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GLOBAL MULTIPLICITY FOR PARAMETRIC ANISOTROPIC NEUMANN (p,q)-EQUATIONS

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Dedicated to the memory of Professors Edward Fadell and Sufian Husseini

Abstract. We consider a Neumann boundary value problem driven by the anisotropic (p,q)-Laplacian plus a parametric potential term. The reaction is "superlinear". We prove a global (with respect to the parameter) multiplicity result for positive solutions. Also, we show the existence of a minimal positive solution and finally, we produce a nodal solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper, we study the following parametric anisotropic Neumann (p,q)-equation:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta_{p(z)}u - \Delta_{q(z)}u + \lambda|u|^{p(z)-2}u = f(z,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega,\ \lambda \in \mathbb{R}. \end{cases}$$

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Given $r \in C(\overline{\Omega})$ with $1 < \min_{\overline{\Omega}} r$, we denote by $\Delta_{r(z)}$ the anisotropic r-Laplace differential operator defined by

$$\Delta_{r(z)} u = \operatorname{div}\big(|Du|^{r(z)-2} Du\big) \quad \text{for all } u \in W^{1,r(z)}(\Omega).$$

In contrast to the isotropic r-Laplacian (that is, if $r(\,\cdot\,)$ is constant), the anisotropic one is not homogeneous and this is a source of difficulties in the study of anisotropic problems. Equation (P_λ) is driven by the sum of two such operators. So, even when the exponents are constant functions (isotropic operators), the differential operator of the problem is not homogeneous. There is also a parametric potential term $u \mapsto \lambda |u|^{p(z)-2}u$ with $\lambda \in \mathbb{R}$ being the parameter. Note that λ need not be positive and so the differential operator is not in general coercive. In the reaction (right-hand side of problem (P_λ)), we have a Carathéodory function $f(z,x)$ (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, the function $x \mapsto f(z,x)$ is continuous), which is $(p_+ - 1)$-superlinear as $x \to \pm\infty$ $\big($here, $p_+ = \max_{\overline{\Omega}} p$ for $p \in C^{0,1}(\overline{\Omega})\big)$. However, we do not use the Ambrosetti–Rabinowitz condition (the AR-condition for short), which is common in the literature when dealing with "superlinear" problems. Our condition on $f(z,\,\cdot\,)$ is less restrictive and incorporates in our framework, also superlinear nonlinearities with "slower" growth near $\pm\infty$.

Our aim is to study the changes in the set of positive solutions as the parameter $\lambda \in \mathbb{R}$ moves on the real line. We prove a global multiplicity result (a bifurcation-type result for large values of the parameter). More precisely, we show the existence of a critical parameter value $\lambda_* > -\infty$ such that

- for all $\lambda > \lambda_*$, problem (P_λ) has at least two positive smooth solutions;
- for $\lambda = \lambda_*$, problem (P_λ) has at least one positive smooth solution;
- for $\lambda \in (0, \lambda_*)$, problem (P_λ) has no positive solution.

We also establish that for all $\lambda \in [\lambda_*, \infty)$, problem (P_λ) has a smallest positive solution. Finally, the extremal constant sign solutions are used to produce a nodal (sign-changing) solution.

Analogous bifurcation type results describing the changes in the set of positive solutions for anisotropic Neumann problems were proved by Fan and Deng [4] and Deng and Wang [1]. They consider problems driven by the $p(z)$-Laplacian and impose restrictive positivity and monotonicity conditions on the reaction $f(z,\,\cdot\,)$ and, in addition, they employ the AR-condition to express the superlinearity of the reaction. We also mention the recent isotropic work of Papageorgiou and Zhang [16] with a parametric boundary condition. Finally, further existence and multiplicity results can be found in [5], [6], [9], [10] and the references therein.

2. Mathematical background and hypotheses

The analysis of problem (P_λ) requires the use of function spaces with variable exponents. A comprehensive presentation of the theory of these spaces can be found in the book of Diening, Harjulehto, Hästo and Ruzička [2]. We also refer to the monograph of Rădulescu and Repovš [17] for the basic variational and topological methods used in the treatment of problems with variable exponent.

Let $M(\Omega)$ be the vector space of all measurable functions $u\colon \Omega \to \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue-null subset of Ω. Given $r \in C(\overline{\Omega})$, we define $r_- = \min\limits_{\overline{\Omega}} r$ and $r_+ = \max_{\overline{\Omega}} r$.

Consider the set $E_1 = \{r \in C(\overline{\Omega}) : 1 < r_-\}$. Then, for $r \in E_1$, we define the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ as follows $L^{r(z)}(\Omega) = \{u \in M(\Omega) : \rho_r(u) < \infty\}$, where $\rho_r(\,\cdot\,)$ is the modular function defined by

$$\rho_r(u) = \int_\Omega |u|^{r(z)}\, dz.$$

We equip the space $L^{r(z)}(\Omega)$ with the so called "Luxemburg norm" defined by

$$\|u\|_{r(z)} = \inf\left\{\vartheta > 0 : \rho_r\left(\frac{u}{\vartheta}\right) \le 1\right\}.$$

Then $L^{r(z)}(\Omega)$ becomes a Banach space which is separable and reflexive (in fact, uniformly convex). For $r \in E_1$, we define the conjugate variable exponent $r'(\,\cdot\,)$ corresponding to $r(\,\cdot\,)$ by

$$r'(z) = \frac{r(z)}{r(z)-1} \quad \text{for all } z \in \overline{\Omega}.$$

Evidently, $r' \in E_1$ and $1/r(z) + 1/r'(z) = 1$ for all $z \in \overline{\Omega}$. We know that $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$ and the following Hölder's inequality is true

$$\int_\Omega |uv|\, dz \le \left(\frac{1}{r_-} + \frac{1}{r'_-}\right)\|u\|_{r(z)}\|v\|_{r'(z)}$$

for all $u \in L^{r(z)}(\Omega)$ and all $v \in L^{r'(z)}(\Omega)$.

Having the variable exponent Lebesgue spaces, we can define the corresponding variable exponent Sobolev spaces. So, if $r \in E_1$, then we define

$$W^{1,r(z)}(\Omega) = \left\{u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega)\right\},$$

with Du being the weak gradient of $u(\,\cdot\,)$. We equip this space with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)} \quad \text{for all } u \in W^{1,r(z)}(\Omega)$$

with $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$. It follows that $W^{1,r(z)}(\Omega)$ is a Banach space which is separable and reflexive (in fact, uniformly convex).

For $r \in E_1$ we introduce the corresponding critical Sobolev variable exponent $r^*(\cdot)$ defined by

$$r^*(z) = \begin{cases} \dfrac{Nr(z)}{N - r(z)} & \text{if } r(z) < N, \\ +\infty & \text{if } N \le r(z), \end{cases} \qquad \text{for all } z \in \overline{\Omega}.$$

Suppose that $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $\tau \in C(\overline{\Omega})$ with $1 \le \tau_-$. We have the following embeddings (anisotropic Sobolev embedding theorem).

Proposition 2.1.

(a) $W^{1,r(z)}(\Omega) \hookrightarrow L^{\tau(z)}(\Omega)$ *continuously if* $\tau(z) \le r^*(z)$ *for all* $z \in \overline{\Omega}$.
(b) $W^{1,r(z)}(\Omega) \hookrightarrow L^{\tau(z)}(\Omega)$ *compactly if* $\tau(z) < r^*(z)$ *for all* $z \in \overline{\Omega}$.

If $u \in W^{1,r(z)}(\Omega)$, then we write $\rho_r(Du) = \rho_r(|Du|)$. There is a close relation between the norm $\|\cdot\|_{r(z)}$ and the modular function $\rho_r(\cdot)$.

Proposition 2.2. *If* $r \in E_1$ *and* $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$, *then*:

(a) $\|u\|_{r(z)} = \vartheta$ *if and only if* $\rho_r(u/\vartheta) = 1$.
(b) $\|u\|_{r(z)} < 1$ (*resp.* $= 1$, > 1) *if and only If* $\rho_r(u) < 1$ (r*esp.* $= 1$, > 1).
(c) *If* $\|u\|_{r(z)} < 1$ *then* $\|u\|_{r(z)}^{r_+} \le \rho_r(u) \le \|u\|_{r(z)}^{r_-}$.
(d) *If* $\|u\|_{r(z)} > 1$ *then* $\|u\|_{r(z)}^{r_-} \le \rho_r(u) \le \|u\|_{r(z)}^{r_+}$.
(e) $\|u_n\|_{r(z)} \to 0$ *if and only if* $\rho_r(u_n) \to 0$.
(f) $\|u_n\|_{r(z)} \to \infty$ *if and only if* $\rho_r(u_n) \to +\infty$.

Let $A_r \colon W^{1,r(z)}(\Omega) \to W^{1,r(z)}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N}\, dz$$

for all $u, h \in W^{1,r(z)}(\Omega)$. This operator has the following properties, see Gasinski and Papageorgiou [7] and Rădulescu and Repovš [17, p. 40].

Proposition 2.3. *The operator* $A_r \colon W^{1,r(z)}(\Omega) \to W^{1,r(z)}(\Omega)^*$ *is bounded* (*that is, maps bounded sets to bounded sets*), *continuous, monotone* (*thus, maximal monotone, too*) *and of type* $(\mathrm{S})_+$, *that is,*

$$\text{“}u_n \xrightarrow{w} u \quad \text{in } W^{1,r(z)}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \le 0$$

imply that $u_n \to u$ *in* $W^{1,r(z)}(\Omega)$”.

We now recall the Weierstrass–Tonelli theorem, which we will use in the sequel. For the convenience of the reader we include also the proof.

Theorem 2.4. *If* X *is a reflexive Banach space and* $\phi \colon X \to \mathbb{R}$ *is sequentially weakly lower semicontinuous and coercive, then there exists* $\widehat{u} \in X$ *such that*

$$\phi(\widehat{u}) = \inf\{\phi(u) : u \in X\}.$$

Proof. The coercivity of ϕ implies that for $R > 0$ big we have

$$\inf_X \varphi = \inf_{\overline{B}_R} \phi,$$

with $\overline{B}_R = \{u \in X : \|u\|_X \le R\}$.

On account of the reflexivity of X and the Eberlein-Smulian theorem, $\overline{B}_R$ is sequentially weakly compact. Since $\varphi(\,\cdot\,)$ is sequentially weakly lower semicontinuous, we conclude that there exists $\widehat{u} \in X$ such that $\phi(\widehat{u}) = \inf_X \phi$. □

On account of the anisotropic regularity theory (see [3, Theorem 1.3] and [19, Corollary 3.1]), we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$, which is defined by

$$D_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Recall that $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$, with $n(\,\cdot\,)$ being the outward unit normal on $\partial\Omega$. If $h_1, h_2 \in M(\Omega)$ with $h_1 \le h_2$, then we define:

$$\begin{aligned}
&[h_1, h_2] = \{u \in W^{1,r(z)}(\Omega) : h_1(z) \le u(z) \le h_2(z) \text{ for a.a. } z \in \Omega\},\\
&[h_1) = \{u \in W^{1,r(z)}(\Omega) : h_1(z) \le u(z) \text{ for a.a. } z \in \Omega\},\\
&\operatorname{int}_{C^1(\Omega)}[h_1, h_2] = \text{ the interior in } C^1(\overline{\Omega}) \text{ of } [h_1, h_2] \cap C^1(\overline{\Omega}).
\end{aligned}$$

Suppose that X is a Banach space and $\varphi \in C^1(X)$. We introduce the following sets

$$\begin{aligned}
K_\varphi &= \{u \in X : \varphi'(u) = 0\} \qquad \text{(the critical set of } \varphi\text{)},\\
\varphi^c &= \{x \in X : \varphi(u) \le c\} \qquad \text{for all } c \in \mathbb{R}.
\end{aligned}$$

We say that $\varphi(\,\cdot\,)$ satisfies the "C-condition", if it has the following property: "Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* admits a strongly convergent subsequence".

