

ELLIPTIC PROBLEMS ON WEIGHTED LOCALLY FINITE GRAPHS

MAURIZIO IMBESI — GIOVANNI MOLICA BISCI — DUŠAN D. REPOVŠ

Dedicated to the memory of Edward Fadell and Sufian Hussein

ABSTRACT. Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph whose finite measure μ has a positive lower bound. Motivated by a wide interest in the current literature, in this paper we study the existence of classical solutions for a class of elliptic equations involving the μ -Laplacian operator on the graph \mathcal{G} , whose analytic expression is given by

$$\Delta_{\mu}u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} w(x, y)(u(y) - u(x)) \quad (\text{for all } x \in V),$$

where $w: V \times V \rightarrow [0, +\infty)$ is a weight symmetric function and the sum on the right-hand side of the above expression is taken on the neighbours vertices $x, y \in V$, that is $x \sim y$ whenever $w(x, y) > 0$. More precisely, by exploiting direct variational methods, we study problems whose simple prototype has the following form

$$\begin{cases} -\Delta_{\mu}u(x) = \lambda f(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0, \end{cases}$$

where D is a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$, the nonlinear term $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy suitable structure conditions and λ is a positive real parameter. By applying a critical point result coming out from a classical Pucci–Serrin theorem in addition to a local minimum result for differentiable functionals due to Ricceri, we are able to prove the existence of at least two solutions for the treated problems. We emphasize the crucial role played by the famous Ambrosetti–Rabinowitz growth condition along the proof of the main theorem and its consequences. Our results improve the general results obtained by A. Grigor'yan, Y. Lin, and Y. Yang in [17].

2020 *Mathematics Subject Classification*. Primary: 35R02; Secondary: 35J91, 35A15.

Key words and phrases. Semi-linear equations on graphs; variational methods; critical point theory.

1. Introduction

Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph and μ be a positive finite measure over V that admits a positive lower bound. The aim of this paper is to study the existence of non-trivial solutions for certain classes of nonlinear elliptic Dirichlet one-parameter problems of the form

$$(1.1) \quad \begin{cases} -\Delta_\mu u(x) = \alpha(x)u(x) + \lambda f(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0, \end{cases}$$

where D is a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$ and λ is a real positive parameter. Moreover, the coefficient $\alpha: V \rightarrow \mathbb{R}$ of the linear term is assumed integrable over D and $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ is the nonlinear part, which is continuous for every $x \in D$ and satisfies some growth restrictions both near zero and at infinity.

There is an extensive theory for the study of nonlinear elliptic equations on Euclidean domains by using Sobolev spaces and related Sobolev embedding results. A natural question arises of how to establish an appropriate framework to cope with (1.1) on graph domains.

In this new setting several difficulties naturally arises. For instance, the standard concept of generalized derivative of a function cannot be used, and so the notion of differential operators such as the Laplacian on graph domains need to be clarified. More precisely, according to the notations and definitions given in Section 2, the μ -Laplacian operator $\Delta_\mu: W_0^{1,2}(D) \rightarrow L^2(D)$ has been defined in the distributional sense

$$\int_D \Delta_\mu u(x)v(x) d\mu = - \int_D \Gamma(u, v)(x) d\mu,$$

for every $v \in W_0^{1,2}(D)$, where the map $\Gamma: W_0^{1,2}(D) \times W_0^{1,2}(D) \rightarrow \mathbb{R}^D$ is the gradient form whose analytical local expression is given below

$$\Gamma(u, v)(x) := \frac{1}{2\mu(x)} \sum_{y \sim x} w(x, y)(u(y) - u(x))(v(y) - v(x)),$$

for every $x \in D$.

Once a Laplacian is defined, we may introduce a Hilbert space structure and then establish compactness theorems allowing (1.1) to be investigated.

In this direction, A. Grigor'yan in [14], [15] and A. Grigor'yan, Y. Lin, and Y. Yang in [17], [18] obtained some embedding theorems that can be applied by studying variational problems settled on graphs for which Poincaré-type inequalities hold; see, for instance, the paper [35]. Moreover, the μ -Laplacian operator or some of its generalizations as described in Section 5 permit to describe Random walks on graphs. A comprehensive discussion can be found in the monograph [22]; see also the papers [6] and [23], [24] for related topics.

Moreover, from the parabolic point of view, M. Barlow, T. Coulhon, and A. Grigor'yan in [3] studied some upper estimates on the long time behavior of the heat kernel on non-compact Riemannian manifolds and infinite graphs, which only depend on a lower bound of the volume growth.

Set

$$(1.2) \quad \lambda_1 := \inf_{u \in C_0(D) \setminus \{0\}} \frac{\int_D |\nabla u|^2(x) d\mu}{\int_D |u(x)|^2 d\mu}.$$

The main result of the present paper is a multiplicity theorem as stated here below.

THEOREM 1.1. *Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph, D be a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$ and let $\mu: D \rightarrow]0, +\infty[$ be a measure on D . Let $\alpha \in L^1(D)$ be a function satisfying either*

$$(1.3) \quad \alpha(x) \leq 0 \quad \text{for every } x \in D$$

or

$$(1.4) \quad \int_D |\alpha(x)| d\mu < \mu_0^2 \lambda_1 \quad \text{with } \mu_0 := \min_{x \in D} \mu(x) > 0$$

and let $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that

$$(1.5) \quad f(x, \cdot) \text{ is continuous in } \mathbb{R} \text{ and } f(x, 0) \neq 0 \text{ for some } x \in D$$

as well as

$$(1.6) \quad \text{there are } \beta > 2 \text{ and } r_0 > 0 \text{ such that}$$

$$tf(x, t) \geq \beta F(x, t) > 0 \text{ for any } |t| \geq r_0 \text{ and every } x \in D,$$

where F is the potential given by

$$(1.7) \quad F(x, t) := \int_0^t f(x, s) ds \quad \text{for any } (x, t) \in D \times \mathbb{R}.$$

Then, for any $\varrho > 0$ and any

$$(1.8) \quad 0 < \lambda < \frac{\varrho}{2 \max_{x \in D, |s| \leq \kappa \sqrt{\varrho}} \left| \int_0^s f(x, t) dt \right|},$$

where

$$(1.9) \quad \kappa := \begin{cases} \frac{1}{\mu_0 \sqrt{\lambda_1}} & \text{if (1.3) holds,} \\ \frac{1}{\mu_0 \sqrt{\lambda_1} \sqrt{1 - \frac{1}{\mu_0^2 \lambda_1} \int_D |\alpha(x)| d\mu}} & \text{if (1.4) holds,} \end{cases}$$

the problem (1.1) admits at least two non-trivial solutions one of which lies in

$$\mathbb{B}_\varrho := \left\{ u \in W_0^{1,2}(D) : \langle u, u \rangle - \int_D \alpha(x)|u(x)|^2 d\mu < \varrho \right\}.$$

We would stress that the maximal interval of λ 's where the conclusion of Theorem 1.1 holds true is given by $]0, \lambda^*[$, where

$$(1.10) \quad \lambda^* := \frac{1}{2} \sup_{\varrho > 0} \frac{\varrho}{\max_{x \in D, |s| \leq \kappa\sqrt{\varrho}} \left| \int_0^s f(x, t) dt \right|} \in]0, +\infty]$$

with κ given as in (1.9).

REMARK 1.2. We emphasize that Theorem 1.1 ensures the existence of one non-trivial solution (instead of two) if we require that

$$(1.11) \quad f(x, \cdot) \text{ is continuous in } \mathbb{R} \text{ and } f(x, 0) = 0 \text{ for any } x \in \overset{\circ}{D},$$

instead of condition (1.5).

The existence result given in Theorem 1.1 is a consequence of [28, Theorem 1] and [31, Theorem 6]; see Theorem 3.2 below. More precisely, the main result is achieved by proving that the geometry of Theorem 3.2 is respected by our abstract framework: for this we develop a functional analytical setting in order to correctly encode the datum in the variational formulation. We emphasize that the compactness condition required by Theorem 3.2 is satisfied in the graph-theoretical setting thanks to the choice of the functional setting we work in; see Section 2. Moreover, Theorem 1.1 improves the existence results obtained in [17, Theorem 2] and it can be viewed as a graph-theoretical counterpart of [30, Theorem 4].

We also observe that, when we deal with partial differential equations driven by the Laplace operator (or, more generally, by uniformly elliptic operators) with homogeneous Dirichlet boundary conditions, assumption (1.6) – namely the Ambrosetti–Rabinowitz condition (briefly (AR)) – is the standard superquadraticity condition on F ; see, among others, the classical papers [1], [29], [34]. This condition is often considered when dealing with superlinear elliptic boundary value problems and its importance is due to the fact that (1.6) assures the boundedness of the Palais–Smale sequences for the energy functional associated with the problem under consideration.

A special case of Theorem 1.1 reads as follows.

THEOREM 1.3. *Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph, D a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$ and let $\mu : D \rightarrow]0, +\infty[$ be a measure on D . Let $\alpha \in L^1(D)$ satisfy (1.3) and let $f : D \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying*

(1.5) and (1.6). Moreover, assume that

(1.12) there are positive constants M_0 and σ such that

$$\max_{(x,s) \in \overset{\circ}{D} \times [-M_0, M_0]} |f(x, s)| \leq \mu_0^2 \frac{M_0}{(\sigma + 1)} \frac{\lambda_1}{2},$$

where $\mu_0 := \min_{x \in D} \mu(x) > 0$. Then, the Dirichlet problem

$$(1.13) \quad \begin{cases} -\Delta_\mu u(x) = \alpha(x)u(x) + f(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0, \end{cases}$$

admits at least two non-trivial solutions one of which lies in the open ball $\mathbb{B}_{\mu_0^2 M_0^2 \lambda_1}$.

