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EXISTENCE AND MULTIPLICITY OF SOLUTIONS INVOLVING THE $p(x)$-LAPLACIAN EQUATIONS: ON THE EFFECT OF TWO NONLOCAL TERMS

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Abstract. We study a class of $p(x)$-Kirchhoff problems which is seldom studied because the nonlinearity has nonstandard growth and contains a bi-nonlocal term. Based on variational methods, especially the Mountain pass theorem and Ekeland's variational principle, we obtain the existence of two nontrivial solutions for the problem under certain assumptions. We also apply the Symmetric mountain pass theorem and Clarke's theorem to establish the existence of infinitely many solutions. Our results generalize and extend several existing results.

1. **Introduction.** The purpose of the present paper is to study the existence and multiplicity of solutions for the following $p(x)$-Kirchhoff equation, with an additional nonlocal term:

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u + f(x,u)\left[\int_\Omega F(x,u)dx\right]^r & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$, $N > p(x) > 1$, $r > 0$ and λ are real parameters, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying certain conditions which will be stated later, and

$$F(x,u) = \int_0^u f(x,t)dt \geq 0.$$

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We consider the $p(x)$-Laplacian operator of the form:

$$\Delta_{p(x)} = div(|\nabla u|^{p(x)-2}\nabla u) = \sum_{i=1}^{N}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right),$$

which is not homogeneous and is related to the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$.

These facts imply some difficulties. For example, some classical theories and methods, including the Lagrange multiplier theorem and the theory of Sobolev spaces, cannot be applied. Problem (1) is called a bi-nonlocal problem because of the presence of the terms

$$\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \text{ and } \left[\int_\Omega F(x,u)dx\right]^r,$$

which implies that the first equation in (1) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties that make the study of such problems particularly interesting.

Besides, such problems have some physical motivations. Indeed, problem (1) is related with a physical model introduced by Kirchhoff [20] as follows:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(a + b\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where ρ, a, b, L are constants. Here,

$$M\left(\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) := a + b\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$$

describes the changes of the tension due to the increment in the length of the strings during the vibrations.

It therefore seems reasonable to be possible to give a realistic meaning for $M(0) = 0$, i.e., when the basic tension of the string is zero. Problem (2) has received a lot of attention only after Lions [22] proposed an abstract framework for this problem. We refer the reader to [6, 7, 9] for the Laplacian operator and [10, 14, 16, 19] for the p-Laplacian operator.

On the other hand, there are only a few papers which deal with nonlocal $p(x)$-Kirchhoff equation via variational approach, we can see [1, 2, 3, 12, 13, 17, 18, 29] and the references therein. Using variational methods, Corrêa-Costa [12] investigated the following nonlocal $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = h(x,u), & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where

$$h(x,u) = \lambda|u|^{q(x)-2}u\left[\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx\right]^r, \quad m_0 \le M(t) \le m_1.$$

Here, m_0 and m_1 are positive constants, and

$$M(t) = t^{\alpha-1}, \quad q^-(r+1) < \alpha p^-, \quad \frac{\alpha(p^+)^\alpha}{(q^-)^{\alpha-1}} < \frac{(q^-)^{r+1}(r+1)}{(q^+)^r}.$$

They proved several results on the existence of positive solutions. Recently, their result was extended in Corrêa-Costa [13] to the general nonlinearities cases: $h(x,u)$ and $M(t)$ were replaced respectively, by

$$f(x,u)\left[\int_\Omega F(x,u)\right]^r, \quad Q_1 t^{\gamma(x)-1} \le f(x,t) \le Q_2 t^{q(x)-1},$$

$$A_0 + At^{\alpha(x)} \le M(t) \le B_0 + Bt^{\beta(x)},$$

where A_0, A, B_0, B, Q_1, Q_2 are positive constants and $\alpha(x), \beta(x), \gamma(x), q(x) \in C_+(\overline{\Omega})$ satisfy the following conditions

$$\alpha(x) \le \beta(x) \ \text{ and } \gamma(x) \le q(x) < p^* = \frac{Np(x)}{N-p(x)}.$$

By using Krasnoselskii's genus, they proved the existence of infinitely many solutions for (3). For a deeper treatment, we refer to [8, 30] and the references therein.