This is a compactness condition of the functional $\varphi(\,\cdot\,)$ which compensates for the fact that the ambient space X is not in general locally compact (being infinite dimensional). Various techniques have been proposed in the literature in order to recover the compactness in several circumstances. We refer to Tang and Cheng [21] who proposed a new approach to restore the compactness of Palais–Smale sequences and to Tang and Chen [20] who introduced an original method to recover the compactness of minimizing sequences.

If $Y_2 \subseteq Y_1 \subseteq X$, then by $H_k(Y_1, Y_2)$ (for $k \in \mathbb{N}_0$), we denote the k^{th} relative singular homology group with integer coefficients. If $u \in K_\varphi$ is isolated, then the k^{th} critical group of φ at u is defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}\big) \quad \text{for all } k \in \mathbb{N}_0,$$

with $c = \varphi(u)$ and U is a neighbourhood of u such that $\varphi^c \cap K_\varphi \cap U = \{u\}$. The excision property of singular homology implies that this notion is well-defined, that is, independent of the choice of the isolating neighborhood U (see [14]).

If $u \in M(\Omega)$, we set

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\}, \quad \text{for all } z \in \Omega.$$

Then, if $u \in W^{1,r(z)}(\Omega)$, we know that

$$u^\pm \in W^{1,r(z)}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ - u^-.$$

Given a Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, we denote by $N_g(\,\cdot\,)$ the corresponding Nemyt'skiĭ operator, that is,

$$N_g(u)(\,\cdot\,) = g(\,\cdot\,, u(\,\cdot\,)) \quad \text{for all } u \in M(\Omega).$$

Since a Carathéodory function is jointly measurable, $N_g(u) \in M(\Omega)$.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and by $\|\cdot\|$ we will denote the norm of the Sobolev space $W^{1,p(z)}(\Omega)$.

Now we will introduce our hypotheses on the data of problem (P_λ).

(H_0) $p, q \in C^{0,1}(\overline{\Omega})$ and $1 < q(z) < p(z)$ for all $z \in \overline{\Omega}$.

(H_1) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z, x)| \le a(z)\big[1 + x^{r(z)-1}\big]$ for almost all $z \in \Omega$, all $x \ge 0$, with $\widehat{a} \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$, $p_+ < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$;

(ii) if
$$F(z, x) = \int_0^x f(z, s)\, ds$$
then
$$\lim_{x \to +\infty} \frac{F(z, x)}{x^{p_+}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) if $e(z, x) = f(z, x)x - p_+ F(z, x)$, then there exists $\mu \in L^1(\Omega)$ such that
$$e(z, x) \le e(z, y) + \mu(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le y;$$

(iv) there exist $\tau \in C(\overline{\Omega})$ and $C_0, \delta_0, \widehat{C} > 0$ such that
$$\begin{aligned} &1 < \tau_+ < q_-, \\ &C_0 x^{\tau(z)-1} \le f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le \delta_0, \\ &-\widehat{C} x^{p(z)-1} \le f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0; \end{aligned}$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the function $x \mapsto f(z,x) + \widehat{\xi}_\rho x^{p(z)-1}$ is nondecreasing on $[0,\rho]$.

REMARK 2.5. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,\infty)$, without any loss of generality we assume that

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \le 0. \tag{2.1}$$

Hypotheses (H_1) (ii), (iii) imply that, for almost all $z \in \Omega$ the mapping $f(z,\cdot)$ is $(p_+ - r)$-superlinear as $x \to +\infty$. However, this superlinearity condition is not expressed using the AR-condition. We recall that the AR-condition (unilateral version due to (2.1)) says that there exist $\vartheta > p_+$ and $M > 0$ such that

$$\begin{aligned} &0 < \vartheta F(z,x) \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M, \\ &0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \end{aligned}$$

These conditions imply that there exists $\widetilde{C} > 0$ such that

$$\widetilde{C} x^{\vartheta-1} \le f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M.$$

So, the AR-condition dictates at least $(\vartheta - 1)$-polynomial growth for $f(z,\cdot)$. Hypotheses (H_1) (ii), (iii) are less restrictive and incorporate in our framework superlinear nonlinearities with "slower" growth as $x \to +\infty$ (see the example below). Also we emphasize that in contrast to the previous works [1], [4], we do not assume that $f \ge 0$ neither that $f(z,\cdot)$ is nondecreasing. These are hypotheses employed by Fan and Deng [4] and Deng and Wang [1]. Moreover, in the aforementioned works the authors assume the AR-condition for $f(z,\cdot)$. Finally, note that also in contrast to the previous works, we do not assume that the parameter λ is strictly positive. Here, $\lambda \in \mathbb{R}$ and so the differential operator (left-hand side) of problem (P_λ) is not in general coercive.

EXAMPLE 2.6. Consider the function

$$f(z,x) = \begin{cases} \vartheta (x^+)^{\tau(z)-1} - \widehat{C}_0 (x^+)^{\gamma(z)-1} & \text{if } x \le 1, \\ x^{p_+-1} \ln x + (\vartheta - \widehat{C}_0) x^{\eta(z)-1} & \text{if } 1 < x \end{cases}$$

with $\tau, \gamma, \eta \in C(\overline{\Omega})$, $\tau_+ < q_-$, $1 < \tau(z) < \gamma(z)$, $1 < \eta(z) < p(z)$ for all $z \in \overline{\Omega}$ with $\widehat{C}_0 > \vartheta > 0$. Then this function satisfies hypotheses (H_1) above but does not satisfy the hypotheses of [1], [4] (the AR-condition fails and $f(z,\cdot)$ is not monotone on $\mathbb{R}_+ = [0,\infty)$).

We introduce the following sets

$$\begin{aligned} \mathcal{L}^+ &= \{\lambda \in \mathbb{R} : \text{problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\}, \\ S_\lambda^+ &= \text{ the set of positive solutions of } (\mathrm{P}_\lambda). \end{aligned}$$

3. Positive solutions

We start by showing that $\mathcal{L}^+$ is nonempty. To this end, let $\eta > 0$ and consider the following auxiliary anisotropic Neumann problem

$$\begin{cases} -\Delta_{p(z)}u - \Delta_{q(z)}u + u^{p(z)-1} = \eta & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \ u \geq 0. \end{cases} \tag{3.1}$$

Proposition 3.1. *If hypotheses* (H_0) *hold, then problem* (3.1) *has a unique positive solution* $\overline{u}_\eta \in \operatorname{int} C_+$ *and* $\overline{u}_\eta \to 0$ *in* $C^1(\overline{\Omega})$ *as* $\eta \to 0^+$.

Proof. Let $K_p\colon L^{p(z)}(\Omega) \to L^{p'(z)}(\Omega)$ be the nonlinear operator defined by

$$K_p(u)(\cdot) = |u(\cdot)|^{p(\cdot)-2}u(\cdot) \quad \text{for all } u \in L^{p(z)}(\Omega).$$

This operator is continuous and strictly monotone, too (see [14, p. 117]). Then we introduce $V\colon W^{1,p(z)}(\Omega) \to W^{1,p(z)}(\Omega)^*$ defined by

$$V(u) = A_p(u) + A_q(u) + K_p(u) \quad \text{for all } u \in W^{1,p(z)}(\Omega).$$

Then $V(\cdot)$ is maximal monotone (see [14, p. 135]), strictly monotone and

$$\langle V(u), u\rangle \geq \rho_p(Du) + \rho_p(u) \text{ for all } u \in W^{1,p(z)}(\Omega) \ \Rightarrow V(\cdot) \text{ is coercive}$$

(see Proposition 2.2). Then Corollary 2.8.7 of [14, p. 135] implies that $V(\cdot)$ is surjective. So, we can find $\overline{u}_\eta \in W^{1,p(z)}(\Omega)$ such that $V(\overline{u}_\eta) = \eta$.

On account of the strict monotonicity of $V(\cdot)$, this solution is unique. Taking duality brackets with $-\overline{u}_\eta^- \in W^{1,p(z)}(\Omega)$, we obtain

$$\rho_p(D\overline{u}_\eta^-) + \rho_p(\overline{u}_\eta^-) \leq \int_\Omega \eta(-\overline{u}_\eta^-)\,dz \leq 0 \ \Rightarrow \ \overline{u}_\eta \geq 0, \ \overline{u}_\eta \neq 0$$

(see Proposition 2.2 and recall that $\eta > 0$).

We have

$$-\Delta_{p(z)}\overline{u}_\eta - \Delta_{q(z)}\overline{u}_\eta + \overline{u}_\eta^{p(z)-1} = \eta \quad \text{in } \Omega, \qquad \frac{\partial \overline{u}_\eta}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

From Winkert and Zacher [22] (see also Papageorgiou, Rădulescu and Zhang [15, Proposition A1]), we have $\overline{u}_\eta \in L^\infty(\Omega)$. Then the anisotropic regularity theory (see Fan [3] and Tan and Fang [19]), we have $\overline{u}_\eta \in C_+ \setminus \{0\}$. From (3.2) we have

$$\Delta_{p(z)}\overline{u}_\eta + \Delta_{q(z)}\overline{u}_\eta \leq \overline{u}_\eta^{p(z)-1} \text{ in } \Omega \ \Rightarrow \ \overline{u}_\eta \in \operatorname{int} C_+$$

(see Papageorgiou, Qin and Rădulescu [11, Proposition 4]).

Now, let $\eta_n \to 0^+$ and let $\overline{u}_n = \overline{u}_{\eta_n} \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. The anisotropic regularity theory (see [3], [19]) implies that there exist $\alpha \in (0,1)$ and $C_1 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, implies that at least for a subsequence we have

$$\overline{u}_n \to \overline{u} \quad \text{in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{3.4}$$

Recall that

$$A_p(\overline{u}_n) + A_q(\overline{u}_n) + K_p(\overline{u}_n) = \eta_n \quad \text{in } W^{1,p(z)}(\Omega \text{ for all } n \in \mathbb{N},$$

that is,

$$\langle A_p(\overline{u}_n), h\rangle + \langle A_q(\overline{u}_n), h\rangle + \int_\Omega |\overline{u}_n|^{p(z)-2}\overline{u}_n h\,dz = \int_\Omega \eta_n h\,dz$$

for all $h \in W^{1,p(z)}(\Omega)$. We pass to the limit as $n \to \infty$. Because of (3.4) and Proposition 2.3 we have

$$A_p(\overline{u}_n) \to A_p(\overline{u}), \quad A_q(\overline{u}_n) \to A_q(\overline{u}) \quad \text{in } W^{1,p(z)}(\Omega)$$

and

$$\int_\Omega |\overline{u}_n|^{p(z)-2}\overline{u}_n h\,dz \to \int_\Omega |\overline{u}|^{p(z)-2}\overline{u} h\,dz.$$

Hence in the limit we have

$$A_p(\overline{u}) + A_q(\overline{u}) + K_p(\overline{u}) = 0 \ \Rightarrow\ \overline{u} = 0.$$

Therefore we obtain (see (3.4))

$$\overline{u}_\eta \to 0 \quad \text{in } C^1(\overline{\Omega}) \text{ as } \eta \to 0^+.$$

□

Using Proposition 3.1, we see that, for $\eta \in (0,1)$ small, we have

$$0 \le \overline{u}_\eta(z) \le 1 \quad \text{for all } z \in \overline{\Omega}. \tag{3.5}$$

For such an $\eta \in (0,1)$, let $m_\eta = \min_{\overline{\Omega}} \overline{u}_\eta > 0$ (recall that $\overline{u}_\eta \in \operatorname{int} C_+$). Then let

$$\widehat{\lambda} = \frac{\|N_f(\overline{u}_\eta)\|_\infty}{m_\eta^{p_+-1}} + 1 > 0 \tag{3.6}$$

(see hypothesis (H_1) (i)). We will show that $\widehat{\lambda} \in \mathcal{L}^+$ and so $\mathcal{L}^+ \neq \emptyset$.