On account of Remark 1.2, Theorem 1.3 ensures that problem (1.1) admits at least one non-trivial solutions requiring (1.11) instead of (1.5). The main novelty here, with respect to the approach considered in [17, Theorem 2], is to avoid the following condition of the nonlinearity term f at zero

$$(1.14) \quad \limsup_{t \rightarrow 0^+} \frac{f(x, t)}{t} < \lambda_1,$$

for every $x \in D$; see Corollary 1.6. Moreover, we also emphasize that Theorems 1 and 2 in [17] are an immediate and direct consequence of Theorem 1.3; see Theorems 1.4 and 1.5. Indeed, let us consider a relaxed version of condition (1.6) as given below

(1.15) there are $\beta > 2$ and $r_0 > 0$ such that

$$tf(x, t) \geq \beta F(x, t) > 0 \text{ for any } t \geq r_0 \text{ and every } x \in D.$$

By using (1.15) the following result holds; see [17, Theorem 1].

THEOREM 1.4. *Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph, D a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$ and let $\mu: D \rightarrow]0, +\infty[$ be a measure on D . Let $\alpha \in L^1(D)$ satisfy (1.3) and let $f: D \times \mathbb{R} \rightarrow [0, +\infty[$ be a function such that (1.11) holds. Moreover, assume that (1.14) and (1.15) hold. Then, problem (1.13) admits at least one non-trivial and non-negative solution.*

A particular form of Theorem 1.4 reads as follows; see [17, Theorem 2].

THEOREM 1.5. *Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph, D a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$ and let $\mu: D \rightarrow]0, +\infty[$ be a measure on D . Let $\gamma, p \in \mathbb{R}$ with $\gamma < \lambda_1$ and $p > 2$. Then, the following problem*

$$(1.16) \quad \begin{cases} -\Delta_\mu u(x) = \gamma u(x) + |u(x)|^{p-2}u(x) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0, \end{cases}$$

admits at least one positive solution.

Suppose that $h: V \rightarrow \mathbb{R}$ is a function satisfying the coercive condition on D , namely there exists some constant $\delta > 0$ such that

$$(1.17) \quad \int_D u(x)(-\Delta_\mu + h)u(x) d\mu \geq \delta \int_D |\nabla u|^2(x) d\mu,$$

for every $u: V \rightarrow \mathbb{R}$. By using (1.17) and the classical (AR) condition, semi-linear elliptic equations of the form

$$-\Delta_\mu u + h(x)u(x) = |u(x)|^{p-2}u(x) \quad \text{for } x \in D$$

on weighted locally finite graphs have been studied in [35, Theorem 1.1]. For the sake of completeness we point out that in Theorem 1.5 we don't require any coercivity assumption as in (1.17).

Finally, an immediate meaningful consequence of Theorem 1.3 is given below; see Remark 4.5 for additional comments and details.

COROLLARY 1.6. *Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph and D be a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$. Furthermore, let $f: \mathbb{R} \rightarrow [0, +\infty[$ be a continuous function and $\mu: D \rightarrow]0, +\infty[$ be a measure on D such that*

$$(1.18) \quad \mu_0 > 2\sqrt{\frac{\max_{t \in [-1,1]} f(t)}{\lambda_1}},$$

where $\mu_0 := \min_{x \in D} \mu(x)$. Then, if $f(x, 0) = 0$ for every $x \in D$ and (1.15) holds, the Dirichlet problem

$$(1.19) \quad \begin{cases} -\Delta_\mu u(x) = f(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0, \end{cases}$$

admits at least a non-trivial and non-negative solution. If $f(x, 0) \neq 0$ for some $x \in \overset{\circ}{D}$ and (1.6) holds, then problem (1.19) admits at least two non-trivial solutions one of which lies in the open ball $\mathbb{B}_{\mu_0^2 \lambda_1}$.

The paper is organized as follows. In Section 2 we recall some basic notions on weighted locally finite graphs. Successively, Section 3 is devoted to the variational methods and abstract framework that we use here. In Section 4 the main existence and multiplicity results have been proved. Finally, in the last section the extension of the main results to Yamabe-type equations of weighted locally finite graphs and involving the (m, p) -Laplacian operator have been considered; see Theorem 5.1.

For the sake of completeness we cite the very recent monograph [15] by A. Grigor'yan as general reference for basic notions used throughout the manuscript. See also the papers [4], [5] for interesting problems and results.

2. Analysis on weighted locally finite graphs

Let V be a nonempty set and let $E \subset V \times V$ be a binary relation. Set

$$x \sim y \text{ in } V \iff xy := (x, y) \in E$$

and assume that

$$xy \in E \iff yx \in E.$$

The couple $\mathcal{G} := (V, E)$ is said to be a non-oriented graph (briefly graph) with set of vertices V a set of edges E . If every $x \in V$ has finitely many neighbours, that is $|\{y \in V : xy \in E\}| < +\infty$, the graph \mathcal{G} is said to be weighted locally finite.

A weight on a weighted locally finite graph \mathcal{G} is a function $w: V \times V \rightarrow [0, +\infty[$ such that

$$w(x, y) = w(y, x) \text{ and } w(x, x) = 0, \text{ (for all } x, y \in V).$$

In such a case

$$x \sim y \text{ in } V \iff w(x, y) > 0,$$

and, for every fixed $x \in V$, its degree, defined by

$$\text{deg}(x) := \sum_{y \in V} w(x, y),$$

is finite.

From now on, let us restrict our attention on a weighted locally finite graph \mathcal{G} and let $\mu: V \rightarrow]0, +\infty[$ be a finite positive measure on V . Moreover, let us define

$$(2.1) \quad \int_V u(x) d\mu := \sum_{x \in V} \mu(x)u(x),$$

for every $u: V \rightarrow \mathbb{R}$. With abuse of notations classical notations for Lebesgue spaces are still used.

Fixed $n \in \mathbb{N}^*$, a path on \mathcal{G} is any finite sequence of vertices $(x_k)_{k=1}^n \subseteq V$ such that

$$x_k x_{k+1} \in E \text{ for every } k = 1, \dots, n - 1.$$

The length of a path on \mathcal{G} is the number of edges in the path. Thus, if $x, y \in V$ we denote by $P(x, y)$ be a path connecting x with y , i.e. a path on \mathcal{G} in which $x_1 = x$ and $x_n = y$. Finally, $\wp(x, y)$ denotes the set of paths connecting x with y , and $\ell(P(x, y))$ is the length of $P(x, y) \in \wp(x, y)$.

We say that \mathcal{G} is connected if, for any two vertices $x, y \in V$, there is a path connecting them. Any connected graph \mathcal{G} has a natural metric structure induced

by the metric $d: V \times V \rightarrow [0, +\infty[$ given by

$$(2.2) \quad d(x, y) := \begin{cases} \min_{P(x,y) \in \wp(x,y)} \{\ell(P(x, y))\} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Consequently, a connected weighted locally finite graph has at most countable many vertices.

As customary, given $\mathcal{G} := (V, E)$ be a weighted locally finite graph, a (proper) subgraph \mathcal{H} of \mathcal{G} is a graph of the form $\mathcal{H} := (D, L)$, where $D \subset V$ and $L \subset E$. Moreover, the vertex boundary ∂D and the vertex interior $\overset{\circ}{D}$ of a connected subgraph $\mathcal{H} := (D, L)$ are defined respectively as

$$\partial D := \{x \in D : \exists y \notin D \text{ such that } xy \in E\},$$

and $\overset{\circ}{D} := D \setminus \partial D$. In addition, we say that D is bounded if it is a bounded subset of V with respect to the usual vertex distance given in (2.2).

For every function $u: V \rightarrow \mathbb{R}$, the μ -Laplacian (or Laplacian for short) of u is defined as

$$(2.3) \quad \Delta_\mu u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} w(x, y)(u(y) - u(x)),$$

for every $x \in V$. Moreover, the associated gradient form has the expression given below

$$(2.4) \quad \Gamma(u, v)(x) := \frac{1}{2\mu(x)} \sum_{y \sim x} w(x, y)(u(y) - u(x))(v(y) - v(x)),$$

for every $x \in V$. Hence, for every $x \in V$, we denote by

$$(2.5) \quad |\nabla u|(x) := \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} w(x, y)(u(y) - u(x))^2 \right)^{1/2},$$

the slope of the function u , where $\Gamma(u) := \Gamma(u, u)$.

By using (2.3) and (2.5), if D is a bounded and connected set of V (briefly a domain) such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$, the Sobolev space associated to D can be defined as follows

$$(2.6) \quad W^{1,2}(D) := \{u: D \rightarrow \mathbb{R} : u \text{ is a real-valued function}\},$$

endowed by the norm

$$(2.7) \quad \|u\|_{W^{1,2}(D)} := \|u\|_{L^2(D)} + \|\nabla u\|_{L^2(D)},$$

for every $u \in W^{1,2}(D)$. Hence, the space $W^{1,2}(D)$ coincides with the set of all real-valued functions defined on the domain D .

Now, it easily seen that $W^{1,2}(D)$ admits a natural structure of Hilbert space in which the inner product is defined as follows

$$(2.8) \quad \langle u, v \rangle := \int_D \Gamma(u, v)(x) d\mu + \int_D u(x)v(x) d\mu, \quad (\text{for all } u, v \in W^{1,2}(D)).$$

Moreover, since D is a finite set, we emphasize that the Hilbert space $W^{1,2}(D)$ is finite dimensional and the finiteness of D also yields

$$(2.9) \quad \mu_0 := \min_{x \in D} \mu(x) > 0,$$

since the measure μ is positive on V .

