1}} \\ &\leq C_2\|\phi_{u_n}\|^{\frac{2p^2}{2p-1}}\Big(\int_\Omega |u_n|^{2p}d\xi\Big)^{\frac{p-1}{2p-1}}.\end{aligned}$$

Hence, $\{\phi_{u_n}|u_n|^{p-2}u_n\}$ is bounded in $L^{\frac{2p}{2p-1}}(\Omega)$. Since

$$\phi_{u_n}|u_n|^{p-2}u_n \to \phi_u|u|^{p-2}u, \text{ a.e. in } \Omega,$$

we get

$$\int_\Omega \phi_{u_n}|u_n|^{p-2}u_n v d\xi \to \int_\Omega \phi_u|u|^{p-2}uvd\xi, \text{ for every } v \in S^{1,p}(\Omega).$$

The proof of Lemma 2.1 is complete.

By similar arguments as in An and Liu [14], we can get the following result.

Lemma 2.2. *Let $\Psi(u) = \phi_u$ for every $u \in S^{1,p}(\Omega)$, where ϕ_u is as in Lemma 2.1, and let*

$$\Upsilon = \{(u,\phi) \in S^{1,p}(\Omega) \times S^{1,p}(\Omega) : J'_\phi(u,\phi) = 0\}.$$

Then Ψ is C^1 and Υ is the graph of Ψ.

We define the corresponding energy functional $I_\lambda(u) = J(u,\phi_u)$ of problem (1.1) by

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{\mu}{2p}\int_\Omega \phi_u|u|^p d\xi - \frac{\lambda}{q}\int_\Omega |u|^q d\xi - \frac{1}{Q^*}\int_\Omega |u|^{Q^*} d\xi, \text{ for every } u \in S^{1,p}(\Omega). \quad (2.5)$$

Based on the definition of J and Lemma 2.2, we can conclude that I_λ is of C^1.

Lemma 2.3. *(see An and Liu [14]) Let $(u,\phi) \in S^{1,p}(\Omega) \times S^{1,p}(\Omega)$. Then (u,ϕ) is a critical point of J if and only if u is a critical point of I_λ and $\phi = \Psi(u)$, where Ψ was defined in Lemma 2.2.*

According to Lemma 2.3, we know that a solution (u,ϕ_u) of problem (1.1) corresponds to a critical point u of the functional I_λ with $\phi = \Psi(u)$ and

$$\begin{aligned}\langle I'_\lambda(u), v\rangle =& a\int_\Omega |\nabla_H u|^{p-2}\nabla_H u \nabla_H v d\xi + b\|u\|^p \int_\Omega |\nabla_H u|^{p-2}\nabla_H u\nabla_H v d\xi \\ &- \mu\int_\Omega \phi_u|u|^{p-2}uvd\xi - \lambda\int_\Omega |u|^{q-2}uvd\xi - \int_\Omega |u|^{Q^*-2}uvd\xi, \text{ for every } v \in S^{1,p}(\Omega).\end{aligned} \quad (2.6)$$

Therefore, based on the above arguments, we shall strive to use critical point theory and some analytical techniques to prove the existence of critical points of functional I_λ.

2.2. $(PS)_c$ condition

In this subsection, our main focus will be on proving that the functional I_λ satisfies the Palais-Smale condition.

Lemma 2.4. *Let $q \in (2p, Q^*)$. Then there exists $\mu_1 > 0$ such that for any $\mu < \mu_1$, the energy functional I_λ satisfies $(PS)_c$ condition, where*

$$c \in \left(0,\ (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}}\right) \tag{2.7}$$

and S is the best Sobolev constant given by (2.3).

Proof. Let us assume that $\{u_n\}_n \subset S^{1,p}(\Omega)$ is a $(PS)_c$ sequence related to the functional I_λ, that is,

$$I_\lambda(u_n) \to c \text{ and } I'_\lambda(u_n) \to 0, \text{as } n \to \infty. \tag{2.8}$$