Proposition 3.2. *If hypotheses H_0, H_1 hold, then $\mathcal{L}^+ \neq \emptyset$ and $S_\lambda^+ \subseteq \operatorname{int} C_+$ for every $\lambda \in \mathbb{R}$.*

Proof. Let $\overline{u}_\eta \in \operatorname{int} C_+$ and $\widehat{\lambda} > 0$ be as above. We have

$$\begin{aligned}
&-\Delta_{p(z)}\overline{u}_\eta - \Delta_{q(z)}\overline{u}_\eta + \widehat{\lambda}\overline{u}_\eta^{p(z)-1} \\
&\quad \ge -\Delta_{p(z)}\overline{u}_\eta - \Delta_{q(z)}\overline{u}_\eta + \frac{f(z,\overline{u}_\eta(z))}{m_\eta^{p_+-1}}\,\overline{u}_\eta^{p_+-1} + \overline{u}_\eta^{p(z)-1} && \text{(see (3.5), (3.6))} \\
&\quad \ge \eta + f(z,\overline{u}_\eta) && \text{(see Proposition 3.1)} \\
&\quad \ge f(z,\overline{u}_\eta) && \text{in } \Omega.
\end{aligned} \tag{3.7}$$

We introduce the Carathéodory function $\widehat{f}(z,x)$ defined by

$$\widehat{f}(z,x)=\begin{cases} f(z,x^+) & \text{if } x\le \overline{u}_\eta(z),\\ f(z,\overline{u}_\eta(z)) & \text{if } \overline{u}_\eta(z)<x. \end{cases} \tag{3.8}$$

We set

$$\widehat{F}(z,x)=\int_0^x \widehat{f}(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}\colon W^{1,p(z)}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}(u)=\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz+\int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz + \widehat{\lambda}\int_\Omega \frac{1}{p(z)}|u|^{p(z)}\,dz-\int_\Omega \widehat{F}(z,u)\,dz$$

for all $u\in W^{1,p(z)}(\Omega)$. We have

$$\widehat{\varphi}(u)\ge \frac{1}{p_+}\big[\rho_p(Du)+\widehat{\lambda}\rho_p(u)\big]-\int_\Omega \widehat{F}(z,u)\,dz \;\Rightarrow\; \widehat{\varphi}(\,\cdot\,) \text{ is coercive}$$

(see (3.8) and Proposition 2.2). Also, using Proposition 2.1 (the anisotropic Sobolev embedding theorem), we see that $\widehat{\varphi}(\,\cdot\,)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\widehat{u}\in W^{1,p(z)}(\Omega)$ such that

$$\widehat{\varphi}(\widehat{u})=\inf\big\{\widehat{\varphi}(u): u\in W^{1,p(z)}(\Omega)\big\}. \tag{3.9}$$

Let $u\in \operatorname{int} C_+$ and choose $t\in(0,1)$ small such that

$$0<tu(z)\le \min\Big\{\delta_0,\min_{\overline{\Omega}}\overline{u}_\eta\Big\}\quad \text{for all } z\in\overline{\Omega}. \tag{3.10}$$

Here $\delta_0>0$ is as in hypothesis (H_1) (iv) and recall that $\overline{u}_\eta\in \operatorname{int} C_+$, so that $\min_{\overline{\Omega}}\overline{u}_\eta>0$. We have

$$\widehat{\varphi}(tu)\le \frac{t^{q_-}}{q_-}\big[\rho_p(Du)+\rho_q(Du)+\widehat{\lambda}\rho_p(u)\big]-\frac{C_0t^{\tau_+}}{\tau_+}\,\rho_\tau(u)$$

(see (3.10), hypothesis (H_1) (iv) and recall that $t\in(0,1)$)

$$=C_2t^{q_-}-C_3t^{\tau_+}\quad \text{for some } C_2,C_3>0.$$

Recall that $\tau_+<q_-$ (see hypothesis (H_1) (iv)). So, choosing $t\in(0,1)$ even smaller if necessary, we have

$$\begin{aligned}\widehat{\varphi}(tu)<0 \;&\Rightarrow\; \widehat{\varphi}(\widehat{u})<0=\widehat{\varphi}(0)\quad (\text{see (3.9)})\\ &\Rightarrow\; \widehat{u}\ne 0.\end{aligned}$$

From (3.9), if $\widehat{\varphi}'(\widehat{u})=0$, then

$$\langle A_p(\widehat{u}),h\rangle+\langle A_q(\widehat{u}),h\rangle+\widehat{\lambda}\int_\Omega |\widehat{u}|^{p(z)-2}\widehat{u}h\,dz=\int_\Omega \widehat{f}(z,\widehat{u})h\,dz \tag{3.11}$$

for all $h \in W^{1,p(z)}(\Omega)$. In (3.11) first we choose $h = -\widehat{u} \in W^{1,p(z)}$. Then we have

$$\rho_p(D\widehat{u}^-) + \rho_q(D\widehat{u}^-) + \widehat{\lambda}\rho_p(\widehat{u}^-) = 0 \quad \text{(see (3.8), (2.1))} \ \Rightarrow\ \widehat{u} \geq 0, \quad \widehat{u} \neq 0.$$

Next, in (3.11) we choose $h = (\widehat{u} - \overline{u}_\eta)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
\langle A_p(\widehat{u}),&(\widehat{u} - \overline{u}_\eta)^+\rangle + \langle A_q(\widehat{u}), (\widehat{u} - \overline{u}_\eta)^+\rangle + \widehat{\lambda}\int_\Omega \widehat{u}^{p(z)-1}(\widehat{u} - \overline{u}_\eta)^+\,dz \\
&= \int_\Omega f(z, \overline{u}_\eta)(\widehat{u} - \overline{u}_\eta)^+\,dz && \text{(see (3.8))} \\
&\leq \langle A_q(\overline{u}_\eta), (\widehat{u} - \overline{u}_\eta)^+\rangle + \langle A_q(\overline{u}_\eta), (\widehat{u} - \overline{u}_\eta)^+\rangle \\
&\quad + \widehat{\lambda}\int_\Omega \overline{u}_\eta^{p(z)-1}(\widehat{u} - \overline{u}_\eta)^+\,dz && \text{(see (3.7))} \\
&\Rightarrow \widehat{u} \leq \overline{u}_\eta && \text{(see Proposition 2.3).}
\end{aligned}$$

So, we have proved that

$$\widehat{u} \in [0, \overline{u}_\eta], \quad \widehat{u} \neq 0. \tag{3.12}$$

From (3.11), (3.12) and (3.8) it follows that $\widehat{u} \in S_\lambda^+$ and so $\widehat{\lambda} \in \mathcal{L}^+ \neq \emptyset$. Moreover, as before, the anisotropic regularity theory (see [3], [19]) and the anisotropic maximum principle (see [11]), imply that $S_\lambda^+ \subseteq \operatorname{int} C_+$ for all $\lambda \in \mathbb{R}$.□

Next, we show that $\mathcal{L}^+$ is connected, more precisely $\mathcal{L}^+$ is an upper half line.

Proposition 3.3. *If hypotheses* (H_0), (H_1) *hold,* $\lambda \in \mathcal{L}$ *and* $\lambda < \mu < \infty$, *then* $\mu \in \mathcal{L}^+$.

Proof. Since by hypothesis $\lambda \in \mathcal{L}^+$, we can find $u_\lambda \in S_\lambda^+ \subseteq \operatorname{int} C_+$. Then we have

$$\begin{aligned}
-\Delta_{p(z)}u_\lambda - \Delta_{q(z)}u_\lambda &+ \mu u_\lambda^{p(z)-1} \\
&\geq -\Delta_{p(z)}u_\lambda - \Delta_{q(z)}u_\lambda + \lambda u_\lambda^{p(z)-1} = f(z, u_\lambda)
\end{aligned} \tag{3.13}$$

in Ω. Let $\vartheta > -\mu$ and consider the Carathéodory function $k(z, x)$ defined by

$$k(z, x) = \begin{cases} f(z, x^+) + \vartheta(x^+)^{p(z)-1} & \text{if } x \leq u_\lambda(z), \\ f(z, u_\lambda(z)) + \vartheta u_\lambda(z)^{p(z)-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.14}$$

Let $K(z, x) = \int_0^x k(z, s)\,ds$ and consider the C^1-functional $\widehat{\varphi}_\mu \colon W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
\widehat{\varphi}_\mu(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz &+ \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz \\
&+ \int_\Omega \frac{\vartheta + \mu}{p(z)}|u|^{p(z)}\,dz - \int_\Omega K(z, u)\,dz
\end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. Since $\vartheta + \mu > 0$ from (3.14) and Proposition 2.2, we see that $\widehat{\varphi}_\mu(\,\cdot\,)$ is coercive. Also using Proposition 2.1 (the anisotropic Sobolev embedding theorem), we infer that $\widehat{\varphi}_\mu(\,\cdot\,)$ is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p(z)}(\Omega)$ such that

$$\widehat{\varphi}_\mu(u_\mu) = \inf\big\{\widehat{\varphi}_\mu(u) : u \in W^{1,p(z)}(\Omega)\big\}. \tag{3.15}$$

As in the proof of Proposition 3.2, via hypothesis (H_1) (iv), we show that

$$\widehat{\varphi}_\mu(u_\mu) < 0 = \widehat{\varphi}_\mu(0) \;\Rightarrow\; u_\mu \neq 0.$$

From (3.15) we have that

$$\langle \widehat{\varphi}'_\mu(u_\mu), h\rangle = 0 \quad \text{for all } h \in W^{1,p(z)}(\Omega). \tag{3.16}$$

As before (see the proof of Proposition 3.2), choosing $h = -u_\mu^- \in W^{1,p(z)}(\Omega)$ and $h = (u_\mu - u_\lambda)^+ \in W^{1,p(z)}(\Omega)$ in (3.16), we show that

$$u_\mu \in [0, \mu_\lambda], \quad u_\mu \neq 0, \quad u_\mu \neq u_\lambda \tag{3.17}$$

(since $\lambda < \mu$). From (3.16), (3.17) and (3.14), we deduce that $u_\mu \in S_\mu^+ \subseteq \operatorname{int} C_+$ and so $\mu \in \mathcal{L}^+$. $\square$

An interesting byproduct of the above proof, is the following corollary.

Corollary 3.4. *If hypotheses* (H_0), (H_1) *hold,* $\lambda \in \mathcal{L}^+$, $u_\lambda \in S_\lambda^+$ *and* $\lambda < \mu < \infty$, *then* $\mu \in \mathcal{L}^+$ *and we can find* $u_\mu \in S_\mu^+$ *such that* $u_\mu \le u_\lambda$.