Finally, $W_0^{1,2}(D)$ is the completion of

$$(2.10) \quad C_0(D) := \{u : D \rightarrow \mathbb{R} : u = 0 \text{ on } \partial D\},$$

with respect to the Sobolev norm $\|\cdot\|_{L^2(D)} + \|\nabla \cdot\|_{L^2(D)}$, i.e.

$$W_0^{1,2}(D) = \overline{C_0(D)}^{\|\cdot\|_{L^2(D)} + \|\nabla \cdot\|_{L^2(D)}}.$$

Now, a sort of integration formula by parts is valid. More precisely,

$$(2.11) \quad \begin{aligned} - \int_D (\Delta_\mu u(x))v(x) d\mu &= - \int_D \left(\frac{\sum_{y \sim x} w(x, y)(u(y) - u(x))}{\mu(x)} \right) v(x) d\mu \\ &= \int_D \frac{1}{2\mu(x)} \sum_{y \sim x} w(x, y)(u(y) - u(x))(v(y) - v(x)) d\mu, \end{aligned}$$

for every $v \in W_0^{1,2}(D)$; see [20] for additional comments and details.

Let us set

$$\|\cdot\|_{L^\infty(D)} := \max_{x \in D} |\cdot| \quad \text{and} \quad \|\cdot\|_{L^\nu(D)} := \left(\int_D |\cdot|^\nu d\mu \right)^{1/\nu},$$

for every $\nu \in [1, +\infty[$ and let λ_1 be the first eigenvalue of $-\Delta_\mu$ with respect to Dirichlet boundary condition as defined in (1.2). Clearly $\lambda_1 \in]0, +\infty[$ since D is finite and thanks to the definition given in (2.1).

The following compact Sobolev embedding result holds.

PROPOSITION 2.1. *Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph and D be a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$. Then $W_0^{1,2}(D)$ is embedded in $L^q(D)$ for every $q \in [1, +\infty[$. In particular*

$$(2.12) \quad \|u\|_{L^\infty(D)} \leq \frac{1}{\mu_0 \sqrt{\lambda_1}} \left(\int_D |\nabla u|^2(x) d\mu \right)^{1/2},$$

and

$$(2.13) \quad \|u\|_{L^\nu(D)} \leq \frac{\mu(D)^{1/\nu}}{\mu_0 \sqrt{\lambda_1}} \left(\int_D |\nabla u|^2(x) d\mu \right)^{1/2},$$

for every $\nu \in [1, +\infty[$ and every $u \in W_0^{1,2}(D)$. Moreover, for every bounded sequence $(u_k)_k$ in $W_0^{1,2}(D)$, up to a subsequence, there exists $u \in L^q(D)$ such that $u_k \rightarrow u$ in $L^q(D)$, namely the embedding $W_0^{1,2}(D) \hookrightarrow L^q(D)$ is compact provided that $q \in [1, +\infty]$.

PROOF. Since D is a finite set, $W_0^{1,2}(D)$ is a finite dimensional space. Consequently $W_0^{1,2}(D)$ is pre-compact, i.e. for every bounded sequence $(u_k)_k$ in $W_0^{1,2}(D)$, up to a subsequence, there exists $u_\infty \in W_0^{1,2}(D)$ such that $u_k \rightarrow u_\infty$ in $W_0^{1,2}(D)$. In addition

$$(2.14) \quad \|u\|_{W_0^{1,2}(D)} = \left(\int_D |\nabla u|^2(x) d\mu \right)^{1/2}$$

is a norm equivalent to (2.7) on $W_0^{1,2}(D)$. Hence, on account of (1.2), one has

$$(2.15) \quad \left(\int_D |u(x)|^2 d\mu \right)^{1/2} = \left(\sum_{x \in D} \mu(x) |u(x)|^2 \right)^{1/2} \\ \leq \frac{1}{\sqrt{\lambda_1}} \left(\int_D |\nabla u|^2(x) d\mu \right)^{1/2},$$

for any $u \in W_0^{1,2}(D)$.

Now, by (2.15) and (2.9) one has that

$$(2.16) \quad \|u\|_{L^\infty(D)} \leq \frac{1}{\mu_0 \sqrt{\lambda_1}} \|u\|_{W_0^{1,2}(D)},$$

for every $u \in W_0^{1,2}(D)$. Moreover, inequality (2.16) immediately yields

$$(2.17) \quad \left(\int_D |u(x)|^\nu d\mu \right)^{1/\nu} \leq \frac{\mu(D)^{1/\nu}}{\mu_0 \sqrt{\lambda_1}} \|u\|_{W_0^{1,2}(D)},$$

for every $\nu \in [1, +\infty[$, where $\mu(D) := \sum_{x \in D} \mu(x)$ denotes the volume of D . Therefore (2.12) and (2.13) hold.

Finally, since $W_0^{1,2}(D)$ is pre-compact, by (2.12) and (2.13) the embedding $W_0^{1,2}(D) \hookrightarrow L^q(D)$ is compact as claimed provided that $q \in [1, +\infty]$. \square

See [17, Theorem 7] for additional comments and remarks. We also emphasize that, since D is a finite set, $(W_0^{1,2}(D), \|\cdot\|)$ is a finite dimensional Hilbert space, where the norm

$$(2.18) \quad \|u\| := \left(\int_D |\nabla u|^2(x) d\mu \right)^{1/2}$$

on $W_0^{1,2}(D)$ is equivalent to the norm $\|\cdot\|_{W^{1,2}(D)}$ given in (2.7) thanks to Proposition 2.1. Moreover, in this finite setting, one has

$$W_0^{1,2}(D) = C_0(D),$$

where $C_0(D)$ is defined in (2.10).

3. Variational methods and abstract framework

The aim of this section is to prove that, under natural assumptions on the nonlinear term f , problem (1.1) admits two non-trivial classical solutions. As we already said, this is done by means of variational techniques.

Indeed, if $\alpha \in L^1(D)$ and $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f(x, \cdot)$ is continuous in \mathbb{R} for every $x \in D$, then problem (1.1) is of variational nature, and its energy functional $\mathcal{J}: W_0^{1,2}(D) \rightarrow \mathbb{R}$ is defined by

$$(3.1) \quad \mathcal{J}_\lambda(u) := \frac{\langle u, u \rangle}{2\lambda} - \frac{1}{2\lambda} \int_D \alpha(x) |u(x)|^2 d\mu - \int_D F(x, u(x)) d\mu,$$

for every $u \in W_0^{1,2}(D)$.

Direct computations ensure that the functional \mathcal{J}_λ is continuously Gâteaux differentiable at $u \in W_0^{1,2}(D)$ and its Gâteaux derivatives at $u \in W_0^{1,2}(D)$ has the form

$$\langle \mathcal{J}'_\lambda(u), v \rangle = \frac{\langle u, v \rangle}{\lambda} - \frac{1}{\lambda} \int_D \alpha(x) u(x) v(x) d\mu - \int_D f(x, u(x)) v(x) d\mu,$$

for any $v \in W_0^{1,2}(D)$. As usual, a function $u \in W_0^{1,2}(D)$ such that $\langle \mathcal{J}'_\lambda(u), v \rangle = 0$, for every $v \in W_0^{1,2}(D)$, is said to be a critical point of the functional \mathcal{J}_λ .

Now, fixed $\lambda > 0$ in \mathbb{R} , a function $u \in W_0^{1,2}(D)$ is a weak solution of the problem (1.1) if

$$(3.2) \quad \frac{\langle u, v \rangle}{\lambda} - \frac{1}{\lambda} \int_D \alpha(x) u(x) v(x) d\mu - \int_D f(x, u(x)) v(x) d\mu = 0,$$

for any $v \in W_0^{1,2}(D)$.

Thus, the critical points of \mathcal{J}_λ are exactly the weak solutions of problem (1.1). Moreover, the following result ensures that every weak solution of problem (1.1) classically solves it, that is

$$(3.3) \quad \begin{cases} -\Delta_\mu u(x) = \alpha(x) u(x) + \lambda f(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0. \end{cases}$$

PROPOSITION 3.1. *Let $\alpha \in L^1(D)$ and $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x, \cdot)$ is continuous in \mathbb{R} for every $x \in D$. If $u \in W_0^{1,2}(D)$ is a weak solution of problem (1.1), then (3.3) holds.*

PROOF. Fixed $\lambda > 0$, let $u \in W_0^{1,2}(D)$ such that (3.2) holds for every $v \in W_0^{1,2}(D)$. Hence, fixed $y \in D$, let us define the function $v_y \in W_0^{1,2}(D)$ as follows

$$(3.4) \quad v_y(x) := \begin{cases} (-\Delta_\mu u(y) - \alpha(y) u(y) - \lambda f(y, u(y))) \delta_y^x & \text{if } x \notin \partial D, \\ 0 & \text{if } x \in \partial D, \end{cases}$$

for every $x \in D$, where as customary δ_y^x denotes the Kronecker delta. Since (3.2) holds for every $v \in W_0^{1,2}(D)$, for $v := v_y$ by (2.8) and on account of (2.11) it

follows that

$$\int_D (-\Delta_\mu u(x) - \alpha(x)u(x) - \lambda f(x, u(x))v_y(x))d\mu = 0.$$

Consequently, definition (2.1) immediately yields

$$\sum_{x \in D} (-\Delta_\mu u(x) - \alpha(x)u(x) - \lambda f(x, u(x))v_y(x))\mu(x) = 0,$$

that is

$$-\Delta_\mu u(y) - \alpha(y)u(y) - \lambda f(y, u(y)) = 0,$$

thanks to (3.4) and the range of the measure μ . The conclusion immediately follows since $y \in D$ is arbitrary. \square

In order to state Theorem 3.2 below we also recall that a C^1 -functional $J: E \rightarrow \mathbb{R}$, where E is a real Banach space with topological dual E^* , satisfies the *Palais–Smale condition* (in short (PS)-condition) when

- Every sequence $(u_k)_k$ in E such that $(J(u_k))_k$ is bounded and

$$\|J'(u_k)\|_{E^*} \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

possesses a convergent subsequence in E .