It follows that

$$\begin{aligned} c + o(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{q} I'_\lambda(u_n)u_n \\ &\geq a(\frac{1}{p} - \frac{1}{q})\|u_n\|^p + b(\frac{1}{2p} - \frac{1}{q})\|u_n\|^{2p} - \mu(\frac{1}{2p} - \frac{1}{q}) \int_\Omega \phi_{u_n}|u_n|^p d\xi + (\frac{1}{q} - \frac{1}{Q^*}) \int_\Omega |u_n|^{Q^*} d\xi \\ &\geq a(\frac{1}{p} - \frac{1}{q})\|u_n\|^p + (b - \mu\hat{C})(\frac{1}{2p} - \frac{1}{q})\|u_n\|^{2p}, \end{aligned} \tag{2.9}$$

where $\hat{C}$ is a positive constant given by Lemma 2.1(ii). Let $\mu_1 = \frac{b}{\hat{C}}$. By (2.9), we know that $(PS)_c$ sequence $\{u_n\}_n \subset S^{1,p}(\Omega)$ is bounded for every $\mu < \mu_1$. Thus, we may assume that $u_n \rightharpoonup u$ weakly in $S^{1,p}(\Omega)$, and $u_n \to u$ in $L^s(\Omega)$ with $1 < s < Q^*$. Furthermore, since $I_\lambda(u_n) = I_\lambda(|u_n|)$, we may also assume that $u_n \geq 0$ and $u \geq 0$. Therefore, invoking the concentration compactness principle on the Heisenberg group (see Vassiliev [23, Lemma 3.5]), we obtain

$$\begin{aligned} &|\nabla_H u_n|^p d\xi \rightharpoonup d\omega \geq |\nabla_H u|^p d\xi + \Sigma_{j\in\Lambda}\omega_j\delta_{x_j}, \\ &|u_n|^{Q^*} d\xi \rightharpoonup d\nu = |u|^{Q^*} d\xi + \Sigma_{j\in\Lambda}\nu_j\delta_{x_j}, \end{aligned} \tag{2.10}$$

where $\{x_j\}_{j\in\Lambda} \subset \Omega$ is the most a countable set of distinct points, ω and ν in $\mathbb{H}^N$ are two positive Radon measures, and $\{\omega_j\}_{j\in\Lambda}$, $\{\nu_j\}_{j\in\Lambda}$ are nonnegative numbers. Moreover, we have

$$\omega_j \geq S\nu_j^{\frac{p}{Q^*}}. \tag{2.11}$$

Next, we shall show that $\Lambda = \emptyset$. Indeed, assume that the hypothesis $\omega_j \neq 0$ holds for some $j \in \Lambda$. Then when $\varepsilon > 0$ is sufficiently small, we can find $0 \leq \psi_{\varepsilon,j} \leq 1$ satisfying the following

$$\begin{cases} \psi_{\varepsilon,j} = 1 & \text{in } B_H(\xi_j, \frac{\varepsilon}{2}), \\ \psi_{\varepsilon,j} = 0 & \text{in } \Omega\backslash B_H(\xi_j, \varepsilon), \\ |\nabla_H\psi_{\varepsilon,j}| \leq \frac{2}{\varepsilon}, \end{cases} \tag{2.12}$$

where $\psi_{\varepsilon,j} \in C_0^\infty(B_H(\xi_j,\varepsilon))$ is a cut-off function. Clearly, $(u_n\psi_{\varepsilon,j})_n$ is bounded in $S^{1,p}(\Omega)$. It follows from (2.8) and the boundedness of $(u_n\psi_{\varepsilon,j})_n$ that

$$\langle I'_\lambda(u_n), u_n\psi_{\varepsilon,j}\rangle \to 0, \text{ as } n \to \infty,$$

that is,

$$\begin{aligned} &a\Big(\int_\Omega |\nabla_H u_n|^p\psi_{\varepsilon,j} + \int_\Omega u_n|\nabla_H u_n|^{p-2}\nabla_H u_n \nabla_H\psi_{\varepsilon,j} d\xi\Big) + b\|u_n\|^p\Big(\int_\Omega |\nabla_H u_n|^p\psi_{\varepsilon,j} \\ &+ \int_\Omega u_n|\nabla_H u_n|^{p-2}\nabla_H u_n\nabla_H\psi_{\varepsilon,j}d\xi\Big) - \mu\int_\Omega \phi_{u_n}|u_n|^p\psi_{\varepsilon,j}d\xi = \lambda\int_\Omega u_n^q\psi_{\varepsilon,j}d\xi + \int_\Omega u_n^{Q^*}\psi_{\varepsilon,j}d\xi + o(1). \end{aligned} \tag{2.13}$$