We can improve the assertion of this corollary as follows.

Proposition 3.5. *If hypotheses* (H_0), (H_1) *hold,* $\lambda \in \mathcal{L}^+$, $u_\lambda \in S_\lambda^+$ *and* $\lambda < \mu < \infty$, *then* $\mu \in \mathcal{L}^+$ *and we can find* $u_\mu \in S_\mu^+$ *such that* $u_\lambda - u_\mu \in D_+$.

Proof. From Corollary 3.4 we already know that $\mu \in \mathcal{L}^+$ and we can find $u_\mu \in S_\mu^+ \subseteq \operatorname{int} C_+$ such that

$$0 \le u_\mu \le u_\lambda. \tag{3.18}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis (H_1) (v). We have

$$\begin{aligned} -\Delta_{p(z)} u_\lambda &- \Delta_{q(z)} u_\lambda + [\lambda + \widehat{\xi}_\rho] u_\lambda^{p(z)-1} = f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p(z)-1} \\ &\ge f(z, u_\mu) + \widehat{\xi}_\rho u_\mu^{p(z)-1} \qquad \text{(see (3.18) and hypothesis } (\mathrm{H}_1)\text{ (v))} \\ &= -\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + \Big[\mu + \widehat{\xi}_\rho\Big] u_\mu^{p(z)-1} \qquad (\text{since } u_\mu \in S_\mu^+) \\ &= -\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + [\lambda + \widehat{\xi}_\rho] u_\mu^{p(z)-1} + (\mu - \lambda) u_\mu^{p(z)-1} \\ &\ge -\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + [\lambda + \widehat{\xi}_\rho] u_\mu^{p(z)-1} \qquad (\text{since } \lambda < \mu). \end{aligned} \tag{3.19}$$

We know that $u_\mu \in \operatorname{int} C_+$. Hence $0 < m_\mu = \min_{\overline{\Omega}} u_\mu$. We set $\widehat{m}_\mu = \min\{m_\mu, 1\} > 0$. Then

$$0 < [\mu - \lambda]\, \widehat{m}_\mu^{p_+ - 1} \le [\mu - \lambda]\, u_\mu^{p(z)-1} \quad \text{for all } z \in \overline{\Omega}.$$

Then from (3.19) and Proposition 5 of Papageorgiou, Qin and Rădulescu [11], we infer that $u_\lambda - u_\mu \in D_+$. $\square$

Let $\lambda_* = \inf \mathcal{L}^+$.

PROPOSITION 3.6. *If hypotheses* (H_0), (H_1) *hold, then* $\lambda_* > -\infty$.

PROOF. Let $\lambda > \lambda_*$. Then on account of Proposition 3.3, we have $\lambda \in \mathcal{L}^+$. So, we can find $u \in S_\lambda^+ \subseteq \operatorname{int} C_+$ and, for all $h \in W^{1,p(z)}(\Omega)$, we have

$$\langle A_p(u), h\rangle + \langle A_q(u), h\rangle + \lambda \int_\Omega u^{p(z)-1} h\, dz = \int_\Omega f(z,u) h\, dz. \tag{3.20}$$

In (3.20) we choose $h \equiv 1 \in W^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} \lambda \int_\Omega u^{p(z)-1}\, dz = \int_\Omega f(z,u)\, dz &\geq -\widehat{C} \int_\Omega u^{p(z)-1}\, dz \\ &\Rightarrow \ \big(\lambda + \widehat{C}\big) \int_\Omega u^{p(z)-1}\, dz \geq 0 \\ &\Rightarrow \ \lambda + \widehat{C} \geq 0 \quad \text{and so} \quad \lambda \geq -\widehat{C}. \end{aligned}$$

So, we conclude that $\lambda_* \geq -\widehat{C} > -\infty$. $\square$

By imposing a sign condition on $f(z,\cdot)$, we can have that $\mathcal{L}^+ \subseteq \mathbb{R}_+ = [0,+\infty]$, that is, $\lambda_* \geq 0$.

The new conditions on the reaction $f(z,x)$ are the following.

(H_1') $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses (H_1') (i)–(iii), (v) are the same as the corresponding hypotheses (H_1) (i)–(iii), (v) and

(iv) there exist $\tau \in C(\overline{\Omega})$ and $C_0, \delta_0 > 0$ such that

$$\begin{aligned} & 1 < \tau_+ < q_-, \\ & C_0 x^{\tau(z)-1} \leq f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta_0, \\ \text{and} \quad & 0 \leq f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \end{aligned}$$

REMARK 3.7. So, the new conditions of $f(z,\cdot)$ require that $f(z,\cdot)\big|_{\mathbb{R}_+}$ is nonnegative (it can not change sign). This was the case with the reactions in the works of Fan and Deng [4] and Deng and Wang [1].

Under the above stronger conditions on the reaction $f(z,\cdot)$ we can show that the set $\mathcal{L}^+$ of admissible parameters is a subset of $\mathbb{R}_+$.

PROPOSITION 3.8. *If hypotheses* (H_0), (H_1') *hold, then* $\lambda_* \geq 0$.

PROOF. Let $\lambda > \lambda_*$. We know that $\lambda \in \mathcal{L}^+$ and so there exists $u \in S_\lambda^+ \subseteq \operatorname{int} C_+$. From (3.20) with $h \equiv 1 \in W^{1,p(z)}(\Omega)$, we have

$$\begin{aligned} \lambda \int_\Omega u^{p(z)-1}\, dz = \int_\Omega f(z,u)\, dz &\geq 0 \quad (\text{see } (\mathrm{H}_1')\,(\mathrm{iv})) \\ &\Rightarrow \ \lambda \geq 0 \quad \text{and so} \quad \lambda_* \geq 0. \end{aligned}$$

The proof is complete. □

On account of hypotheses (H_1) (i), (iv), we see that we can find $C_4 > 0$ such that

$$f(z,x) \geq C_0 x^{\tau(z)-1} - C_4 x^{r(z)-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.21}$$

Let $\eta > 0$ and let $\widehat{\lambda}_\eta = \lambda_* + \eta$. Evidently, $\widehat{\lambda}_\eta \in \mathcal{L}^+$ (see Corollary 3.4). The unilateral growth condition in (3.21) leads to the following auxiliary anisotropic Neumann problem

$$\begin{cases} -\Delta_{p(z)} u - \Delta_{q(z)} u + \widehat{\lambda}_\eta |u|^{p(z)-1} = C_0 u^{\tau(z)-1} - C_4 u^{r(z)-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \ u \geq 0. \end{cases} \tag{3.22}$$

Proposition 3.9. *If hypotheses* (H_0) *hold, then problem* (3.22) *has a unique positive solution* $u_\eta^* \in \operatorname{int} C_+$.

Proof. Let $\lambda \in (\lambda_*, \widehat{\lambda}_\eta]$. We know that $\lambda \in \mathcal{L}^+$ (see Proposition 3.3) and so we can find $u \in S_\lambda^+ \subseteq \operatorname{int} C_+$. Let $\vartheta > -\widehat{\lambda}_\eta$ and consider the Carathéodory function

$$\beta(z,x) = \begin{cases} C_0 (x^+)^{\tau(z)-1} - C_4 (x^+)^{r(z)-1} + \vartheta (x^+)^{p(z)-1} & \text{if } x \leq u(z), \\ C_0 u(z)^{\tau(z)-1} - C_4 u(z)^{r(z)-1} + \vartheta u(z)^{p(z)-1} & \text{if } u(z) < x. \end{cases} \tag{3.23}$$

We set

$$B(z,x) = \int_0^x \beta(z,s)\, ds$$

and consider the C^1-functional $\Psi \colon W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \Psi(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz \\ &\quad + \int_\Omega \frac{\vartheta + \widehat{\lambda}_\eta}{p(z)} |u|^{p(z)}\, dz - \int_\Omega B(z,u)\, dz \end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. From (3.23) and since $\vartheta > -\widehat{\lambda}_\eta$ we see that $\Psi(\,\cdot\,)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_\eta^* \in W^{1,p(z)}(\Omega)$ such that

$$\Psi(u_\eta^*) = \inf \big\{ \Psi(u) : u \in W^{1,p(z)}(\Omega) \big\}. \tag{3.24}$$

Since $\tau_+ < q_- < p_+ < r_-$, if $v \in \operatorname{int} C_+$ and $t \in (0,1)$ is small (at least so that $tv(z) \leq u(z)$ for all $z \in \overline{\Omega}$), then

$$\Psi(tv) < 0 \ \Rightarrow \ \Psi(u_\eta^*) < 0 = \Psi(0) \quad \text{(see (3.24))} \ \Rightarrow \ u_\eta^* \neq 0.$$

From (3.24) we have

$$\Psi'(u_\eta^*) = 0 \;\Rightarrow\; \langle A_p(u_\eta^*), h\rangle + \langle A_q(u_\eta^*), h\rangle + \big(\vartheta + \widehat{\lambda}_\eta\big) \int_\Omega |u_\eta^*|^{p(z)-2} u_\eta^* h\, dz = \int_\Omega \beta(z, u_\eta^*) h\, dz \tag{3.25}$$

for all $h \in W^{1,p(z)}(\Omega)$. In (3.25) first we choose $h = -(u_\eta^*)^- \in W^{1,p(z)}(\Omega)$. Using (3.23) we obtain

$$\begin{aligned}&\rho_p(D(u_\eta^*)^-) + \rho_q((u_\eta^*)^-) + \big[\vartheta + \widehat{\lambda}_\eta\big]\rho_p((u_\eta^*)^-) = 0\\ &\qquad\Rightarrow\; u_\eta^* \geq 0,\; u_\eta^* \neq 0 \quad (\text{recall that } \vartheta > -\widehat{\lambda}_\eta).\end{aligned}$$

Next, in (3.25) we choose $(u_\eta^* - u)^+ \in W^{1,p(z)}(\Omega)$. Then we have

$$\begin{aligned}&\langle A_p(u_\eta^*), (u_\eta^* - u)^+\rangle + \langle A_q(u_\eta^*), (u_\eta^* - u)^+\rangle + (\vartheta + \widehat{\lambda}_\eta) \int_\Omega |u_\eta^*|^{p(z)-2} u_\eta^* h\, dz\\ &\quad= \int_\Omega \big[C_0 u^{\tau(z)-1} - C_4 u^{r(z)-1} + \vartheta u^{p(z)-1}\big](u_\eta^* - u)^+ \qquad (\text{see } (3.23))\\ &\quad\leq \int_\Omega \big[f(z,u) + \vartheta u^{p(z)-1}\big](u_\eta^* - u)^+\, dz \qquad (\text{see } (3.21))\\ &\quad\leq \langle A_p(u), (u_\eta^* - u)^+\rangle + \langle A_q(u), (u_\eta^* - u)^+\rangle\\ &\qquad + (\vartheta + \widehat{\lambda}_\eta) \int_\Omega u^{p(z)-1}(u_\eta^* - u)^+\, dz \quad (\text{since } u \in S_\lambda \text{ and } \lambda \leq \widehat{\lambda}_\eta)\\ &\quad\Rightarrow u_\eta^* \leq u \qquad (\text{see Proposition } 2.3).\end{aligned}$$

So, we have proved that

$$u_\eta^* \in [0, u], \quad u_\eta^* \neq 0. \tag{3.26}$$

From (3.25), (3.26) and (3.23) it follows that

$$u_\eta^* \text{ is a positive solution of problem (3.22)}.$$

As before, the anisotropic regularity theory ([3], [19]) and the anisotropic maximum principle (see [11]), imply that $u_\eta^* \in \operatorname{int} C_+$.