The abstract tool used along the present paper in order to prove the existence of weak solutions for (1.1) is the following theorem; see [30, Theorem 6].

THEOREM 3.2. *Let E be a reflexive real Banach space and let $\Phi, \Psi: E \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

- Φ is sequentially weakly lower semicontinuous and coercive in E ,
- Ψ is sequentially weakly continuous in E .

In addition, assume that for each $\mu > 0$ the functional $J_\mu := \mu\Phi - \Psi$ satisfies the (PS)-condition. Then, for each $\varrho > \inf_E \Phi$ and each

$$\mu > \inf_{u \in \Phi^{-1}(]-\infty, \varrho])} \frac{\sup_{v \in \Phi^{-1}(]-\infty, \varrho])} \Psi(v) - \Psi(u)}{\varrho - \Phi(u)},$$

the following alternative holds: either the functional J_μ has a strict global minimum which lies in $\Phi^{-1}(]-\infty, \varrho])$, or J_μ has at least two critical points one of which lies in $\Phi^{-1}(]-\infty, \varrho])$.

The abstract result given in Theorem 3.2 is an interplay between the celebrated three critical point theorem given by P. Pucci and J. Serrin in [28] and a quoted local minimum result obtained by B. Ricceri in [31]. Some applications of Ricceri’s variational principle are contained in [25], [32], [33]. We also cite [21] for related topics on the variational methods used in this paper. Classical notions can be found in [2].

The (PS)-condition is one of the main compactness assumption required on the energy functional when considering critical point theorem. In order to simplify its proof, in the sequel we will perform the following result, which is valid for the energy functional \mathcal{J}_λ given in (3.1).

PROPOSITION 3.3. *Let $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x, \cdot)$ is continuous for every $x \in W$ and $\alpha \in L^1(D)$. Moreover, fixed $\lambda > 0$, let \mathcal{J}_λ be the energy functional defined in (3.1). If the sequence $(u_k)_k$ is bounded in $W_0^{1,2}(D)$ and*

$$\|\mathcal{J}'_\lambda(u_k)\|_{(W_0^{1,2}(D))^*} \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

then $(u_k)_k$ has a Cauchy subsequence in $W_0^{1,2}(D)$ and so $(u_k)_k$ has a convergent subsequence.

PROOF. Let $(u_k)_k$ be a bounded sequence in $W_0^{1,2}(D)$ and let $\lambda > 0$ be fixed. By Proposition 2.1 there exist a subsequence, which we still denote by $(u_k)_k$ and a function $u_\infty \in L^\infty(D)$ such that $u_k \rightarrow u_\infty$ in $L^\infty(D)$ as $k \rightarrow +\infty$. By (2.12), since $|u(x)| \leq \|u\|/\mu_0\sqrt{\lambda_1}$ for every $x \in D$, it follows that

$$\begin{aligned} \|u_j - u_l\| &= \sup_{\|v\| \leq 1} |\langle u_j - u_l, v \rangle| \\ &= \sup_{\|v\| \leq 1} \left| \lambda \mathcal{J}'_\lambda(u_j)(v) - \lambda \mathcal{J}'_\lambda(u_l)(v) + \int_D \alpha(x)(u_j(x) - u_l(x))v(x) d\mu \right. \\ &\quad \left. - \lambda \int_D (f(x, u_j(x)) - f(x, u_l(x)))v(x) d\mu \right| \\ &\leq \lambda \|\mathcal{J}'_\lambda(u_j)\| + \lambda \|\mathcal{J}'_\lambda(u_l)\| + \frac{\max_{x \in D} |u_j(x) - u_l(x)|}{\mu_0\sqrt{\lambda_1}} \int_D |\alpha(x)| d\mu \\ &\quad + \frac{\lambda}{\mu_0\sqrt{\lambda_1}} \int_D |f(x, u_j(x)) - f(x, u_l(x))| d\mu \rightarrow 0 \end{aligned}$$

as $j, l \rightarrow +\infty$, since $\mathcal{J}'_\lambda(u_k) \rightarrow 0$ as $k \rightarrow +\infty$ and by using the classical Lebesgue Dominated Convergence Theorem. \square

Before proving Theorem 1.1 it will be useful to define another norm on $W_0^{1,2}(D)$ as follows:

$$(3.5) \quad \|u\|_\alpha := \sqrt{\langle u, u \rangle - \int_D \alpha(x)|u(x)|^2 d\mu},$$

where α is the function satisfying the assumptions stated in Theorem 1.1 and $\langle \cdot, \cdot \rangle$ is defined in (2.8). It is easy to see that $\|\cdot\|_\alpha$ is a norm on $W_0^{1,2}(D)$ equivalent to the usual one given in (2.18). More precisely, if α satisfies condition (1.3) we have that

$$(3.6) \quad \|u\|_\alpha^2 = \langle u, u \rangle - \int_D \alpha(x)|u(x)|^2 d\mu \geq \langle u, u \rangle = \|u\|^2,$$

and, by (2.12), we get

$$\begin{aligned} \|u\|_\alpha^2 &= \langle u, u \rangle - \int_D \alpha(x) |u(x)|^2 d\mu \\ &\leq \langle u, u \rangle - \frac{1}{\mu_0^2 \lambda_1} \|u\|^2 \int_D \alpha(x) d\mu = \left(1 + \frac{1}{\mu_0^2 \lambda_1} \int_D |\alpha(x)| d\mu\right) \|u\|^2. \end{aligned}$$

On the other hand, if α verifies condition (1.4) we have that

$$\begin{aligned} (3.7) \quad \|u\|_\alpha^2 &= \langle u, u \rangle - \int_D \alpha(x) |u(x)|^2 d\mu \geq \langle u, u \rangle - \int_D |\alpha(x)| |u(x)|^2 d\mu \\ &\geq \langle u, u \rangle - \frac{1}{\mu_0^2 \lambda_1} \|u\|^2 \int_D |\alpha(x)| d\mu = \left(1 - \frac{1}{\mu_0^2 \lambda_1} \int_D |\alpha(x)| d\mu\right) \|u\|^2 \end{aligned}$$

and

$$\begin{aligned} \|u\|_\alpha^2 &= \langle u, u \rangle - \int_D \alpha(x) |u(x)|^2 d\mu \leq \langle u, u \rangle - \int_D |\alpha(x)| |u(x)|^2 d\mu \\ &\leq \langle u, u \rangle + \frac{1}{\mu_0^2 \lambda_1} \|u\|^2 \int_D |\alpha(x)| d\mu \leq 2\|u\|^2, \end{aligned}$$

thanks to (1.4).

4. Existence and multiplicity results

This section is devoted to proving our main results.

PROOF OF THEOREM 1.1. The idea of the proof consists in applying Theorem 3.2 to the functional

$$\mathcal{J}_\lambda(u) = \frac{1}{2\lambda} \Phi(u) - \Psi(u), \quad \text{where } \Phi(u) := \|u\|_\alpha^2,$$

as well as

$$\Psi(u) := \int_D F(x, u(x)) d\mu, \quad \text{for any } u \in W_0^{1,2}(D).$$

Note that here we perform Theorem 3.2 by taking the parameter $\mu = 1/(2\lambda)$.

First of all, let us consider the regularity assumptions required on Φ and Ψ . It is easily shown that the functional Φ is sequentially weakly lower semicontinuous and coercive in $W_0^{1,2}(D)$.

Now, let us prove that the functional Ψ is sequentially weakly continuous in $W_0^{1,2}(D)$. For this purpose, let $(u_k)_k$ be a sequence in the Sobolev space $W_0^{1,2}(D)$ such that

$$u_k \rightarrow u_\infty \quad \text{weakly in } W_0^{1,2}(D)$$

as $k \rightarrow +\infty$, for some $u_\infty \in W_0^{1,2}(D)$. Consequently, Proposition 2.1 yields

$$u_k \rightarrow u_\infty \quad \text{in } L^\infty(D),$$

that is

$$(4.1) \quad \|u_k - u_\infty\|_\infty \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Now, by (4.1) there exists a real constant $c > 0$ such that

$$(4.2) \quad \|u_k\|_\infty \leq c \quad \text{and} \quad \|u_\infty\|_\infty \leq c \quad \text{for any } k \in \mathbb{N}.$$

Then, by (4.2) and keeping in mind that $f(x, \cdot)$ is continuous in \mathbb{R} for every $x \in W$, one has

$$(4.3) \quad \begin{aligned} |\Psi(u_k) - \Psi(u_\infty)| &= \left| \int_D F(x, u_k(x)) \, d\mu - \int_D F(x, u_\infty(x)) \, d\mu \right| \\ &\leq \int_D |F(x, u_k(x)) - F(x, u_\infty(x))| \, d\mu = \int_D \left| \int_{u_k(x)}^{u_\infty(x)} f(x, t) \, dt \right| \, d\mu \\ &\leq \int_D \left| \int_{u_k(x)}^{u_\infty(x)} |f(x, t)| \, dt \right| \, d\mu \leq \int_D \left| \int_{u_k(x)}^{u_\infty(x)} \max_{|t| \leq c} |f(x, t)| \, dt \right| \, d\mu \\ &= \int_D \max_{|t| \leq c} |f(x, t)| |u_k(x) - u_\infty(x)| \, d\mu \\ &\leq \max_{x \in D, |t| \leq c} |f(x, t)| \|u_k - u_\infty\|_\infty \mu(D). \end{aligned}$$