It follows from the dominated convergence theorem that

$$\int_{B_H(\xi_j,\varepsilon)} |u_n|^q\psi_{\varepsilon,j}d\xi \to \int_{B_H(\xi_j,\varepsilon)} |u|^q\psi_{\varepsilon,j}d\xi, \text{ as } n \to \infty.$$

Hence, letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_H(\xi_j,\varepsilon)} |u_n|^q\psi_{\varepsilon,j}d\xi = 0. \tag{2.14}$$

By Lemma 2.1,

$$\lim_{n\to\infty}\int_\Omega \phi_{u_n}|u_n|^{p-2}u_n u d\xi = \int_\Omega \phi_u|u|^p d\xi, \tag{2.15}$$

and since $u_n \to u$ in $L^s(\Omega)$ with $1 < s < Q^*$, one has

$$\begin{aligned} \int_\Omega (\phi_{u_n}|u_n|^p - \phi_{u_n}|u_n|^{p-1}u)d\xi &\le \int_\Omega |\phi_{u_n}||u_n|^{p-1}|u_n - u|d\xi \\ &\le \left(\int_\Omega |\phi_{u_n}|u_n|^{p-1}|^{\frac{p}{p-1}}d\xi\right)^{\frac{p-1}{p}} |u_n - u|_p \to 0. \end{aligned} \tag{2.16}$$

Combining (2.15) with (2.16), we obtain that

$$\lim_{n\to\infty}\int_\Omega \phi_{u_n}|u_n|^p d\xi = \int_\Omega \phi_u|u|^p d\xi,$$

thus

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_H(\xi_j,\varepsilon)} \phi_{u_n}|u_n|^p\psi_{\varepsilon,j}d\xi = \lim_{\varepsilon\to 0}\int_{B_H(\xi_j,\varepsilon)} \phi_u|u|^p\psi_{\varepsilon,j}d\xi = 0. \tag{2.17}$$

Since

$$\int_{B_H(\xi_j,\varepsilon)} d\xi = \int_{B_H(0,\varepsilon)} d\xi = |B_H(0,1)|\varepsilon^Q,$$

applying the Hölder inequality, we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n|\nabla_H u_n|^{p-1}\nabla_H\psi_{\varepsilon,j}d\xi \le \lim_{\varepsilon\to 0}\lim_{n\to\infty}(\int_{B_H(\xi_j,\varepsilon)}|\nabla_H u_n|^p d\xi)^{\frac{p-1}{p}}(\int_{B_H(\xi_j,\varepsilon)}|u_n\nabla_H\psi_{\varepsilon,j}|^p d\xi)^{\frac{1}{p}}$$
$$\le C\lim_{\varepsilon\to 0}(\int_{B_H(\xi_j,\varepsilon)}|u_n|^p|\nabla_H\psi_{\varepsilon,j}|^p d\xi)^{\frac{1}{p}}$$
$$\le C\lim_{\varepsilon\to 0}(\int_{B_H(\xi_j,\varepsilon)}|u_n|^{Q^*}d\xi)^{\frac{1}{Q^*}}(\int_{B_H(\xi_j,\varepsilon)}|\nabla_H\psi_{\varepsilon,j}|^Q d\xi)^{\frac{1}{Q}} = 0.$$

By (2.10), we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla_H u_n|^p\psi_{\varepsilon,j}d\xi \ge \lim_{\varepsilon\to 0}\left(\omega_j + \int_{B_H(\xi_j,\varepsilon)}|\nabla_H u|^p\psi_{\varepsilon,j}d\xi\right) = \omega_j \tag{2.18}$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n^{Q^*}\psi_{\varepsilon,j}d\xi = \lim_{\varepsilon\to 0}\left(\nu_j + \int_{B_H(\xi_j,\varepsilon)}u_n^{Q^*}\psi_{\varepsilon,j}d\xi\right) = \nu_j. \tag{2.19}$$