Next, we show that this positive solution of (3.22) is unique. To this end, we consider the integral functional $j\colon L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \displaystyle\int_\Omega \frac{1}{p(z)} \big|Du^{1/q_-}\big|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} \big|Du^{1/q_-}\big|^{q(z)}\, dz & \\ & \text{if } u \geq 0,\; u^{1/q_-} \in W^{1,p(z)}(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

From Theorem 2.2 of Takač and Giacomoni [18], we know that $j(\,\cdot\,)$ is convex. Let $\operatorname{dom} j = \big\{u \in L^1(\Omega) : j(u) < \infty\big\}$ (the effective domain of $j(\,\cdot\,)$). Suppose y_η^* is another positive solution of problem (3.22). Again, we have $y_\eta^* \in \operatorname{int} C_+$.

On account of Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovš [14, p. 274], we have

$$\frac{u_\eta^*}{y_\eta^*} \in L^\infty(\Omega) \quad \text{and} \quad \frac{y_\eta^*}{u_\eta^*} \in L^\infty(\Omega). \tag{3.27}$$

Let $h = (u_\eta^*)^{q_-}(y_\eta^*)^{q_-}$. From (3.27) and for $|t| < 1$ small, we have

$$(u_\eta^*) + th \in \operatorname{dom} j, \qquad (y_\eta^*)^{q_-} + th \in \operatorname{dom} j.$$

Exploiting the convexity of $j(\,\cdot\,)$ and using the chain rule, we see that $j(\,\cdot\,)$ is Gâteaux differentiable at $(u_\eta^*)^{q_-}$ and at $(y_\eta^*)^{q_-}$ in the direction h. Moreover, via Green's identity, we have

$$\begin{aligned}
j'\big((u_\eta^*)^{q_-}\big)(h) &= \frac{1}{q_-}\int_\Omega \frac{-\Delta_{p(z)}u_\eta^* - \Delta_{q(z)}u_\eta^*}{(u_\eta^*)^{q_- - 1}}\, h\, dz \\
&= \frac{1}{q_-}\int_\Omega \bigg[\frac{C_0}{(u_\eta^*)^{q_- - \tau(z)}} - C_4(u_\eta^*)^{r(z)-q_-} - \widehat{\lambda}_\eta (u_\eta^*)^{p(z)-q_-}\bigg] h\, dz, \\
j'\big((y_\eta^*)^{q_-}\big)(h) &= \frac{1}{q_-}\int_\Omega \frac{-\Delta_{p(z)}y_\eta^* - \Delta_{q(z)}y_\eta^*}{(y_\eta^*)^{q_- - 1}}\, h\, dz \\
&= \frac{1}{q_-}\int_\Omega \bigg[\frac{C_0}{(y_\eta^*)^{q_- - \tau(z)}} - C_4(y_\eta^*)^{r(z)-q_-} - \widehat{\lambda}_\eta (y_\eta^*)^{p(z)-q_-}\bigg] h\, dz.
\end{aligned}$$

The convexity of $j(\,\cdot\,)$ implies that $j'(\,\cdot\,)$ is monotone. Then

$$\begin{aligned}
0 \le \int_\Omega & C_0\bigg[\frac{1}{(u_\eta^*)^{q_- - \tau(z)}} - \frac{1}{(y_\eta^*)^{q_- - \tau(z)}}\bigg] h\, dz \\
&- \int_\Omega C_4\big[(u_\eta^*)^{r(z)-q_-} - (y_\eta^*)^{r(z)-q_-}\big] h\, dz \\
&- \widehat{\lambda}_\eta \int_\Omega \big[(u_\eta^*)^{p(z)-q_-} - (y_n^*)^{p(z)-q_-}\big] h\, dz \le 0 \\
\Rightarrow u_\eta^* &= y_\eta^* \quad (\text{recall that } \tau_+ < q_- < p_-).
\end{aligned}$$

This proves the uniqueness of the positive solution $u_\eta^* \in \operatorname{int} C_+$ of problem (3.22). $\square$

This unique positive solution of problem (3.22) provides a lower bound for the elements of S_λ^+ locally in $\lambda > \lambda_*$.

Proposition 3.10. *If hypotheses* (H_0), (H_1) *hold,* $\eta > 0$ *and* $\lambda \in (\lambda_*, \widehat{\lambda}_\eta = \lambda_* + \eta]$, *then* $u_\eta^* \le u$ *for all* $u \in S_\lambda^+$.

Proof. Let $u \in S_\lambda^+$, $\vartheta > -\widehat{\lambda}_\eta$ and consider the Carathéodory function $k(z,x)$ defined by

$$k(z,x) = \begin{cases} C_0(x^+)^{\tau(z)-1} - C_4(x^+)^{r(z)-1} + \vartheta(x^+)^{p(z)-1} & \text{if } x \le u(z), \\ C_0 u(z)^{\tau(z)-1} - C_4 u(z)^{r(z)-1} + \vartheta u(z)^{p(z)-1} & \text{if } u(z) < x. \end{cases} \tag{3.28}$$

We set

$$K(z, x) = \int_0^x k(z, s)\, ds$$

and consider the C^1-functional $\sigma\colon W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz + \int_\Omega \frac{\vartheta + \widehat{\lambda}_\eta}{p(z)} |u|^{p(z)}\, dz - \int_\Omega K(z, u) dz$$

for all $u \in W^{1,p(z)}(\Omega)$. From (3.28) and since $\vartheta > -\widehat{\lambda}_\eta$, we see that $\sigma(\,\cdot\,)$ is coercive. Also, using the anisotropic Sobolev embedding theorem (see Proposition 2.1), we see that $\sigma(\,\cdot\,)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\widehat{u}_\eta^* \in W^{1,p(z)}(\Omega)$ such that

$$\sigma(\widehat{u}_\eta^*) = \min \big\{ \sigma(u) : u \in W^{1,p(z)}(\Omega) \big\}. \tag{3.29}$$

Since $\tau_+ < q_- \le q(z) < p(z)$ for all $z \in \overline{\Omega}$, we see that, if $v \in \operatorname{int} C_+$ and $t \in (0, 1)$ is small (at least so that $tv \le u$), we have

$$\sigma(tv) < 0 \;\Rightarrow\; \sigma(\widehat{u}_\eta^*) < 0 = \sigma(0) \text{ (see (3.29))} \;\Rightarrow\; \widehat{u}_\eta^* \neq 0.$$

From (3.29) we have $\sigma'(u_\eta^*) = 0$, thus, for all $h \in W^{1,p(z)}(\Omega)$,

$$\langle A_p(\widehat{u}_\eta^*), h\rangle + \langle A_q(\widehat{u}_\eta^*), h\rangle + \int_\Omega (\vartheta + \widehat{\lambda}_\eta) |\widehat{u}_\eta^*|^{p(z)-2} \widehat{u}_\eta^* h\, dz = \int_\Omega k(z, \widehat{u}_\eta^*) h\, dz. \tag{3.30}$$

In (3.30) first we choose $h = -(\widehat{u}_\eta^*)^- \in W^{1,p(z)}(\Omega)$. Using (3.28), we obtain

$$\rho_p(D(\widehat{u}_\eta^*)^-) + \rho_q(D(\widehat{u}_\eta^*)^-) + \int_\Omega (\vartheta + \widehat{\lambda}_\eta) \left((\widehat{u}_\eta^*)^- \right)^{p(z)} dz = 0$$
$$\Rightarrow\; \widehat{u}_\eta^* \ge 0,\; \widehat{u}_\eta^* \neq 0 \quad (\text{recall that } \vartheta > -\widehat{\lambda}_\eta).$$

Next, in (3.30) we choose $h = (\widehat{u}_\eta^* - u)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle A_p(\widehat{u}_\eta^*), (\widehat{u}_\eta^* - u)^+\rangle + \langle A_q(\widehat{u}_\eta^*), (\widehat{u}_\eta^* - u)^+\rangle + \int_\Omega (\vartheta + \widehat{\lambda}_\eta) (\widehat{u}_\eta^*)^{p(z)-1} (\widehat{u}_\eta^* - u)^+\, dz \\
&= \int_\Omega \big[C_0 u^{\tau(z)-1} - C_4 u^{r(z)-1} + \vartheta u^{p(z)-1} \big] (\widehat{u}_\eta^* - u)^+\, dz \quad \text{(see (3.28))} \\
&\le \int_\Omega \big[f(z, u) + \vartheta u^{p(z)-1} \big] (\widehat{u}_\eta^* - u)^+ dz \quad \text{(see (3.21))} \\
&\le \langle A_p(u), (\widehat{u}_\eta^* - u)^+\rangle + \langle A_q(u), (\widehat{u}_\eta^* - u)^+\rangle \\
&\quad + \int_\Omega (\vartheta + \widehat{\lambda}_\eta) u^{p(z)-1} (\widehat{u}_\eta^* - u)^+\, dz \quad (\text{since } u \in S_\lambda^+,\ \lambda \le \widehat{\lambda}_\eta) \\
&\Rightarrow \widehat{u}_\eta^* \le u.
\end{aligned}$$

So, we have proved that

$$\widehat{u}_\eta^* \in [0,u], \quad \widehat{u}_\eta^* \neq 0. \tag{3.31}$$

From (3.30), (3.31), (3.28) and Proposition 3.9, we conclude that

$$\widehat{u}_\eta^* = u_\eta^* \ \Rightarrow\ u_\eta^* \leq u \quad \text{for all } u \in S_\lambda^+, \text{ all } \lambda \in (\lambda_*, \widehat{\lambda}_\eta = \lambda_* + \eta]. \qquad \square$$

Remark 3.11. This proposition reveals that if hypotheses (H_1') hold, then $\lambda_* > 0$.

Next, we show that for all $\lambda > \lambda_*$, we have at least two positive solutions.

Proposition 3.12. *If hypotheses* (H_0), (H_1) *hold and* $\lambda > \lambda_*$, *then problem* (P_λ) *has at least two positive solutions* u_0, $\widehat{u} \in \operatorname{int} C_+$, $u_0 \neq \widehat{u}$.