Hence, by (4.1) and (4.3), we obtain that

$$|\Psi(u_k) - \Psi(u_\infty)| \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

so that the functional Ψ is sequentially weakly continuous in $W_0^{1,2}(D)$ as claimed. Now, we observe that

$$(4.4) \quad \text{the functional } \mathcal{J}_\lambda \text{ is unbounded from below in } W_0^{1,2}(D).$$

Indeed, assumption (1.6) implies that there exist two positive constants b_1 and b_2 such that

$$(4.5) \quad F(x, t) \geq b_1 |t|^\beta - b_2 \quad \text{for any } x \in D \text{ and } t \in \mathbb{R}.$$

To prove (4.5) let $r_0 > 0$ be as in (2.9): then, for every $x \in D$ and for any $t \in \mathbb{R}$ with $|t| \geq r_0 > 0$

$$\frac{t f(x, t)}{F(x, t)} \geq \beta.$$

Suppose $t > r_0$. Dividing by t and integrating both terms in $[r_0, t]$ we obtain

$$F(x, t) \geq \frac{F(x, r_0)}{r_0^\beta} |t|^\beta.$$

Using the same arguments it is easy to prove that if $t < -r_0$ then it results

$$F(x, t) \geq \frac{F(x, -r_0)}{r_0^\beta} |t|^\beta,$$

so that for any $t \in \mathbb{R}$ with $|t| \geq r_0$ we get

$$(4.6) \quad F(x, t) \geq m(x) |t|^\beta,$$

where

$$m(x) := \frac{\min\{F(x, r_0), F(x, -r_0)\}}{r_0^\beta}.$$

Since the function $t \mapsto F(\cdot, t)$ is continuous in \mathbb{R} , by the Weierstrass Theorem, it is bounded for any $t \in \mathbb{R}$ such that $|t| \leq r_0$, say

$$(4.7) \quad |F(x, t)| \leq \widetilde{M}(x) \quad \text{in } \{|t| \leq r_0\},$$

where $\widetilde{M}(x) := \max\{|F(x, t)| : |t| \leq r_0\}$. Formula (4.5) follows from (4.6) and (4.7) by taking

$$b_1 := \min_{x \in D} m(x) \quad \text{and} \quad b_2 := \max_{x \in D} (\widetilde{M}(x) + m(x)r_0^\beta).$$

Thus, by (4.5), for any $u \in W_0^{1,2}(D)$ one has

$$(4.8) \quad \int_D F(x, u(x)) \, d\mu \geq b_1 \int_D |u(x)|^\beta \, d\mu - b_2 \mu(D).$$

Let $u_0 \in W_0^{1,2}(D)$ with $\int_D |u_0(x)|^\beta \, d\mu > 0$. Then, by (4.8), we have that

$$\begin{aligned} \mathcal{J}_\lambda(tu_0) &= \frac{t^2}{2\lambda} \|u_0\|_\alpha^2 - \int_D F(x, tu_0(x)) \, d\mu \\ &\leq \frac{t^2}{2\lambda} \|u_0\|_\alpha^2 - b_1 |t|^\beta \int_D |u_0(x)|^\beta \, d\mu + b_2 \mu(D) \rightarrow -\infty, \end{aligned}$$

as $t \rightarrow +\infty$, since $\beta > 2$ by assumption (1.6) and $\int_D |u_0(x)|^\beta \, d\mu > 0$. The proof of (4.4) is completed.

Now, it remains to see that the functional \mathcal{J}_λ verifies the (PS)-condition. To this goal, it is sufficient to employ Proposition 3.3. More precisely, suppose that there exists a real constant $c > 0$ such that $|\mathcal{J}_\lambda(u_k)| \leq c$ for every $k \in \mathbb{N}$ and $\mathcal{J}'_\lambda(u_k) \rightarrow 0$ as $k \rightarrow +\infty$. Now, by (3.1), it follows that

$$\begin{aligned} (4.9) \quad c + \beta^{-1} \|u_k\| &\geq \lambda \mathcal{J}_\lambda(u_k) - \lambda \beta^{-1} \mathcal{J}'_\lambda(u_k)(u_k) \\ &= \left(\frac{1}{2} - \frac{1}{\beta}\right) \|u_k\|^2 + \lambda \beta^{-1} \int_D (f(x, u_k(x)) u_k(x) - \beta F(x, u_k(x))) \, d\mu \\ &= \left(\frac{1}{2} - \frac{1}{\beta}\right) \|u_k\|^2 \\ &\quad + \lambda \beta^{-1} \int_{\{x \in D: |u_k(x)| < r_0\}} (f(x, u_k(x)) u_k(x) - \beta F(x, u_k(x))) \, d\mu \\ &\quad + \lambda \beta^{-1} \int_{\{x \in D: |u_k(x)| \geq r_0\}} (f(x, u_k(x)) u_k(x) - \beta F(x, u_k(x))) \, d\mu, \end{aligned}$$

for every $k \in \mathbb{N}$ sufficiently large.

By (1.6), the third term in (4.9) is non-negative while the second term is bounded by a positive constant independent of k . Since $\beta > 2$, (4.9) implies that $(u_k)_k$ is bounded in $W_0^{1,2}(D)$. Therefore, by Proposition 3.3, the energy functional \mathcal{J}_λ satisfies the (PS) compactness condition.

Moreover, let $\varrho > 0$ and

$$\chi(\varrho) := \inf_{u \in \mathbb{B}_\varrho} \frac{\sup_{v \in \mathbb{B}_\varrho} \Psi(v) - \Psi(u)}{\varrho - \|u\|_\alpha^2},$$

where $\mathbb{B}_\varrho = \{v \in W_0^{1,2}(D) : \|v\|_\alpha < \sqrt{\varrho}\}$.

The definition of χ yields that for every $u \in \mathbb{B}_\varrho$

$$\chi(\varrho) \leq \frac{\sup_{v \in \mathbb{B}_\varrho} \Psi(v) - \Psi(u)}{\varrho - \|u\|_\alpha^2}$$

thus, being $0 \in \mathbb{B}_\varrho$, we obtain that

$$\begin{aligned} (4.10) \quad \chi(\varrho) &\leq \frac{1}{\varrho} \sup_{v \in \mathbb{B}_\varrho} \Psi(v) \leq \frac{1}{\varrho} \sup_{v \in \mathbb{B}_\varrho} \left| \int_D F(x, v(x)) \, d\mu \right| \\ &\leq \frac{1}{\varrho} \sup_{v \in \mathbb{B}_\varrho} \int_D |F(x, v(x))| \, d\mu. \end{aligned}$$

Now, assume that the function α satisfies assumption (1.3). Then, if $v \in \overline{\mathbb{B}}_\varrho$, by (2.12) and (3.6) we get that

$$(4.11) \quad |v(x)| \leq \frac{1}{\mu_0 \sqrt{\lambda_1}} \|v\| \leq \frac{1}{\mu_0 \sqrt{\lambda_1}} \|v\|_\alpha \leq \frac{\sqrt{\varrho}}{\mu_0 \sqrt{\lambda_1}} \quad \text{for any } x \in D,$$

which, together with the continuity of F and the finiteness of D , gives for any $x \in D$

$$(4.12) \quad |F(x, v(x))| \leq \max_{x \in D, |s| \leq \sqrt{\varrho}/(\mu_0 \sqrt{\lambda_1})} \left| \int_0^s f(x, t) \, dt \right|.$$

Therefore, inequality (4.12) yields

$$(4.13) \quad \int_D |F(x, v(x))| \, d\mu \leq \mu(D) \max_{x \in D, |s| \leq \sqrt{\varrho}/(\mu_0 \sqrt{\lambda_1})} \left| \int_0^s f(x, t) \, dt \right|$$

for any $v \in \overline{\mathbb{B}}_\varrho$. By (4.10) and (4.13) we have that

$$\chi(\varrho) \leq \frac{1}{\varrho} \max_{x \in D, |s| \leq \sqrt{\varrho}/(\mu_0 \sqrt{\lambda_1})} \left| \int_0^s f(x, t) \, dt \right| < \frac{1}{2\lambda},$$

provided λ satisfies condition (5.7).

On the other hand, if the function α satisfies assumption (1.4), we can argue in the same way, just replacing (4.11) with the following inequality

$$\begin{aligned} (4.14) \quad |v(x)| &\leq \frac{1}{\mu_0 \sqrt{\lambda_1}} \|v\| \leq \frac{1}{\mu_0 \sqrt{\lambda_1} \sqrt{1 - \frac{1}{\mu_0^2 \lambda_1} \int_D |\alpha(x)| \, d\mu}} \|v\|_\alpha \\ &\leq \frac{1}{\mu_0 \sqrt{\lambda_1} \sqrt{1 - \frac{1}{\mu_0^2 \lambda_1} \int_D |\alpha(x)| \, d\mu}} \sqrt{\varrho} \end{aligned}$$

for any $x \in D$, and thanks to (3.7).