Therefore, by (2.13)–(2.19), one gets $\nu_j \ge a\omega_j$. It follows from (2.11) that

$$\nu_j = 0 \quad \text{or} \quad \nu_j \ge (aS)^{\frac{Q}{p}}.$$

In fact, if $\nu_j \ge (aS)^{\frac{Q}{p}}$ holds, therefore by (2.8) and (2.10), for every $\mu < \mu_1$, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\{I_\lambda(u_n) - \frac{1}{q}I'_\lambda(u_n)u_n\} \ge \lim_{n\to\infty}(\frac{1}{q} - \frac{1}{Q^*})\int_\Omega |u_n|^{Q^*}d\xi \\ &\ge (\frac{1}{q} - \frac{1}{Q^*})\nu_j \ge (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}}, \end{aligned} \tag{2.20}$$

which contradicts (2.7). Thus, $\Lambda = \emptyset$. By (2.10) and $\Lambda = \emptyset$, we have

$$\int_\Omega |u_n|^{Q^*}d\xi \to \int_\Omega |u|^{Q^*}d\xi. \tag{2.21}$$

Let

$$\lim_{n\to\infty}\|u_n\|^p = A.$$

If $A = 0$, then $u_n \to 0$ in $S^{1,p}(\Omega)$. So assume now that $A > 0$. By (2.8), we get

$$\begin{aligned} &a\int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n\nabla_H v d\xi + bA\int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n\nabla_H v d\xi \\ &- \mu\int_\Omega \phi_{u_n}|u_n|^{p-2}u_n v d\xi - \lambda\int_\Omega |u_n|^{q-2}u_n v d\xi - \int_\Omega |u_n|^{Q^*-2}u_n v d\xi = o(1). \end{aligned} \tag{2.22}$$

Let $v = u$ in (2.22). Then

$$a\|u\|^p + bA\|u\|^p - \mu\int_\Omega \phi_u|u|^p d\xi - \lambda\int_\Omega |u|^q d\xi - \int_\Omega |u|^{Q^*}d\xi = 0. \tag{2.23}$$

By (2.8), (2.10), (2.21) and Lemma 2.1, one also has

$$\lim_{n\to\infty} a\|u_n\|^p + bA\|u_n\|^p - \mu\int_\Omega \phi_u|u|^p d\xi - \lambda\int_\Omega |u|^q d\xi - \int_\Omega |u|^{Q^*}d\xi = 0. \tag{2.24}$$

Thus, combining (2.23) and (2.24), we get $\lim_{n\to\infty}\|u_n\|^p = \|u\|^p$. Thus, we see that $u_n \to u$ in $S^{1,p}(\Omega)$ by the uniform convexity of $S^{1,p}(\Omega)$. This completes the proof of Lemma 2.4.

3. Proof of Theorem 1.1

3.1. Existence of a nontrivial weak solutions

We need the following auxiliary lemmas to prove our main result.

Lemma 3.1. *Let $q \in (2p, Q^*)$ and $\mu \in (0, \mu_1)$. Then functional I_λ satisfies the mountain pass geometry, that is,*
(i) There exist constants $\rho, \alpha > 0$ satisfying $I_\lambda(u)|_{\partial B_\rho} \geq \alpha$, for every $u \in S^{1,p}(\Omega)$;
(ii) There exists $e \in S^{1,p}(\Omega)\backslash\overline{B_\rho}$ satisfying $I_\lambda(e) < 0$.

Proof. First, applying the Hölder inequality, we get

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{\mu}{2p}\int_\Omega \phi_u|u|^p d\xi - \frac{\lambda}{q}\int_\Omega |u|^q d\xi - \frac{1}{Q^*}\int_\Omega |u|^{Q^*} d\xi \\ &\geq \frac{a}{p}\|u\|^p + \frac{b-\mu\hat{C}}{2p}\|u\|^{2p} - \frac{\lambda}{q}\int_\Omega |u|^q d\xi - \frac{1}{Q^*}\int_\Omega |u|^{Q^*} d\xi \\ &\geq \frac{a}{p}\|u\|^p - \frac{\lambda}{q}S^{-\frac{q}{p}}|\Omega|^{\frac{Q^*-q}{Q^*}}\|u\|^q - \frac{1}{Q^*}S^{-\frac{Q^*}{p}}\|u\|^{Q^*} \\ &= \|u\|^p\{\frac{a}{p} - \frac{\lambda}{q}S^{-\frac{q}{p}}|\Omega|^{\frac{Q^*-q}{Q^*}}\|u\|^{q-p} - \frac{1}{Q^*}S^{-\frac{Q^*}{p}}\|u\|^{Q^*-p}\}. \end{aligned} \tag{3.1}$$