Proof. Let $\eta,\mu \in (\lambda_*,\infty)$ such that $\lambda_* < \eta < \lambda < \mu$. We know that $\eta,\mu \in \mathcal{L}^+$ (see Proposition 3.3). Moreover, on account of Proposition 3.5 we can find $u_\eta \in S_\eta^+$, $u_0 \in S_\lambda^+$ and $u_\mu \in S_\mu^+$ such that

$$u_\eta - u_0 \in D_+ \quad \text{and} \quad u_0 - u_\mu \in D_+ \Rightarrow\ u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[u_\mu, u_\eta]. \tag{3.32}$$

Let $\vartheta > -\lambda$ and consider the Carathéodory functions $\widehat{g}(z,x)$ and $g(z,x)$ defined by

$$\widehat{g}(z,x) = \begin{cases} f(z,u_\mu(z)) + \vartheta u_\mu(z)^{p(z)-1}, \\ f(z,x) + \vartheta x^{p(z)-1}, \\ f(z,u_\eta(z)) + \vartheta u_\eta(z)^{p(z)-1}, \end{cases} \tag{3.33}$$

and

$$g(z,x) = \begin{cases} f(z,u_\mu(z)) + \vartheta u_\mu(z)^{p(z)-1} & \text{if } x < u_\mu(z), \\ f(z,x) + \vartheta x^{p(z)-1} & \text{if } u_\mu(z) \leq x. \end{cases} \tag{3.34}$$

We set

$$\widehat{G}(z,x) = \int_0^x \widehat{g}(z,s)\,ds, \qquad G(z,x) = \int_0^x g(z,s)\,ds$$

and consider the C^1-functionals $\widehat{\gamma}_\lambda, \gamma_\lambda \colon W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \widehat{\gamma}_\lambda(u) &= \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz \\ &\quad + \int_\Omega \frac{\vartheta+\lambda}{p(z)}|u|^{p(z)}\,dz - \int_\Omega \widehat{G}(z,u)\,dz, \\ \gamma_\lambda(u) &= \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz \\ &\quad + \int_\Omega \frac{\vartheta+\lambda}{p(z)}|u|^{p(z)}\,dz - \int_\Omega G(z,u)\,dz \end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. Using (3.33) and (3.34), we show easily that

$$K_{\widehat{\gamma}_\lambda} \subseteq [u_\mu, u_\eta] \cap \operatorname{int} C_+ \quad \text{and} \quad K_{\gamma_\lambda} \subseteq [u_\mu) \cap \operatorname{int} C_+. \tag{3.35}$$

It is clear from (3.33) and (3.34) that

$$\gamma_\lambda\big|_{[u_\mu, u_\eta]} = \widehat{\gamma}_\lambda\big|_{[u_\mu, u_\eta]}. \tag{3.36}$$

Then, from (3.35) and (3.36), we see that we may assume that

$$K_{\widehat{\gamma}_\lambda} = \{u_0\}. \tag{3.37}$$

Otherwise, we already have a second positive solution for problem (P_λ) and so we are done. From (3.33) and since $\vartheta > -\lambda$, we see that $\widehat{\gamma}(\,\cdot\,)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, $\widehat{\gamma}_\lambda(\,\cdot\,)$ has a global minimizer on account of (3.37) this global minimizer is u_0. From (3.32) and (3.36) it follows that

$$\begin{aligned} &u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \gamma_\lambda(\,\cdot\,) \\ &\qquad\qquad \Rightarrow\; u_0 \text{ is a local } W^{1,p(z)}(\Omega)\text{-minimizer of } \gamma_\lambda(\,\cdot\,) \end{aligned} \tag{3.38}$$

(see [7]). From (3.35) and (3.34), we see that we may assume that K_{γ_λ} is finite, otherwise we already have a sequence of distinct positive smooth solutions for (P_λ) and so we are done. Then from (3.38) and Theorem 5.7.6 of [14, p. 367], we see that there exists $\rho \in (0,1)$ small such that

$$\gamma_\lambda(u_0) < \inf\big\{\gamma_\lambda(u) : \|u - u_0\| = \rho\big\} = m_\lambda. \tag{3.39}$$

Note that if $u \in \operatorname{int} C_+$, then on account of hypothesis (H_1) (ii), we have

$$\gamma_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.40}$$

Claim. $\gamma_\lambda(\,\cdot\,)$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ such that

$$|\gamma_\lambda(u_n)| \le C_5 \qquad \text{for some } C_5 > 0,\ \text{all } n \in \mathbb{N}, \tag{3.41}$$

$$(1 + \|u_n\|)\gamma_\lambda'(u_n) \to 0 \quad \text{in } W^{1,p(z)}(\Omega)^* \text{ as } n \to \infty. \tag{3.42}$$

From (3.42), we have

$$\begin{aligned} &\Big|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle \\ &\qquad + \int_\Omega (\vartheta + \lambda)|u_n|^{p(z)-2}u_n h\,dz - \int_\Omega g(z, u_n)h\,dz\Big| \le \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \end{aligned} \tag{3.43}$$

for all $h \in W^{1,p(z)}(\Omega)$ and with $\varepsilon_n \to 0^+$. In (3.43) we choose $h = -u_n^- \in W^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} &\rho_p(Du_n^-) + \rho_q(Du_n^-) + [\vartheta + \lambda]\rho_q(u_n^-) \le C_6\|u_n^-\| \\ &\qquad \text{for some } C_6 > 0, \text{ all } n \in \mathbb{N} \quad (\text{see } (3.34)) \\ &\Rightarrow\; \{u_n^-\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded} \\ &\qquad (\text{see Proposition 2.2 and recall that } \vartheta > -\lambda). \end{aligned} \tag{3.44}$$

To show that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded, we need to show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded (see (3.44)). Arguing by contradiction, suppose that at least for a subsequence we have

$$\|u_n^+\| \to \infty. \tag{3.45}$$

Let $v_n = u_n^+/\|u_n^+\|$ for $n \in \mathbb{N}$. Then $\|v_n\| = 1$, $v_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } W^{1,p(z)}(\Omega), \qquad v_n \to v \quad \text{in } L^{r(z)}(\Omega),\ v \geq 0. \tag{3.46}$$

Let $\Omega_+ = \{z \in \Omega :\ v(z) > 0\}$. First we assume that $|\Omega_+|_N > 0$ (that is, $v \neq 0$). Then we have

$$\begin{aligned}
&u_n^+ \to +\infty && \text{for a.a. } z \in \Omega_+\\
&\Rightarrow \frac{F(z,u_n^+(z))}{u_n^+(z)^{p_+}} \to \infty && \text{for a.a. } z \in \Omega_+\\
& && \text{(see hypothesis (H}_1\text{) (ii))}\\
&\Rightarrow \int_{\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz \to +\infty && \text{(by Fatou's lemma).}
\end{aligned} \tag{3.47}$$

Note that hypotheses (H$_1$) (i), (ii) imply that

$$-C_7 \leq F(z,x) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \geq 0,\ \text{some } C_7 > 0.$$

Hence we have

$$\begin{aligned}
\int_{\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz &= \int_{\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz + \int_{\Omega\setminus\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz\\
&\geq \int_{\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz - \frac{C_7|\Omega|_N}{\|u_N^+\|^{p_+}}
\end{aligned}$$

$$\Rightarrow \int_{\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz \to +\infty \quad \text{(see (3.47) and (3.45)).} \tag{3.48}$$

On the other hand, from (3.34), (3.42) and (3.44), we can say that, for some $C_8 > 0$ and all $n \in \mathbb{N}$,

$$\begin{aligned}
-\frac{1}{q_-}\bigg[\int_{\Omega} \frac{1}{\|u_n^+\|^{p_+-p(z)}}|Dv_n|^{p(z)}\,dz + \int_{\Omega} \frac{1}{\|u_n^+\|^{p_+-q(z)}}|Dv_n|^{q(z)}\,dz\\
+ \int_{\Omega} \frac{\vartheta+\lambda}{\|u_n^+\|^{p_+-p(z)}}|v_n|^{p(z)}\,dz + \int_{\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz \leq C_8
\end{aligned}$$

implies

$$\int_{\Omega} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}\,dz \leq C_9 \quad \text{for some } C_9 > 0,\ \text{all } n \in \mathbb{N}. \tag{3.49}$$

Comparing (3.49) and (3.47), we have a contradiction.

Next, we assume that $|\Omega_+|_N = 0$, (that is, $v \equiv 0$). For all $n \in \mathbb{N}$, let $t_n \in [0,1]$ be such that

$$\gamma_\lambda\big(t_n u_n^+\big) = \max\big\{\gamma_\lambda\big(t u_n^+\big) : 0 \le t \le 1\big\}. \tag{3.50}$$

For $\xi > 1$, let $y_n = \xi^{1/p_-} v_n$ for all $n \in \mathbb{N}$. Then $y_n \xrightarrow{w} 0$ in $W^{1,p(z)}(\Omega)$. It follows that

$$\int_\Omega \frac{\vartheta+\lambda}{p(z)}\, y_n^{p(z)}\,dz \to 0, \qquad \int_\Omega G(z,y_n)\,dz \to 0. \tag{3.51}$$

On account of (3.45), we can find $n_0 \in \mathbb{N}$ such that

$$\frac{\xi^{1/p_-}}{\|u_n^+\|} \in (0,1] \quad \text{for all } n \ge n_0. \tag{3.52}$$

Then, for $n \ge n_0$, we have

$$\begin{aligned}
\gamma_\lambda(t_n u_n^+) &= \gamma_\lambda(y_n) \quad \text{(see (3.52), (3.50))}\\
&= \int_\Omega \frac{1}{p(z)}\,\xi^{p(z)/p_-}|Dv_n|^{p(z)}\,dz\\
&\quad + \int_\Omega \frac{\vartheta+\lambda}{p(z)}\,\xi^{p(z)/p_-} v_n^{p(z)}\,dz - \int_\Omega G(z,y_n)\,dz\\
&\ge \frac{\xi}{p_+}\big[\rho_p(Dv_n) + (\vartheta+\lambda)\rho_p(v_n)\big] - \int_\Omega G(z,y_n)\,dz\\
&\quad \text{(recall that } \xi > 1)\\
\Rightarrow \gamma_\lambda(t_n u_n^+) &\ge \frac{C_{10}\xi}{2p_+} \qquad (3.53)\\
&\quad \text{for some } C_{10} > 0, \text{ all } n \ge n_1 \ge n_0 \text{ (see (3.51))}.
\end{aligned}$$

Since $\xi > 1$ is arbitrary, from (3.53) we infer that

$$\gamma_\lambda(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{3.54}$$

For some $C_{11} > 0$ and all $n \in \mathbb{N}$ (see (3.34), (3.41)) we have

$$\gamma_\lambda(0) = 0 \quad \text{and} \quad \gamma_\lambda(u_n^+) \le C_{11}. \tag{3.55}$$

From (3.54) and (3.55), it follows that we can find $n_2 \in \mathbb{N}$ such that

$$\begin{aligned}
& t_n \in (0,1) \quad \text{for all } n \ge n_2 \;\Rightarrow\; \frac{d}{dt}\gamma_\lambda(t u_n^+)\Big|_{t=t_1} = 0 \quad \text{(see (3.50))}\\
&\Rightarrow \langle \gamma_\lambda'(t_n u_n^+), t_n u_n^+\rangle = 0 \quad \text{for all } n \ge n_2 \quad \text{(by the chain rule).} \qquad (3.56)
\end{aligned}$$

Then, for $n \geq n_2$, we have

$$
\begin{aligned}
\gamma_\lambda(t_n u_n^+) &= \gamma_\lambda(t_n u_n^+) - \frac{1}{p_+}\langle \gamma_\lambda'(t_n u_n^+), t_n u_n^+\rangle \quad \text{(see (3.56))} \\
&\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right] |Du_n^+|^{p(z)}\,dz \\
&\quad + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right] |Du_n^+|^{q(z)}\,dz \\
&\quad + \frac{1}{p_+}\int_\Omega \big[g(z, t_n u_n^+)(t_n u_n^+) - p_+ G(z, t_n u_n^+)\big]dz \\
&\quad \text{(since } t_n \in (0,1)) \\
&\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right] |Du_n^+|^{p(z)} + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right] |Du_n^+|^{q(z)}\,dz \\
&\quad + \frac{1}{p_+}\int_\Omega \big[g(z, u_n^+)u_n^+ - p_+ G(z, u_n^+)\big]dz + C_{12} \\
&\quad \text{for some } C_{12} > 0 \text{ (see (H}_1\text{) (iii) and (3.34))} \\
&= \gamma_\lambda(u_n^+) - \frac{1}{p_+}\langle \gamma_\lambda'(u_n^+), u_n^+\rangle + C_{12} \leq C_{13}
\end{aligned} \tag{3.57}
$$

for some $C_{13} > 0$ and all $n \in \mathbb{N}$ (see (3.41), (3.42), (3.44)).