In both cases, owing to Theorem 3.2 and considering (1.5) and (4.4), we conclude that problem (1.1) admits at least two non-trivial weak solutions one of which lies in \mathbb{B}_ϱ . The proof of Theorem 1.1 is finally complete on account of Proposition 3.1. \square

REMARK 4.1. First of all, we notice that condition (2.12) plays a crucial role in the proof of Theorem 1.1, whereas the Sobolev embedding theorems are employed in the classical case of bounded domains; see, among others, the papers [1], [29], [34]. The proof of Theorem 1.1 relies on the same arguments as used in [26, Theorem 1].

REMARK 4.2. Note that the trivial function is a weak solution of problem (1.1) if and only if $f(\cdot, 0) = 0$. Hence, condition (1.5) assures that all the solutions of problem (1.1), if any, are non-trivial. In the case when $f(\cdot, 0) = 0$, in order to get the existence of multiple solutions for (1.1) some extra assumptions on the nonlinear term f are necessary.

PROOF OF THEOREM 1.3. The conclusion is achieved by using Theorem 1.1. Indeed, as it is easily seen, condition (1.12) yields

$$(4.15) \quad \max_{(x,s) \in D \times [-M_0, M_0]} \left| \int_0^s f(x, t) dt \right| \leq \mu_0^2 \frac{M_0^2}{(\sigma + 1)} \frac{\lambda_1}{2}.$$

Thus, if we set

$$\theta := \begin{cases} \frac{\mu_0^2 M_0^2 \lambda_1}{2 \max_{(x,s) \in D \times [-M_0, M_0]} \left| \int_0^s f(x, t) dt \right|} & \text{if } \max_{(x,s) \in D \times [-M_0, M_0]} \left| \int_0^s f(x, t) dt \right| > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

by the fact that $\sigma > 0$, one has

$$(4.16) \quad 1 < \sigma + 1 \leq \theta \leq \lambda^*,$$

since (4.15) holds and

$$\lambda^* := \sup_{\varrho > 0} \frac{\varrho}{\max_{x \in D, |s| \leq \kappa \sqrt{\varrho}} \left| \int_0^s f(x, t) dt \right|} = \frac{1}{\kappa^2} \sup_{z > 0} \frac{z^2}{\max_{x \in D, |s| \leq z} \left| \int_0^s f(x, t) dt \right|},$$

on account of the definition given in (1.10). Then, thanks to (4.16), Theorem 1.1 ensures that for $\lambda = 1$ problem (1.13) admits at least two non-trivial weak solutions one of which lies in the open ball $\mathbb{B}_{\mu_0^2 M_0^2 \lambda_1}$. Finally, Proposition 3.1 ensures that every weak solution of problem (1.13) is also classical and this concludes the proof of Theorem 1.3. \square

REMARK 4.3. We notice that a condition similar to (1.12) has been used previously in the literature by K.J. Falconer and J. Hu by studying elliptic problems on the gasket; see [7, Theorem 3.5].

PROOF OF THEOREM 1.4. In order to prove the existence of a non-negative solution of problem (1.13) it is enough to introduce the functions

$$F_+(x, t) := \int_0^t f_+(x, \tau) d\tau,$$

with

$$f_+(x, t) = \begin{cases} f(x, t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

for every $x \in D$. Note that f_+ satisfy condition (1.12), while assumption (1.6) is verified by f_+ and F_+ for every $x \in D$ and for any $t > r_0$. Thus, let $\mathcal{J}_+ : W_0^{1,2}(D) \rightarrow \mathbb{R}$ be the functional defined as follows

$$\begin{aligned} \mathcal{J}_+(u) &:= \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_D \alpha(x) |u(x)|^2 d\mu - \int_D F_+(x, u(x)) d\mu \\ &= \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_D \alpha(x) |u(x)|^2 d\mu - \int_D F(x, u^+(x)) d\mu, \end{aligned}$$

where $u^+(x) := \max\{0, u(x)\}$ for every $x \in D$.

It is easy to see that the energy functional \mathcal{J}_+ is well defined and Fréchet differentiable in $u \in W_0^{1,2}(D)$. More precisely,

$$(4.17) \quad \langle \mathcal{J}'_+(u), v \rangle = \langle u, v \rangle - \int_D \alpha(x) u(x) v(x) d\mu - \int_D f_+(x, u(x)) v(x) d\mu,$$

for any $v \in W_0^{1,2}(D)$.

Now, by (1.14), it follows that

(4.18) there are positive constants M_0 and σ such that

$$\max_{(x,s) \in \overset{\circ}{D} \times [-M_0, M_0]} f_+(x, s) \leq \mu_0^2 \frac{M_0}{(\sigma + 1)} \frac{\lambda_1}{2}.$$

Since (1.15) holds, as in the proof of Theorem 1.3 there exists $u_\infty \in W_0^{1,2}(D) \setminus \{0\}$ that classically solves the following Dirichlet problem

$$(4.19) \quad \begin{cases} -\Delta_\mu u(x) = \alpha(x) u(x) + f_+(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ u|_{\partial D} = 0. \end{cases}$$

We claim that u_∞ is non-negative in D . Indeed, as usual, let $u_\infty^- := \max\{-u_\infty, 0\}$ be the negative part of u_∞ . Since

$$\Gamma(u_\infty^-, u_\infty) = \Gamma(u_\infty^-) + \Gamma(u_\infty^-, u_\infty^+) \geq |\nabla u_\infty^-|^2,$$

and $\alpha \in L^1(D)$ satisfy (1.3), on account of (2.11) it follows that

$$\begin{aligned} \int_D |\nabla u_\infty^-|^2(x) d\mu &\leq \int_D \Gamma(u_\infty^-)(x) d\mu + \int_D \Gamma(u_\infty^-, u_\infty^+)(x) d\mu \\ &= \int_D \Gamma(u_\infty^-, u_\infty)(x) d\mu = - \int_D u_\infty^-(x) \Delta_\mu u_\infty(x) d\mu \\ &= \int_\Omega u_\infty^-(x) f_+(x, u_\infty^+(x)) dx = 0. \end{aligned}$$

This implies that $u_\infty^- \equiv 0$ and thus $u_\infty \geq 0$ in D . Since $f^+ = f$ in $\mathbb{R}_0^+ := [0, +\infty[$ the function $u_\infty \in W_0^{1,2}(D) \setminus \{0\}$ classically solves (1.13) as claimed. \square

REMARK 4.4. We notice that the functional \mathcal{J}_+ defined along the proof of Theorem 1.4 satisfies the compactness (PS) condition, so that classical variational arguments can be applied by studying problem (4.19). To this goal let us observe that, arguing as in the proof of Theorem 1.1, assumption (1.15) implies that there exist two positive constants b_1 and b_2 such that

$$(4.20) \quad F(x, t) \geq b_1 t^\beta - b_2 \quad \text{for any } t \geq r_0 \text{ and every } x \in D.$$

Let $c \in \mathbb{R}$ and let $(u_k)_k$ be a sequence in $W_0^{1,2}(D)$ such that

$$(4.21) \quad \mathcal{J}_+(u_k) \rightarrow c,$$

and

$$(4.22) \quad \sup \left\{ \left| \langle \mathcal{J}'_+(u_k), v \rangle \right| : v \in W_0^{1,2}(D), \|v\| = 1 \right\} \rightarrow 0,$$

as $k \rightarrow +\infty$. Without loss of generality, we only consider the case of a (definitely) non-trivial sequence. For any $k \in \mathbb{N}$, by (4.22) and (4.21), it easily follows that there exists $\kappa > 0$ such that

$$(4.23) \quad \left| \left\langle \mathcal{J}'_+(u_k), \frac{u_k}{\|u_k\|} \right\rangle \right| \leq \kappa,$$

and

$$(4.24) \quad |\mathcal{J}_+(u_k)| \leq \kappa,$$

for every $k \in \mathbb{N}$. Moreover, by (1.11), it follows that

$$\begin{aligned} (4.25) \quad &\left| \int_{\{x \in D: 0 \leq u_k^+(x) \leq r_0\}} \left(F(x, u_k^+(x)) - \frac{1}{\beta} f(x, u_k^+(x)) u_k^+(x) \right) dx \right| \\ &\leq \sum_{x \in \{x \in D: 0 \leq u_k^+(x) \leq r_0\}} \mu(x) \left(F(x, u_k^+(x)) - \frac{1}{\beta} f(x, u_k^+(x)) u_k^+(x) \right) \\ &\leq \sum_{x \in \{x \in D: 0 \leq u_k^+(x) \leq r_0\}} \mu(x) \max_{(x,t) \in D \times [0, r_0]} \left(F(x, t) - \frac{1}{\beta} f(x, t) t \right) \\ &\leq \mu(D) \max_{(x,t) \in D \times [0, r_0]} \left(F(x, t) - \frac{1}{\beta} f(x, t) t \right) < +\infty. \end{aligned}$$

Also, thanks to (2.9) and (4.25), we get

$$\begin{aligned}
 (4.26) \quad & \mathcal{J}_+(u_k) - \frac{1}{\beta} \langle \mathcal{J}'_+(u_k), u_k \rangle \\
 &= \left(\frac{1}{2} - \frac{1}{\beta}\right) \|u_k\|^2 - \frac{1}{\beta} \int_D (\beta F_+(x, u_k(x)) - f_+(x, u_k(x)) u_k(x)) dx \\
 &\geq \left(\frac{1}{2} - \frac{1}{\beta}\right) \|u_k\|^2 \\
 &\quad - \int_{\{x \in D: 0 \leq u_k(x) \leq r_0\}} \left(F(x, u_k^+(x)) - \frac{1}{\beta} f(x, u_k^+(x)) u_k^+(x)\right) dx \\
 &\geq \left(\frac{1}{2} - \frac{1}{\beta}\right) \|u_k\|^2 - \mu(D) \max_{(x,t) \in D \times [0, r_0]} \left(F(x, t) - \frac{1}{\beta} f(x, t) t\right).
 \end{aligned}$$

As a consequence of (4.23) and (4.24) we also have

$$\mathcal{J}_+(u_k) - \frac{1}{\beta} \langle \mathcal{J}'_+(u_k), u_k \rangle \leq \kappa (1 + \|u_k\|)$$

so that, by (4.26) for any $k \in \mathbb{N}$,

$$\|u_k\|^2 \leq \kappa_* (1 + \|u_k\|)$$

for a suitable constant $\kappa_* > 0$. The above inequality immediately yields that the sequence $(u_k)_k$ is bounded in $W_0^{1,2}(D)$. Therefore, by Proposition 3.3, the energy functional \mathcal{J}_+ satisfies the (PS) condition as claimed.