Let

$$f(t) = \frac{a}{p} - \frac{\lambda}{q}S^{-\frac{q}{p}}|\Omega|^{\frac{Q^*-q}{Q^*}}t^{q-p} - \frac{1}{Q^*}S^{-\frac{Q^*}{p}}t^{Q^*-p}, \text{ for every } t \geq 0.$$

We now show that there exists a constant $\rho > 0$ satisfying $f(\rho) \geq \frac{a}{p}$. We see that f is a continuous function on $[0, +\infty)$ and $\lim_{t\to 0^+} f(t) = \frac{a}{p}$. Hence there exists ρ such that $f(t) \geq \frac{a}{p} - \varepsilon_1$, for every $0 \leq t \leq \rho$, where ρ is small enough such that $\|u\| = \rho$. If we choose $\varepsilon_1 = \frac{a}{2p}$, we have $f(t) \geq \frac{a}{2p}$, for every $0 \leq t \leq \rho$. In particular, $f(\rho) \geq \frac{a}{2p}$ and we obtain $I_\lambda(u) \geq \frac{a}{2p}\rho^p = \alpha$ for $\|u\| = \rho$. Hence assertion (i) of Lemma 3.1 holds.

Next, we shall show that assertion (ii) of Lemma 3.1 also holds:

$$\begin{aligned} I_\lambda(su) &= \frac{as^p}{p}\|u\|^p + \frac{bs^{2p}}{2p}\|u\|^{2p} - \frac{\mu s^{2p}}{2p}\int_\Omega \phi_u|u|^p d\xi - \frac{\lambda s^q}{q}\int_\Omega |u|^q d\xi - \frac{s^{Q^*}}{Q^*}\int_\Omega |u|^{Q^*} d\xi \\ &\leq \frac{as^p}{p}\|u\|^p + \frac{bs^{2p}}{2p}\|u\|^{2p} - \frac{s^{Q^*}}{Q^*}\int_\Omega |u|^{Q^*} d\xi \to -\infty \text{ as s} \to +\infty. \end{aligned} \tag{3.2}$$

Thus, we can deduce that $I_\lambda(s_0u) < 0$ and $s_0\|u\| > \rho$, for every s_0 large enough. Let $e = s_0u$. Then e is the desired function and the proof of (ii) of Lemma 3.1 is complete.

Proof of Theorem 1.1(I). We claim that

$$0 < c_\lambda = \inf_{h\in\Gamma}\max_{0\leq s\leq 1} I_\lambda(h(s)) < (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}}, \tag{3.3}$$

where

$$\Gamma = \{h \in C([0,1], S^{1,p}(\Omega)) : h(0) = 1, h(1) = e\}.$$

Indeed, we can choose $v_1 \in S^{1,p}(\Omega)\backslash\{0\}$ with $\|v_1\| = 1$. From (3.2), we have $\lim_{s\to+\infty} I_\lambda(sv_1) = -\infty$. Then

$$\sup_{s\geq 0} I_\lambda(sv_1) = I_\lambda(s_\lambda v_1), \quad \text{for some } s_\lambda > 0.$$

So s_λ satisfies

$$as_\lambda^p\|v_1\|^p + bs_\lambda^{2p}\|v_1\|^{2p} = \mu s_\lambda^{2p}\int_\Omega \phi_{v_1}|v_1|^p d\xi + \lambda s_\lambda^q \int_\Omega |v_1|^q d\xi + s_\lambda^{Q^*}\int_\Omega |v_1|^{Q^*} d\xi. \tag{3.4}$$

Next, we shall prove that $\{s_\lambda\}_{\mu>0}$ is bounded. In fact, suppose that the following hypothesis $s_\lambda \geq 1$ is satisfied for every $\lambda > 0$. Then it follows from (3.4) that

$$\begin{aligned}(a+b)s_\lambda^{2p} &\geq as_\lambda^p\|v_1\|^p + bs_\lambda^{2p}\|v_1\|^{2p} = \mu s_\lambda^{2p}\int_\Omega \phi_{v_1}|v_1|^p d\xi + \lambda s_\lambda^q \int_\Omega |v_1|^q d\xi + s_\lambda^{Q^*}\int_\Omega |v_1|^{Q^*} d\xi \\ &\geq s_\lambda^{Q^*}\int_\Omega |v_1|^{Q^*} d\xi.\end{aligned} \tag{3.5}$$

Since $2p < q < Q^*$, we can deduce that $\{s_\lambda\}_{\lambda>0}$ is bounded.