Comparing (3.57) and (3.54), we have a contradiction. Therefore $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded, hence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded (see (3.44)).

We may assume that

$$
u_n \xrightarrow{w} u \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{r(z)}(\Omega). \tag{3.58}
$$

In (3.43) we choose $h = u_n - u \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.58). Then

$$
\begin{aligned}
&\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0 \\
&\quad \Rightarrow \limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle] \leq 0 \\
&\quad\quad\ \text{(since } A_q(\,\cdot\,) \text{ is monotone)} \\
&\quad \Rightarrow \limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leq 0 \qquad \text{(see (3.58))} \\
&\quad \Rightarrow u_n \to u \text{ in } W^{1,p(z)}(\Omega) \quad \text{(see Proposition 2.3).}
\end{aligned}
$$

Therefore $\gamma_\lambda(\,\cdot\,)$ satisfies the C-condition and we have proved the Claim. Then (3.39), (3.40) and the Claim permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p(z)}(\Omega)$ such that

$$
\begin{cases}
\widehat{u} \in K_{\gamma_\lambda} \subseteq [u_\mu) \cap \operatorname{int} C_+ & \text{(see (3.35))} \\
m_\lambda \leq \gamma_\lambda(\widehat{u}) & \text{(see (3.39)).}
\end{cases} \tag{3.59}
$$

From (3.59), (3.34), (3.39), we infer that

$$\widehat{u} \in S_\lambda^+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \widehat{u} \neq u_0 \quad (\text{for } \lambda \in (\lambda_*, +\infty)). \qquad \square$$

We have to determine what happens with the critical parameter λ_*. We show that λ_* is also admissible and so $\mathcal{L}^+ = [\lambda_*, +\infty)$.

Proposition 3.13. *If hypotheses* (H_0), (H_1') *hold, then* $\lambda_* \in \mathcal{L}^+$.

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}^+$ be such that $\lambda_n \downarrow \lambda_*$. For each $n \in \mathbb{N}$, let $\eta \in (\lambda_*, \lambda_n)$. From Proposition 3.3 we know that $\eta \in \mathcal{L}^+$ and so we can find $u_\eta \in S_\eta^+ \subseteq \operatorname{int} C_+$. From Corollary 3.4, we know that we can find $u_n = u_{\lambda_n} \in S_{\lambda_n}^+ \subseteq \operatorname{int} C_+$ with $u_\eta - u_n \in D_+$.

Consider the energy functional $\varphi_\lambda \colon W^{1,p(z)}(\Omega) \to \mathbb{R}$ for problem (P_λ) defined by

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz + \int_\Omega \frac{\lambda}{p(z)} |u|^{p(z)}\, dz - \int_\Omega F(z,u)\, dz$$

for all $u \in W^{1,p(z)}(\Omega)$. We know that $\varphi_\lambda \in C^1\big(W^{1,p(z)}(\Omega)\big)$ and from the first part of this proof (see also the proof of Proposition 3.3), we have

$$\varphi_{\lambda_n}(u_n) < \varphi_{\lambda_n}(0) = 0 \quad \text{for all } n \in \mathbb{N},$$

then, for all $n \in \mathbb{N}$,

$$\int_\Omega \frac{p_+}{p(z)} |Du_n|^{p(z)}\, dz + \int_\Omega \frac{p_+}{q(z)} |Du_n|^{q(z)}\, dz + \int_\Omega \frac{\lambda_n p_+}{p(z)} |u_n|^{p(z)}\, dz - \int_\Omega p_+ F(z,u_n)\, dz \leq 0. \tag{3.60}$$

On the other hand, since $u_n \in S_n^+$ for all $n \in \mathbb{N}$, we have

$$\varphi_{\lambda_n}'(u_n) = 0 \quad \text{for all } n \in \mathbb{N},$$

then, for all $n \in \mathbb{N}$,

$$\rho_p(Du_n) + \rho_q(Du_n) + \lambda_n \rho_p(u_n) = \int_\Omega f(z,u_n) u_n\, dz \tag{3.61}$$

From (3.60) and (3.61), as in the proof of Proposition 3.12 (see the Claim), via a contradiction argument, we show that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega) \quad \text{is bounded.}$$

We may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W^{1,p(z)}(\Omega), \qquad u_n \to u_* \quad \text{in } L^{r(z)}(\Omega). \tag{3.62}$$

We know that $\langle \varphi_{\lambda_n}'(u_n), h\rangle = 0$ for all $h \in W^{1,p(z)}(\Omega)$ and all $n \in \mathbb{N}$. Choosing $h = u_n - u_* \in W^{1,p(z)}(\Omega)$, passing to the limit as $n \to \infty$ and using (3.62), as before (see the proof of Proposition 2.3), exploiting the $(\mathrm{S})_+$-property of $A_p(\cdot)$ (see Proposition 2.3), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\langle A_{p(z)}(u_n),u_n-u_*\rangle\le 0,\\ &\Rightarrow\ u_n\to u_*\quad\text{in } W^{1,p(z)}(\Omega).\end{aligned}\tag{3.63}$$

Let $\mu>\lambda_n$ for all $n\in\mathbb{N}$. Then $\mu\in\mathcal{L}^+$ and using Proposition 3.10, we can find $u_\mu^*\in\operatorname{int}C_+$ such that

$$u_\mu^*\le u_n\quad\text{for all } n\in\mathbb{N}\ \Rightarrow\ u_\mu^*\le u_*.\tag{3.64}$$

From (3.63) it follows that

$$\begin{aligned}&\langle\varphi_\lambda'(u_*),h\rangle=0\quad\text{for all } h\in W^{1,p(z)}(\Omega)\\ &\qquad\Rightarrow\ u_*\in S_\lambda^+\quad\text{and so } \lambda_*\in\mathcal{L}^+ \text{ (see (3.64))}.\end{aligned}$$

The proof is now complete. □

So, we have proved that $\mathcal{L}=[\lambda_*,\infty)$.

Summarizing, we can state the following global (with respect to the parameter $\lambda\in\mathbb{R}$) multiplicity theorem for problem (P_λ) (a bifurcation-type theorem).

Theorem 3.14. *If hypotheses* (H_0), (H_1) *hold, then there exists* $\lambda_*\in\mathbb{R}$ *such that*:

(a) *for all* $\lambda>\lambda_*$, *problem* (P_λ) *has at least two positive solutions* $u_0,\widehat{u}\in\operatorname{int}C_+$, $u_0\ne\widehat{u}$;
(b) *for* $\lambda=\lambda_*$, *problem* (P_λ) *has at least one positive solution* $u_*\in\operatorname{int}C_+$;
(c) *if* $\lambda<\lambda_*$, *problem* (P_λ) *has no positive solutions.*

4. Minimal positive solution

In this section we show that for every $\lambda\in\mathcal{L}^+=[\lambda_*,\infty)$, problem (P_λ) has a smallest positive solution.

Proposition 4.1. *If hypotheses* (H_0), (H_1) *hold and* $\lambda\in\mathcal{L}^+=[\lambda_*,+\infty)$, *then problem* (P_λ) *has a smallest positive solution* $\widehat{u}_\lambda^*$ *(that is,* $\widehat{u}_\lambda^*\in S_\lambda^+$ *and* $\widehat{u}_\lambda^*\le u$ *for all* $u\in S_\lambda^+$*).*

Proof. From Papageorgiou, Rădulescu and Repovš [13, Proposition 7] we know that S_λ^+ is downward directed (that is, if $u_1,u_2\in S_\lambda^+$, then we can find $u\in S_\lambda^+$ such that $u\le u_1$, $u\le u_2$). Invoking Lemma 3.10 of Hu and Papageorgiou [8, p. 178], we can find $\{u_n\}_{n\in\mathbb{N}}\subseteq S_\lambda^+$ decreasing such that $\inf S_\lambda^+=\inf\limits_{n\in\mathbb{N}}u_n$.

We have that

$$\langle\varphi_\lambda'(u_n),h\rangle=0\quad\text{for all } h\in W^{1,p(z)}(\Omega),\ \text{all } n\in\mathbb{N}.\tag{4.1}$$

Choosing $h = u_n \in W^{1,p(z)}(\Omega)$, we obtain

$$\begin{aligned}
&\rho_p(Du_n) + \rho_q(Du_n) + \lambda\rho_p(u_n) \\
&= \int_\Omega f(z,u_n)u_n\,dz \le \int_\Omega |f(z,u_n)|u_1\,dz \quad (\text{since } 0 \le u_n \le u_1 \text{ for all } n \in \mathbb{N}) \\
&\le C_{14} \quad \text{for some } C_{14} > 0,\ \text{all } n \in \mathbb{N} \quad (\text{see hypothesis } (\mathrm{H}_1)\,(\mathrm{i})) \\
&\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega) \quad \text{is bounded} \quad (\text{see Proposition 2.2}).
\end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} \widehat{u}_\lambda^* \quad \text{in } W^{1,p(z)}(\Omega), \qquad u_n \to \widehat{u}_\lambda^* \quad \text{in } L^{r(z)}(\Omega). \tag{4.2}$$

Choosing $h = u_n - \widehat{u}_\lambda^* \in W^{1,p(z)}(\Omega)$ in (4.1), passing to the limit as $n \to \infty$ and using (4.2) and the $(\mathrm{S})_+$-property of $A_p(\,\cdot\,)$ we obtain $u_n \to \widehat{u}_\lambda^*$ in $W^{1,p(z)}(\Omega)$ hence

$$\langle \varphi'(\widehat{u}_\lambda^*), h\rangle = 0 \quad \text{for all } h \in W^{1,p(z)}(\Omega) \tag{4.3}$$

(see (4.1)). Also, if $\mu > \lambda$, then we have

$$u_\mu^* \le \widehat{u}_\lambda^* \tag{4.4}$$

(see Proposition 3.10). From (4.3) and (4.4), we infer that

$$\widehat{u}_\lambda^* \in S_\lambda^+ \subseteq \operatorname{int} C_+, \qquad \widehat{u}_\lambda^* = \inf S_\lambda^+. \qquad \square$$

5. Nodal solutions

In this section we prove the existence of a nodal solution (sign-changing solution) for problem (P_λ).

If the conditions on $f(z,\cdot\,)$ are bilateral (that is, they are valid for all $x \in \mathbb{R}$ and not only for $x \ge 0$ as in (H_1), then we can have similar results for the negative solutions of (P_λ). So, now we impose the following conditions of $f(z,\cdot\,)$:

(H_1'') $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z,x)| \le a(z)\left[1 + |x|^{r(z)-1}\right]$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$, $p_+ < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$;

(ii) if

$$F(z,x) = \int_0^x f(z,s)\,ds,$$

then

$$\lim_{x\to\pm\infty} \frac{F(z,x)}{x^{p_+}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) if $e(z,x) = f(z,x)x - p_+F(z,x)$, then there exists $\mu \in L^1(\Omega)$ such that

$$e(z,x) \le e(z,y) + \mu(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \le x \le y \text{ and } y \le x \le 0;$$

(iv) there exist $\tau \in C(\overline{\Omega})$ and $C_0, \delta_0, \widehat{C} > 0$ such

$$\begin{aligned} &1 < \tau_+ < q_-, \\ &C_0|x|^{\tau(z)} \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta_0, \\ &-\widehat{C}|x|^{p(z)} \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}; \end{aligned}$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the function

$$x \mapsto f(z,x) + \widehat{\xi}_\rho |x|^{p(z)-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Let $\mathcal{L}^-$ be the set of admissible parameters for negative solutions and let S_λ^- be the set of negative solutions. Then as in Section 3, we can establish the existence of a critical parameter value $\lambda^* > -\infty$ such that

$$\mathcal{L}^- = [\lambda^*, +\infty) \quad \text{and} \quad \emptyset \ne S_\lambda^- \subseteq -\operatorname{int} C_+ \quad \text{for all } \lambda \in \mathcal{L}^-.$$

We have a global multiplicity result for negative solutions (see Theorem 3.14). Moreover, for every $\lambda \in \mathcal{L}^- = [\lambda_*, +\infty)$ there exists a maximal negative solution $\widehat{v}_\lambda^* \in S_\lambda^- \subseteq \operatorname{int} C_+$ (that is, $\widehat{v}_\lambda^* \le v$ for all $v \in S_\lambda^-$).