PROOF OF THEOREM 1.5. Since $p > 2$, one clearly has

$$\limsup_{t \rightarrow 0^+} \frac{|t|^{p-2}t}{t} = 0.$$

Moreover, it easily seen that condition (1.15) is also verified. Consequently, arguing as in the proof of Theorem 1.4, there exists a function $u_\infty \in W_0^{1,2}(D) \setminus \{0\}$ such that

$$(4.27) \quad -\Delta_\mu u_\infty(x) - \gamma u_\infty(x) = (u_\infty^+(x))^{p-1} \quad \text{in } \overset{\circ}{D}.$$

Let us prove that $u_\infty \geq 0$ in D . To this aim, let $u_\infty^-(x) := \min\{u_\infty(x), 0\}$ for every $x \in D$. Since $u_\infty^-(x)u_\infty^+(x) = u_\infty^-(x)(u_\infty^+(x))^{p-1} = 0$ for every $x \in D$, we have

$$- \int_D u_\infty^-(x) \Delta_\mu u_\infty(x) d\mu - \gamma \int_D (u_\infty^-(x))^2 d\mu = 0.$$

Since $u_\infty = u_\infty^+ + u_\infty^-$, the above equation leads to

$$\begin{aligned}
 (4.28) \quad & \frac{\gamma}{\lambda_1} \int_D |\nabla u_\infty^-|^2(x) d\mu \geq \gamma \int_D (u_\infty^-(x))^2 d\mu \\
 &= \int_D |\nabla u_\infty^-|^2(x) d\mu - \int_D u_\infty^-(x) \Delta_\mu u_\infty^+(x) d\mu.
 \end{aligned}$$

Note that

$$\begin{aligned}
 (4.29) \quad - \int_D u_\infty^-(x) \Delta_\mu u_\infty^+(x) d\mu &= - \sum_{x \in \overset{\circ}{D}} u_\infty^-(x) \sum_{y \sim x} w(x, y) (u_\infty^+(y) - u_\infty^+(x)) \\
 &= - \sum_{x \in \overset{\circ}{D}} \sum_{y \sim x} w(x, y) u_\infty^-(x) u_\infty^+(y) \geq 0.
 \end{aligned}$$

Inserting (4.29) into (4.28) and recalling that $\gamma < \lambda_1$, we obtain

$$\int_D |\nabla u_\infty^-|^2(x) d\mu = 0,$$

which implies that $u_\infty^- \equiv 0$ in D . Whence $u_\infty \geq 0$ in D and (4.27) becomes

$$(4.30) \quad \begin{cases} -\Delta_\mu u_\infty(x) - \gamma u_\infty(x) = u_\infty(x)^{p-1} & \text{for } x \in \overset{\circ}{D}, \\ u_\infty(x) \geq 0 & \text{for } x \in \overset{\circ}{D}, \\ u_\infty|_{\partial D} = 0. \end{cases}$$

Finally, suppose that $u_\infty(x_0) = 0 = \min_{x \in \overset{\circ}{D}} u_\infty(x)$ for some $x_0 \in \overset{\circ}{D}$. If $y \in D$ is adjacent to x_0 then $\Delta_\mu u_\infty(x_0) = 0$ by (4.30). The definition of the μ -Laplacian immediately yields $u_\infty(y) = 0$. Therefore we conclude that $u_\infty \equiv 0$ in D , which is absurd. Thus $u_\infty > 0$ in $\overset{\circ}{D}$ as claimed. \square

REMARK 4.5. We notice that condition (1.18) is a special case of (1.12). Consequently, it is easily seen that Theorems 1.3 and 1.4 immediately yield the conclusions of Corollary 1.6 given in Introduction.

5. Yamabe-type problems on graphs

Let $m \in \mathbb{N}$ and $p \in \mathbb{R}$ with $p > 1$. Denote by $W_0^{m,p}(D)$ be the completion of the space

$$C_0^m(D) := \{u : D \rightarrow \mathbb{R} : |\nabla^j u| = 0 \text{ on } \partial D, \text{ for every } j = 0, \dots, m - 1\}$$

with respect to the Sobolev norm

$$(5.1) \quad \|u\| := \sum_{k=0}^m \|\nabla^k u\|_{L^p(D)},$$

where, for every $x \in D$,

$$|\nabla^k u|(x) := \begin{cases} |\nabla \Delta^{(m-1)/2} u|(x) & \text{if } k \in 2\mathbb{N} + 1, \\ |\Delta^{m/2} u|(x) & \text{if } k \in 2\mathbb{N}, \end{cases}$$

with $k = 0, \dots, m$, and

$$\Delta_\mu^\ell u(x) := \begin{cases} u(x) & \text{if } \ell = 0, \\ \frac{1}{\mu(x)} \sum_{y \sim x} w(x, y) (\Delta_\mu^{\ell-1} u(y) - \Delta_\mu^{\ell-1} u(x)) & \text{if } \ell > 0, \end{cases}$$

for every $x \in D$. By [17, Theorem 27] the space $(W_0^{m,p}(D), \|u\|_{W_0^{m,p}(D)})$ is a finite dimensional Banach space, where

$$\|u\|_{W_0^{m,p}(D)} := \|\nabla^m u\|_{L^p(D)}$$

is a norm on $W_0^{m,p}(D)$ equivalent to $\|\cdot\|$ given in (5.1).

With the above notations, a general version of Theorem 1.1 proved in Section 3 can be achieved for the following Yamabe-type problem

$$(5.2) \quad \begin{cases} \mathcal{L}_{m,p}u(x) = \lambda f(x, u(x)) & \text{for } x \in \overset{\circ}{D}, \\ |\nabla^j u| = 0 & \text{on } \partial D, \text{ with } j = 0, \dots, m-1, \end{cases}$$

where λ is a real positive parameter, $f : D \times \mathbb{R} \rightarrow \mathbb{R}$ is a suitable continuous nonlinear term and $\mathcal{L}_{m,p} : W_0^{m,p}(D) \rightarrow L^p(D)$ denotes the (m, p) -Laplacian operator defined in the distributional sense for every $u \in W_0^{m,p}(D)$ as

$$(5.3) \quad \mathcal{L}_{m,p}u := \begin{cases} \int_D |\nabla^m u|^{p-2}(x) \Gamma(\Delta^{(m-1)/2}u, \Delta^{(m-1)/2}v)(x) d\mu & \text{if } m \in 2\mathbb{N} + 1, \\ \int_D |\nabla^m u|^{p-2}(x) \Delta^{m/2}u(x) \cdot \Delta^{m/2}v(x) d\mu & \text{if } m \in 2\mathbb{N}, \end{cases}$$

for any $v \in W_0^{m,p}(D)$.

The (m, p) -Laplacian $\mathcal{L}_{m,p}$ can be explicitly computed at any point of $x \in D$. In particular, $\mathcal{L}_{m,2}$ is the poly-Laplacian operator $(-\Delta_\mu)^m$ that, for $p = 2$, reduces to the μ -Laplacian defined in (2.3). We emphasize that p -Laplacian equations on graphs also in connection with geometrical analysis problems have been considered in the literature; see, among others the papers [8]–[11] as well as [16], [36]–[38]. Moreover, existence and convergence of solutions for nonlinear biharmonic equations on graphs is proved in [19] as well as the 1-Yamabe equation on the graph-theoretical setting has been investigated in [13].

Thanks to (5.3) the solutions of (5.2) are the critical points of the energy functional $\mathcal{J} : W_0^{m,p}(D) \rightarrow \mathbb{R}$ defined by

$$(5.4) \quad \mathcal{J}_\lambda(u) := \frac{1}{p} \|u\|_{W_0^{m,p}(D)}^p - \lambda \int_D F(x, u(x)) d\mu,$$

for every $u \in W_0^{m,p}(D)$.

Similarly to (1.2), let us define the real constant

$$(5.5) \quad \lambda_{m,p} := \inf_{u \in C_0^m(D) \setminus \{0\}} \frac{\int_D |\nabla^m u|^p(x) d\mu}{\int_D |u(x)|^p d\mu}.$$

With the above notations the next result holds.