Next, we shall demonstrate that $s_\lambda \to 0$, as $\lambda \to \infty$. Suppose to the contrary, that there exist $s_\lambda > 0$ and a sequence $(\lambda_n)_n$ with $\lambda_n \to \infty$, as $n \to \infty$, satisfying $s_{\lambda_n} \to s_\lambda$, as $n \to \infty$. Invoking the Lebesgue dominated convergence theorem, we see that

$$\int_\Omega |s_{\lambda_n} v_1|^q d\xi \to \int_\Omega |s_\lambda v_1|^q d\xi, \text{ as } n \to \infty.$$

It now follows that

$$\lambda_n \int_\Omega |s_\lambda v_1|^q d\xi \to \infty, \text{ as } n \to \infty.$$

Thus, invoking (3.4), we can show that this cannot happen. Therefore, $s_\lambda \to 0$, as $\lambda \to \infty$.

Furthermore, (3.4) implies that

$$\lim_{\lambda\to\infty} \lambda \int_\Omega |s_\lambda v_1|^q d\xi = 0$$

and

$$\lim_{\lambda\to\infty} \int_\Omega |s_\lambda v_1|^{Q^*} d\xi = 0.$$

Hence based on the definition of I_λ and $s_\lambda \to 0$, as $\lambda \to \infty$, we get that

$$\lim_{\lambda\to\infty}(\sup_{s\geq 0} I_\lambda(sv_1)) = \lim_{\lambda\to\infty} I_\lambda(s_\lambda v_1) = 0.$$

So, there is $\lambda_1 > 0$, satisfying for every $\lambda > \lambda_1$,

$$\sup_{s\geq 0} I_\lambda(sv_1) < (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}}.$$

Letting $e = t_1 v_1$ with t_1 large enough for $I_\lambda(e) < 0$, we get

$$0 < c_\lambda \leq \max_{0\leq s\leq 1} I_\lambda(h(s)), \text{ where } h(s) = st_1 v_1.$$

Therefore

$$0 < c_\lambda \leq \sup_{s\geq 0} I_\lambda(sv_1) < (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}},$$

for λ large enough. This completes the proof of Theorem 1.1(I).

3.2. Existence of infinitely many nontrivial weak solutions

In this subsection, we shall use the Krasnoselskii genus theory to prove Theorem 1.1(II). To this end, let E be a Banach space and denote by Λ the class of all closed subsets $A \subset E\backslash\{0\}$ that are symmetric with respect to the origin, that is, $u \in E$ implies $-u \in E$. Moreover, suppose that X is k-dimensional and $X = \mathrm{span}\{z_1, \cdots, z_k\}$. For every $n \geq k$, inductively select $z_{n+1} \notin X_n = \mathrm{span}\{z_1, \cdots, z_n\}$. Let $R_n = R(Z_n)$ and $\Upsilon_n = B_{R_n} \bigcap Z_n$. Define

$$W_n = \{\varphi \in C(\Upsilon_n, E) : \varphi|\partial_{B_{R_n} \bigcap Z_n} = id \text{ and } \varphi \text{ is odd}\}$$

and

$$\Gamma_i = \{\varphi(\overline{\Upsilon_n \setminus V}) : \varphi \in W_n, n \geq i, V \in \Lambda, \Lambda \text{ is closed}, \gamma(V) \leq n - i\},$$

where $\gamma(V)$ is the Krasnoselskii genus of V.

Theorem 3.1. *(see Rabinowitz [24, Theorem 9.12]) Let $I \in C^1(E, \mathbb{R})$ be even with $I(0) = 0$ and let E be an infinite-dimensional Banach space. Assume that X is a finite-dimensional space, $E = X \oplus Y$ and that I satisfies the following properties:*
(i) There exists $\theta > 0$ such that I satisfies $(PS)_c$ condition, for every $c \in (0, \theta)$;
(ii) There exist $\rho, \alpha > 0$ satifying $I(u) \geq \alpha$, for every $u \in \partial B_\rho \bigcap Y$;
(iii) For every finite-dimensional subspace $\tilde{E} \subset E$, there exists $R = R(\tilde{E}) > \rho$ such that $I(u) \leq 0$ on $\tilde{E} \setminus B_R$.
For every $i \in N$, let $c_i = \inf_{X\in\Gamma_i} \max_{u\in Z} I(u)$, hence, $0 \leq c_i \leq c_{i+1}$ and $c_i < \theta$, for every $i > k$. Then every c_i is a critical value of I. Moreover, if $c_i = c_{i+1} = \cdots = c_{i+p} = c < \theta$ for $i > k$, then $\gamma(K_c) \geq p + 1$, where