Motivated by the above results, we are interested in the existence and multiplicity of solutions for the $p(x)$-bi-nonlocal type problem (1). We first state the following conditions for the Kirchhoff function M:

(M_1): $M : [0,+\infty) \to [0,+\infty)$ is a continuous function such that there exist $t_0 \ge 0$ and $\gamma \in (1, (p^*)_-/p_+)$ satisfying

$$tM(t) \le \gamma \widehat{M}(t), \quad \text{for all } t \ge t_0, \text{ where } \widehat{M}(t) = \int_0^t M(z)dz.$$

(M_2): There exist positive constants α, A and C such that

$$\widehat{M}(t) \ge Ct^\alpha \text{ for } t \ge A \ge 1 \text{ with } \alpha p^- > p^+.$$

A typical prototype of M is given by

$$M(t) = a + bt^{\alpha-1}, \text{ for all } t \ge 0, \text{ where } a, b \ge 0,\ b > 0 \text{ and } \alpha > 1. \tag{4}$$

When $M(t) > 0$ for all $t \ge 0$, Kirchhoff problems are said to be nondegenerate and this happens for example if $a > 0$ and $b > 0$ in the model case (4). Otherwise, if $M(0) = 0$ and $M(t) > 0$ for all $t > 0$, the Kirchhoff problems are called degenerate and this occurs in the model case (4) when $a = 0$ and $b > 0$.

Moreover, we assume that f is a continuous function which satisfies the following conditions:

(H_1): The subcritical growth condition holds:

$$|f(x,s)| \le C(1+|s|^{q(x)-1}), \text{for all } (x,s) \in \Omega\times\mathbb{R}, \text{where } C > 0,\ p(x) < q(x) < p^*(x);$$

(H_2): The Ambrosetti-Rabinowitz (abbreviated as (AR)) condition holds:

$$F(x,s) = \int_0^s f(x,t)dt$$

is θ-super-homogeneous at infinity, i.e. there exists $s_A > 0$ such that

$$0 < \theta F(x,s) \le sf(x,s), \quad \text{for all } |s| \ge s_A,\ x \in \Omega, \text{ where } \theta > \frac{\gamma p^+}{r+1};$$

(H_3): The following holds uniformly in $x \in \Omega$:

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p(x)-2}s} = 0;$$

(H_4): $f(x,-s) = -f(x,s)$, for all $(x,s) \in \Omega \times \mathbb{R}$.

Remark 1. An example of our conditions being satisfied is given by the following functions:

$$M(t) = bt^{\alpha-1}, \quad \text{where} \quad \alpha > 1, b > 0,$$

and

$$f(x,t) = |t|^{q(x)-1}t, \quad \text{where} \quad p(x) < q(x) < p^*(x).$$

Remark 2. The Ambrosetti-Rabinowitz superlinearity condition was originally introduced by Ambrosetti and Rabinowitz [4] and is still used in many works. This condition depicts a superquadratic growth and is used to ensure the boundedness of Palais-Smale sequences of the energy functional and hence in obtaining the mountain pass geometry. We note that the Palais–Smale condition on the functional is relevant in establishing critical point results and their applications (see also the discussion in [5]).

Now we are in position to state our main results.

Theorem 1.1. *Suppose that function $p \in C(\overline{\Omega})$ satisfies $\gamma p^+ < (r+1)p^-$. Then there exists $\lambda_0 > 0$ such that for every $\lambda < \lambda_0$, with conditions (M_1), (M_2), (H_1), (H_2) and (H_3) satisfied, problem (1) has at least two nontrivial weak solutions.*

Theorem 1.2. *Suppose that function $p \in C(\overline{\Omega})$ satisfies $\gamma p^+ < (r+1)p^-$. Then there exists $\lambda_0 > 0$ such that for every $\lambda < \lambda_0$, with conditions (M_1), (M_2), (H_1), (H_2), (H_3) and (H_4) satisfied, problem (1) has infinitely many solutions in $W_0^{1,p(x)}(\Omega)$.*

Theorem 1.3. *Suppose that conditions (M_1), (M_2), (H_1), (H_2), (H_3) and (H_4) are satisfied. Then for every $\lambda \in \mathbb{R}$, problem (1) has infinitely many solutions in $W_0^{1,p(x)}(\Omega)$.*

We conclude with an outline of the structure of the paper. In Section 2, we introduce some preliminary results concerning Lebesque and generalized Sobolev spaces and we recall some results that will be used later. In Section 3, we study the Palais-Smale condition. Section 4 is devoted to the proof of Theorem 1.1. In Section 5, we prove Theorem 1.2. Finally, Section 6 is dedicated to the proof of Theorem 1.3.

2. **Preliminaries.** In this section, we recall some definitions and basic properties of the generalized Lebesgue space and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. For this purpose, let consider Ω be a bounded domain of $\mathbb{R}^N$ and denote

$$C_+(\overline{\Omega}) = \left\{h \in C(\overline{\Omega}) \mid h(x) > 1, \text{ for all } x \in \overline{\Omega}\right\},$$

$$h^+ = \max_{x\in\overline{\Omega}} h(x), \quad h^- = \min_{x\in\overline{\Omega}} h(x), \quad h \in C(\overline{\Omega}).$$

The generalized Lebesgue space is defined as

$$L^{p(x)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function, } \int_\Omega |u|^{p(x)}\,dx < \infty\right\}$$

and it is equipped by the following norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Thus $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. Let us recall now some results which will be used later.