We set $\widetilde{\lambda}_0 = \max\{\lambda_*, \lambda^*\}$. For every $\lambda \ge \widetilde{\lambda}_0$ the problem has extremal constant sign solutions

$$\widehat{u}_\lambda^* \in S_\lambda^* \subseteq \operatorname{int} C_+, \qquad \widehat{v}_\lambda^* \subseteq -\operatorname{int} C_+.$$

Let $\lambda \ge \widetilde{\lambda}_0$ and $\vartheta > -\lambda$. We introduce the Carathéodory function $\widehat{k}(z,x)$ defined by

$$\widehat{k}(z,x) = \begin{cases} f\big(z, \widehat{v}_\lambda^*(z)\big) + \vartheta \big|\widehat{v}_\lambda^*(z)\big|^{p(z)-2} \widehat{v}_\lambda^*(z) & \text{if } x < \widehat{v}_\lambda^*(z), \\ f(z,x) + \vartheta |x|^{p(z)-2}x & \text{if } \widehat{v}_\lambda^*(z) \le x \le \widehat{u}_\lambda^*(z), \\ f\big(z, \widehat{u}_\lambda^*(z)\big) + \vartheta \widehat{u}_\lambda^*(z)^{p(z)-1} & \text{if } \widehat{u}_\lambda^*(z) < x. \end{cases} \tag{5.1}$$

We also consider the positive and negative truncations of $\widehat{k}(z,\cdot)$, namely the Carathéodory functions

$$\widehat{k}_\pm(z,x) = \widehat{k}\big(z, \pm x^\pm\big). \tag{5.2}$$

We set

$$\widehat{K}(z,x) = \int_0^x \widehat{k}(z,s)\,ds \quad \text{and} \quad \widehat{K}_\pm(z,x) = \int_0^x \widehat{k}_\pm(z,s)\,ds$$

and introduce the C^1-functionals $\widehat{w}_\lambda, \widehat{w}_\lambda^\pm \colon W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \widehat{w}_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\,dz, \\ + \int_\Omega \frac{\vartheta + \lambda}{p(z)} |u|^{p(z)}\,dz - \int_\Omega \widehat{K}(z,u)\,dz \end{aligned}$$

$$\begin{aligned}\widehat{w}_\lambda^\pm(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz \\ + \int_\Omega \frac{\vartheta+\lambda}{p(z)}|u|^{p(z)}\,dz - \int_\Omega \widehat{K}_\pm(z,u)\,dz,\end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. Using (5.1) and (5.2), we can show easily that

$$K_{\widehat{w}_\lambda} \subseteq [\widehat{v}_\lambda^*, \widehat{u}_\lambda^*] \cap C^1(\overline{\Omega}), \quad K_{\widehat{w}_\lambda^+} \subseteq [0, \widehat{u}_\lambda^*] \cap C_+, \quad K_{\widehat{w}_\lambda^-} \subseteq [\widehat{v}_\lambda^*, 0] \cap (-C_+).$$

The extremality of $\widehat{u}_\lambda^*$, $\widehat{v}_\lambda^*$ implies that

$$K_{\widehat{w}_\lambda} \subseteq [\widehat{v}_\lambda^*, \widehat{u}_\lambda^*] \cap C^1(\overline{\Omega}), \quad K_{\widehat{w}_\lambda^+} = \{0, \widehat{u}_\lambda^*\}, \quad K_{\widehat{w}_\lambda^-} = \{0, \widehat{v}_\lambda^*\}. \tag{5.3}$$

Working with these functionals, we produce a nodal (sign-changing) solution.

PROPOSITION 5.1. *If hypotheses* (H_0), (H_1'') *hold and* $\lambda \geq \widetilde{\lambda}_0$, *then problem* (P_λ) *admits a nodal solution* $y_0 \in [\widehat{v}_\lambda^*, \widehat{u}_\lambda^*] \cap C^1(\overline{\Omega})$.

PROOF. First we show that $\widehat{u}_\lambda^* \in \operatorname{int} C_+$ and $\widehat{v}_\lambda^* \in -\operatorname{int} C_+$ are local minimizers of the functional $\widehat{w}_\lambda(\,\cdot\,)$. From (5.1) and (5.2), we see that $\widehat{w}_\lambda^+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_\lambda^* \in W^{1,p(z)}(\Omega)$ such that

$$\widehat{w}_\lambda^+(\overline{u}_\lambda^*) = \inf\big\{\widehat{w}_\lambda^+(u) : u \in W^{1,p(z)}(\Omega)\big\}. \tag{5.4}$$

If $u \in \operatorname{int} C_+$ and we choose $t \in (0,1)$ small so that at least we have $tu \leq \widehat{u}_\lambda^*$ (recall that $\widehat{u}_\lambda^* \in \operatorname{int} C_+$). Then on account of hypothesis (H_1'') (iv) and, since $\tau_+ < q_-$, for $t \in (0,1)$ even smaller, we have

$$\begin{aligned}\widehat{w}_\lambda^+(tu) < 0 \ \Rightarrow\ \widehat{w}_\lambda^+(\overline{u}_\lambda^*) < 0 = \widehat{w}_\lambda^+(0) \quad (\text{see } (5.4)) \\ \Rightarrow\ \overline{u}_\lambda^* \neq 0.\end{aligned}$$

Since $\overline{u}_\lambda^* \in K_{w_\lambda^+}$ (see (5.4)), from (5.3) we infer that $\overline{u}_\lambda^* = \widehat{u}_\lambda^* \in \operatorname{int} C_+$. It is clear from (5.1) and (5.2) that

$$\begin{aligned}\widehat{w}_\lambda\big|_{C_+} = \widehat{w}_\lambda^+\big|_{C_+} \ &\Rightarrow\ \widehat{u}_\lambda^* \quad \text{is a local } C^1(\overline{\Omega})\text{-minimizer of } w_\lambda(\,\cdot\,) \\ &\Rightarrow\ \widehat{u}_\lambda^* \quad \text{is a local } W^{1,p(z)}(\Omega)\text{-minimizer of } w_\lambda(\,\cdot\,) \quad (\text{see [7]}).\end{aligned}$$

Similarly we show that $\widehat{v}_\lambda^* \in -\operatorname{int} C_+$ is a local minimizer of $\widehat{w}(\,\cdot\,)$. This time we work with $\widehat{w}_\lambda^-(\,\cdot\,)$. We may assume that

$$K_{\widehat{w}_\lambda} \quad \text{is finite.} \tag{5.5}$$

Otherwise, on account of (5.3) and the extremality of $\widehat{u}_\lambda^*$ and $\widehat{v}_\lambda^*$, we have a whole sequence of distinct nodal solutions and so we are done. We may assume that

$$\widehat{w}_\lambda(\widehat{v}_\lambda^*) \leq \widehat{w}_\lambda(\widehat{u}_\lambda^*).$$

The reasoning is similar, if the opposite inequality holds. From the fact that $\widehat{u}_\lambda^*$ is a local minimizer of $w_\lambda(\,\cdot\,)$, from (5.5) and by using Theorem 5.7.6 of

Papageorgiou, Rădulescu and Repovš [14, p. 449], we can find $\rho \in (0,1)$ small such that

$$(5.6)\qquad \begin{cases} \widehat{w}_\lambda(\widehat{v}_\lambda^*) \le \widehat{w}_\lambda(\widehat{u}_\lambda^*) < \inf\big\{\widehat{w}(u) : \|u-\widehat{u}_\lambda^*\| = \rho\big\} = \widehat{m}_\lambda, \\ \|\widehat{v}_\lambda^* - \widehat{u}_\lambda^*\| > \rho. \end{cases}$$

Evidently, the functional $\widehat{w}_\lambda(\,\cdot\,)$ is coercive (see (5.1) and recall that $\vartheta > -\lambda$). So, it satisfies the C-condition (see Proposition 5.1.15 of [14, p. 369]). Then, using also (5.6), we see that we can apply the mountain pass theorem and produce $y_0 \in W^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} &y_0 \in K_{w_\lambda} \subseteq [\widehat{v}_\lambda^*, \widehat{u}_\lambda^*] \cap C^1(\overline{\Omega}) &&\text{(see (5.3))},\\ &\widehat{w}_\lambda(\widehat{v}_\lambda^*) \le \widehat{w}_\lambda(\widehat{u}_\lambda^*) < \widehat{m}_\lambda \le \widehat{w}_\lambda(y_0) &&\text{(see (5.6))}. \end{aligned}$$

From the above we see that $y_0 \notin \big\{\widehat{u}_\lambda^*, \widehat{v}_\lambda^*\big\}$. From Theorem 6.5.8 of [14, p. 527] we have

$$(5.7)\qquad C_1(\widehat{w}_\lambda, y_0) \ne 0.$$

On the other hand, hypothesis (H_1) (iv) and Proposition 3.7 of Papageorgiou and Rădulescu [12], imply that

$$(5.8)\qquad C_k(\widehat{w}_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Comparing (5.7) and (5.8), we conclude that

$$(5.9)\qquad y_0 \ne 0 \;\Rightarrow\; y_0 \notin \{0, \widehat{u}_\lambda^*, \widehat{v}_\lambda^*\}.$$

Since $y_0 \in [\widehat{v}_\lambda^*, \widehat{u}_\lambda^*] \cap C^1(\overline{\Omega})$, the extremality of $\widehat{u}_\lambda^*$, $\widehat{v}_\lambda^*$ and (5.9) imply that $y_0 \in C^1(\overline{\Omega})$ is a nodal solution of (P_λ). □

So, we can state the following multiplicity theorem for our problem.

Theorem 5.2. *If hypotheses* (H_0), (H_1'') *hold, then there exists* $\widetilde{\lambda}_0 \in \mathbb{R}$ *such that*

(a) *for* $\lambda = \widetilde{\lambda}_0$, *problem* (P_λ) *has at least three nontrivial solutions*

$$\begin{gathered} u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+, \\ y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \quad \textit{nodal}. \end{gathered}$$

(b) *for all* $\lambda > \widetilde{\lambda}_0$, *problem* (P_λ) *has at least five nontrivial solutions*

$$\begin{gathered} u_0, \widehat{u} \in \operatorname{int} C_+, \qquad u_0 \le \widehat{u}, \qquad u_0 \ne \widehat{u}, \\ v_0, \widehat{v} \in -\operatorname{int} C_+, \qquad \widehat{v} \le v_0, \qquad v_0 \ne \widehat{v}, \\ y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \quad \textit{nodal}. \end{gathered}$$

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