$$K_c = \{u \in E : I(u) = c \text{ and } I'(u) = 0\}.$$

Lemma 3.2. *There is a nondecreasing sequence $\{s_n\}$ of positive real numbers, independent of λ, such that for every $\lambda > 0$, we have*

$$c_n^\lambda = \inf_{W\in\Gamma_n} \max_{u\in W} I_\lambda(u) < s_n,$$

where Γ_n was defined in Theorem 3.1.

Proof. By the definition of Γ_n, one has

$$c_n^\lambda \leq \inf_{W\in\Gamma_n} \max_{u\in W}\{\frac{a}{p}||u_n||^p + \frac{b}{2p}||u_n||^{2p} - \frac{\mu}{2p}\int_\Omega \phi_{u_n} u_n^p d\xi - \frac{1}{Q^*}\int_\Omega |u_n|^{Q^*} d\xi\} = s_n,$$

therefore $s_n < \infty$ and $s_n \leq s_{n+1}$.

Proof of Theorem 1.1(II). We note that I_λ satisfies $I_\lambda(0) = 0$ and $I_\lambda(-u) = I_\lambda(u)$. In the sequel, we shall divide the proof into the following three steps:
Step 1. We shall prove that I_λ satisfies hypothesis (ii) of Theorem 3.1. Indeed, similar to the proof of (i) in Lemma 3.1, we can easily prove that the energy functional I_λ satisfies the hypothesis (*ii*) of Theorem 3.1.
Step 2. We shall prove that I_λ satisfies hypothesis (iii) of Theorem 3.1. Indeed, let Y be a finite-dimensional subspace of $S^{1,p}(\Omega)$. Since all norms in finite-dimensional space are equivalent, it follows that for every $u \in Y$, we have

$$
\begin{aligned}
I_\lambda(u) &\le \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{Q^*}\int_\Omega |u|^{Q^*} d\xi \\
&\le \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{Q^*}C\|u\|^{Q^*},
\end{aligned} \tag{3.6}
$$

for some positive constant $C > 0$. Also, because of $2p < Q^*$, we can choose a large $R > 0$ such that $I_\lambda(u) \le 0$ on $S^{1,p}(\Omega) \setminus B_R$. This fact implies that the energy functional I_λ satisfies the hypothesis (iii) of Theorem 3.1.

Step 3. We shall prove that problem (1.1) has infinitely many nontrivial weak solutions. Indeed, applying the argument in Wei and Wu [25], we can choose a_1 large enough so that for every $a > a_1$,

$$\sup s_n < (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}},$$

that is,

$$c_n^\lambda < s_n < (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}}.$$

Thus, one has

$$0 < c_1^\lambda \le c_2^\lambda \le \cdots \le c_n^\lambda < s_n < (\frac{1}{q} - \frac{1}{Q^*})(aS)^{\frac{Q}{p}}.$$

From Lemma 2.4, we know that I_λ satisfies $(PS)_{c_i^\lambda}(i = 1, 2, \cdots, n)$ condition. This fact implies that the levels $c_1^\lambda \le c_2^\lambda \le \cdots \le c_n^\lambda$ are critical values of I_λ, which be guaranteed by an application of the Rabinowitz result [24, Proposition 9.30].

If $c_i^\lambda = c_{i+1}^\lambda$ where $i = 1, 2, \cdots, k-1$, then applying the Ambrosetti and Rabinowitz result [26, Remark 2.12 and Theorem 4.2], we see that the set $K_{c_i^\lambda}$ consists of infinite number of different points, so problem (1.1) has infinite number of weak solutions. Hence, problem (1.1) has at least k pairs of solutions. Since k is arbitrary, we can conclude that problem (1.1) has infinitely many solutions. This completes the proof of Theorem 1.1(II).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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