Proposition 1 ([28]). *(1)* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is a separable, uniformly convex Banach space and has conjugate space* $L^{q(x)}(\Omega)$, *where* $1/q(x) + 1/p(x) = 1$. *For every* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\,dx\right| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

(2) The inclusion between Lebesgue spaces also generalizes the classical framework, namely, if $0 < |\Omega| < \infty$ *and* p_1, p_2 *are variable exponents such that* $p_1 \leq p_2$ *in* Ω, *then there exists a continuous embedding* $L^{p_2(x)}(\Omega) \to L^{p_1(x)}(\Omega)$.

An important role in working with the generalized Lebesgue–Sobolev spaces is played by the $m(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space, which is the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,dx.$$

For more details about these variable exponent Lebesgue spaces see [23, 25].

Lemma 2.1 ([15]). *Denote*

$$\Lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

Then $\Lambda(u) \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ *and the derivative operator* Λ' *of* Λ *is*

$$\langle \Lambda'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx, \quad \textit{for all } u, v \in W_0^{1,p(x)}(\Omega),$$

and the following holds:

1. Λ *is a convex functional;*
2. $\Lambda' : W_0^{1,p(x)}(\Omega) \to (W^{-1,p'(x)}(\Omega)) = \left(W_0^{1,p(x)}(\Omega)\right)^*$ *is a bounded homeomorphism and strictly monotone operator, and the conjugate exponent satisfies* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$;
3. Λ' *is a mapping of type* (S_+), *namely,* $u_n \rightharpoonup u$ *and* $\limsup\langle \Lambda'(u_n), u_n - u\rangle \leq 0$, *imply* $u_n \to u$ *(strongly) in* $W_0^{1,p(x)}(\Omega)$.

Definition 2.2. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1), if

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx - \lambda\int_\Omega |u|^{p(x)-2}uv dx =$$
$$\left[\int_\Omega F(x,u)dx\right]^r \int_\Omega f(x,u)v dx, \text{ where } v \in W_0^{1,p(x)}(\Omega).$$

The energy functional $J_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ associated with problem (1)

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx - \lambda\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx - \frac{1}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} \\ &:= \Phi(u) - E_\lambda(u) - \Psi(u), \text{ for all } u \in W_0^{1,p(x)}(\Omega), \end{aligned} \tag{5}$$

is well-defined and of C^1-class on $W_0^{1,p(x)}(\Omega)$. Moreover, we have

$$\langle J'_\lambda(u), v\rangle = M\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx - \lambda\int_\Omega |u|^{p(x)-2}uv dx$$

$$-\left[\int_\Omega F(x,u)dx\right]^r\int_\Omega f(x,u)vdx,\ \text{for all } u,v\in W_0^{1,p(x)}(\Omega). \quad (6)$$

Hence, we can observe that the critical points of functional J_λ are the weak solutions for problem (1). In order to simplify the presentation we will denote the norm of $W_0^{1,p(x)}(\Omega)$ by $\|.\|$ instead of $\|\cdot\|_{W_0^{1,p(x)}(\Omega)}$. For simplicity, we use $C_i, i=1,2,...$ to denote general positive constants whose exact values may change from one place to another.

3. The Palais-Smale compactness condition.

Definition 3.1. Let $(W_0^{1,p(x)}(\Omega),\ \|.\|)$ be a Banach space and $J_\lambda\in C^1(W_0^{1,p(x)}(\Omega))$. Given $c\in\mathbb{R}$, we say that J_λ satisfies the Palais–Smale condition at the level $c\in\mathbb{R}$ ("$(PS)_c$ condition" for short) if any sequence $(u_n)\in W_0^{1,p(x)}(\Omega)$ satisfying

$$J_\lambda(u_n)\to c \text{ and } J'_\lambda(u_n)\to 0 \text{ in } W^{-1,p'(x)}(\Omega) \text{ as } n\to\infty, \quad (7)$$

has a convergent subsequence.

Lemma 3.2. *Assume that conditions (M_1),(M_2),(H_1) and (H_2) are satisfied. Then functional J_λ satisfies the $(PS)_c$ condition for any $c\neq 0$.*

Proof. We proceed in two steps.