THEOREM 5.1. Let $\mathcal{G} := (V, E)$ be a weighted locally finite graph, D be a bounded domain of V such that $\overset{\circ}{D} \neq \emptyset$ and $\partial D \neq \emptyset$ and let $\mu: D \rightarrow]0, +\infty[$ be a measure on D . Let $\mu_0 := \min_{x \in D} \mu(x) > 0$ and let $f: D \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that (1.5) holds as well as

$$(5.6) \quad \text{there are } \beta > p \text{ and } r_0 > 0 \text{ such that} \\ tf(x, t) \geq \beta F(x, t) > 0 \quad \text{for any } |t| \geq r_0 \text{ and every } x \in D,$$

where F is the potential given by (1.7). Then, for any $\varrho > 0$ and any

$$(5.7) \quad 0 < \lambda < \frac{\varrho}{2 \max_{x \in D, |s| \leq \kappa_{m,p} \varrho^{1/p}} \left| \int_0^s f(x, t) dt \right|},$$

where

$$(5.8) \quad \kappa_{m,p} := \frac{1}{\mu_0 \lambda_{m,p}^{1/p}},$$

the problem (1.1) admits at least two non-trivial solutions one of which lies in

$$\mathbb{B}_\varrho^{(m,p)} := \{u \in W_0^{m,p}(D) : \|u\|_{W_0^{m,p}(D)} < \varrho^{1/p}\}.$$

Yamabe equations on infinite graphs have been studied in several papers also in connection with the uniqueness result for a Kazdan–Warner type problem on bounded domains; see, among others, the papers [12], [13], [17] and very recently in [27, Corollary 5.3].

We emphasize that the existence result given in Theorem 5.1 is in the same spirit of [27, Theorem 1.1]; see also Theorem 1.5.

Acknowledgements. The manuscript was realized under the auspices of the Italian MIUR project *Variational methods, with applications to problems in mathematical physics and geometry* (2015KB9WPT 009) and the Slovenian Research Agency grants (P1-0292, N1-0114, N1-0083).

REFERENCES

- [1] A. AMBROSETTI AND P.H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
- [2] H. BRÉZIS, *Analyse Fonctionnelle. Théorie et Applications*, Masson, Paris, 1983.
- [3] M. BARLOW, T. COULHON AND A. GRIGOR'YAN, *Manifolds and graphs with slow heat kernel decay*, Invent. Math. **144** (2001), no. 3, 609–649.
- [4] F. CHUNG, A. GRIGOR'YAN AND S.-T. YAU, *Eigenvalues and diameters for manifolds and graphs*, Tsing Hua Lectures on Geometry and Analysis (Hsinchu, 1990), Int. Press, Cambridge, MA, 1997, pp. 79–105.
- [5] F. CHUNG, A. GRIGOR'YAN AND S.-T. YAU, *Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs*, Comm. Anal. Geom. **8** (2000), no. 5, 969–1026.
- [6] T. COULHON AND A. GRIGOR'YAN, *Random walks on graphs with regular volume growth*, Geom. Funct. Anal. **8** (1998), no. 4, 656–701.
- [7] K.J. FALCONER AND J. HU, *Nonlinear elliptic equations on the Sierpiński gasket*, J. Math. Anal. Appl. **240** (1999), 552–573.

- [8] H. GE, *A p -th Yamabe equation on graph*, Proc. Amer. Math. Soc. **146** (2018), no. 5, 2219–2224.
- [9] H. GE, *The p -th Kazdan–Warner equation on graphs*, Commun. Contemp. Math. **22** (2020), no. 6, 1950052, 17.
- [10] H. GE, B. HUA AND W. JIANG, *A note on Liouville type equations on graphs*, Proc. Amer. Math. Soc. **146** (2018), no. 11, 4837–4842.
- [11] H. GE AND W. JIANG, *Kazdan–Warner equation on infinite graphs*, J. Korean Math. Soc. **55** (2018), no. 5, 1091–1101.
- [12] H. GE AND W. JIANG, *Yamabe equations on infinite graphs*, J. Math. Anal. Appl. **460** (2018), no. 2, 885–890.
- [13] H. GE AND W. JIANG, *The 1-Yamabe equation on graphs*, Commun. Contemp. Math. **21** (2019), no. 8, 1850040, 10.
- [14] A. GRIGOR'YAN, *Heat kernels on manifolds, graphs and fractals*, European Congress of Mathematics, vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 393–406.
- [15] A. GRIGOR'YAN, *Introduction to analysis on graphs*, University Lecture Series, vol. 71, American Mathematical Society, Providence, RI, 2018.
- [16] A. GRIGOR'YAN, Y. LIN AND Y. YANG, *Kazdan–Warner equation on graph*, Calc. Var. Partial Differential Equations **55** (2016), no. 4, Art. 92, 13.
- [17] A. GRIGOR'YAN, Y. LIN, AND Y. YANG, *Yamabe type equations on graphs*, J. Differential Equations **261** (2016), no. 9, 4924–4943.
- [18] A. GRIGOR'YAN, Y. LIN AND Y. YANG, *Existence of positive solutions to some nonlinear equations on locally finite graphs*, Sci. China Math. **60** (2017), no. 7, 1311–1324.
- [19] X. HAN, M. SHAO AND L. ZHAO, *Existence and convergence of solutions for nonlinear biharmonic equations on graphs*, J. Differential Equations **268** (2020), no. 7, 3936–3961.
- [20] S. HAESELER AND M. KELLER, *Generalized solutions and spectrum for Dirichlet forms on graphs*. In [21], 181–200.
- [21] A. KRISTÁLY, V. RĂDULESCU AND CS. VARGA, *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, No. **136**, Cambridge University Press, Cambridge, 2010.
- [22] D. LENZ, F. SOBIECZKY AND W. WOESS (eds.), *Random walks, boundaries and spectra*, Proceedings of the Workshop on Boundaries, Graz, Austria, June 29–July 3, 2009 and the Alp-Workshop, Sankt Kathrein, Austria, July 4–5, 2009, Progress in Probability, vol. 64, Birkhäuser, Basel, 2011.
- [23] Y. LIN AND Y. WU, *The existence and nonexistence of global solutions for a semilinear heat equation on graphs*, Calc. Var. Partial Differential Equations **56** (2017), no. 4, paper no. 102, 22.
- [24] S. LIU AND Y. YANG, *Multiple solutions of Kazdan–Warner equation on graphs in the negative case*, Calc. Var. Partial Differential Equations **59** (2020), no. 5, paper no. 164, 15.
- [25] G. MOLICA BISCI AND D. REPOVŠ, *Yamabe-type equations on Carnot groups*, Potential Anal. **46** (2017), no. 2, 369–383.
- [26] G. MOLICA BISCI, D. REPOVŠ AND R. SERVADEI, *Nonlinear problems on the Sierpinski gasket*, J. Math. Anal. Appl. **452** (2017), 883–895.
- [27] A. PINAMONTI AND G. STEFANI, *Existence and uniqueness theorems for some semi-linear equations on locally finite graphs*, Proc. Amer. Math. Soc. **150** (2022), 4757–4770.
- [28] P. PUCCI AND J. SERRIN, *A mountain pass theorem*, J. Differential Equations **60** (1985), 142–149.
- [29] P.H. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math., vol. 65, American Mathematical Society, Providence, RI, 1986.

- [30] B. RICCERI, *On a classical existence theorem for nonlinear elliptic equations*, *Experimental, Constructive and Nonlinear Analysis* (M. Théra, ed.), CMS Conf. Proc., vol. 27, Canad. Math. Soc., 2000, pp. 275–278.
- [31] B. RICCERI, *A general variational principle and some of its applications*, *J. Comput. Appl. Math.*, special issue on Fixed Point Theory with Applications in Nonlinear Analysis **113** (2000), 401–410.
- [32] B. RICCERI, *Nonlinear eigenvalue problems*, *Handbook of Nonconvex Analysis and Applications* (D.Y. Gao and D. Motreanu, eds.), International Press, 2010, pp. 543–595.
- [33] B. RICCERI, *A new existence and localization theorem for Dirichlet problem*, *Dynam. Systems Appl.* **22** (2013), 317–324.
- [34] M. STRUWE, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 3, Springer-Verlag, Berlin, Heidelberg, 1990.
- [35] D. ZHANG, *Semi-linear elliptic equations on graphs*, *J. Partial Differential Equations* **30** (2017), no. 3, 221–231.
- [36] X. ZHANG AND Y. CHANG, *p -th Kazdan–Warner equation on graph in the negative case*, *J. Math. Anal. Appl.* **466** (2018), no. 1, 400–407.
- [37] X. ZHANG AND A. LIN, *Positive solutions of p -th Yamabe type equations on graphs*, *Front. Math. China* **13** (2018), no. 6, 1501–1514.
- [38] X. ZHANG AND A. LIN, *Positive solutions of p -th Yamabe type equations on infinite graphs*, *Proc. Amer. Math. Soc.* **147** (2019), no. 4, 1421–1427.

Manuscript received March 19, 2022

accepted July 18, 2022

MAURIZIO IMBESI

 <https://orcid.org/0000-0002-1128-5203>

Dipartimento di Scienze Matematiche e Informatiche
Scienze Fisiche e Scienze della Terra
Università degli Studi di Messina
Viale F. Stagno d'Alcontres 31
Messina I 98166, ITALY

E-mail address: maurizio.imbesi@unime.it

GIOVANNI MOLICA BISCI

 <https://orcid.org/0000-0003-4802-1727>

Dipartimento di Scienze Pure e Applicate
Università degli Studi di Urbino 'Carlo Bo'
Piazza della Repubblica 13
Urbino I 61029, ITALY

E-mail address: giovanni.molicabisci@uniurb.it

DUŠAN D. REPOVŠ

 <https://orcid.org/0000-0002-6643-1271>

Faculty of Education
and Faculty of Mathematics and Physics
University of Ljubljana
and Institute of Mathematics, Physics and Mechanics
Ljubljana SI 1000, SLOVENIA

E-mail address: dusan.repovs@guest.arnes.si