Step 1. We prove that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Let $(u_n)\subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence for any $c\neq 0$. By (M_1), for $\|u\|$ large enough,

$$\gamma p^+\Phi(u)=\gamma p^+\widehat{M}(\Lambda(u))\geq p^+M(\Lambda(u))\Lambda(u)\geq M(\Lambda(u))\int_\Omega|\nabla u|^{p(x)}dx=\Phi'(u)u. \quad (8)$$

By (H_2) we can see that there exists $C_1>0$ such that

$$-C_1\leq\theta\int_\Omega F(x,u)dx\leq\int_\Omega f(x,u)udx+C_1,\ \text{for all } u\in W_0^{1,p(x)}(\Omega),$$

and thus, given any $\varepsilon\in(0,\theta)$, there exists $A_\varepsilon\geq A$ such that

$$(\theta-\varepsilon)\int_\Omega F(x,u)dx\leq\int_\Omega f(x,u)udx \text{ if } \int_\Omega F(x,u)dx\geq A_\varepsilon. \quad (9)$$

We may assume $A_\varepsilon>\frac{C_1}{\theta}$. Note that in this case the inequality $\int_\Omega F(x,u)dx\geq A_\varepsilon$ is equivalent to $\left|\int_\Omega F(x,u)dx\right|\geq A_\varepsilon$, because

$$\int_\Omega F(x,u)dx\geq-\frac{C_1}{\theta},\quad \text{for all } u\in W_0^{1,p(x)}(\Omega).$$

We claim that there exists $C_\varepsilon>0$ such that

$$\Psi'(u)u-(r+1)(\theta-\varepsilon)\Psi(u)\geq-C_\varepsilon,\ \text{for all } u\in W_0^{1,p(x)}(\Omega). \quad (10)$$

Indeed, if $\left|\int_\Omega F(x,u)dx\right|\leq A_\varepsilon$, then the validity of (10) is obvious. When

$$\left|\int_\Omega F(x,u)dx\right|\geq A_\varepsilon,\ \text{i.e.}\ \int_\Omega F(x,u)dx\geq A_\varepsilon,$$

it follows by (9) that

$$(r+1)(\theta-\varepsilon)\Psi(u)=(\theta-\varepsilon)\left(\int_\Omega F(x,u)dx\right)^{r+1}$$

$$= (\theta - \varepsilon) \left(\int_\Omega F(x,u)dx \right)^r \int_\Omega F(x,u)dx$$
$$\leq \left(\int_\Omega F(x,u)dx \right)^r \int_\Omega f(x,u)udx = \Psi'(u)u,$$

and so (10) holds.

Now let $(u_n) \subset W_0^{1,p(x)}(\Omega)\backslash\{0\}$, $J'_\lambda(u_n) \to 0$ and $J_\lambda(u_n) \to c$ with $c \neq 0$. Since $\gamma p^+ < (r+1)\theta$, there exists $\varepsilon > 0$ small enough so that $\gamma p^+ < (r+1)(\theta - \varepsilon)$. Then, since (u_n) is a $(PS)_c$ sequence, applying (8), (10) and (M_2), for sufficiently large n we have

$$(r+1)(\theta-\varepsilon)c + 1 + \|u_n\| \geq (r+1)(\theta-\varepsilon)J_\lambda(u_n) - J'_\lambda(u_n)u_n$$
$$\geq \left((r+1)(\theta-\varepsilon) - \gamma p^+\right)\Phi(u_n) + \left(\gamma p^+\Phi(u_n) - \Phi'(u_n)u_n\right)$$
$$+ \left(\Psi'(u_n)u_n - (r+1)(\theta-\varepsilon)\Psi(u_n)\right) - \lambda(r+1)(\theta-\varepsilon)\int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}dx$$
$$+\lambda\int_\Omega |u_n|^{p(x)}dx \geq C_2\|u_n\|^{\alpha p^-} - C_3 - C_\varepsilon - \lambda\int_\Omega \left(\frac{(r+1)(\theta-\varepsilon)}{p(x)} - 1\right)|u_n|^{p(x)}dx$$
$$\geq \begin{cases} C_2\|u_n\|^{\alpha p^-} - C_3 - C_\varepsilon, & \text{if } \lambda \leq 0 \\ C_2\|u_n\|^{\alpha p^-} - C_3 - C_\varepsilon - \lambda\left(\frac{(r+1)(\theta-\varepsilon)}{p^-} - 1\right)C_4\|u_n\|^{p^+}, & \text{if } \lambda > 0. \end{cases}$$

Since $\alpha p^- > p^+ > 1$, the above inequalities imply that (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$.

Step 2. Now we claim that (u_n) has a strongly convergent subsequence. To complete the argument we need the following proposition.

Proposition 2. (i) *Functional* $\Phi : X := W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *is sequentially weakly lower semi-continuous,* $\Psi, E_\lambda : X \to \mathbb{R}$ *are sequentially weakly continuous, and thus* J_λ *is sequentially weakly lower semi-continuous.*

(ii) *Mappings* $\Psi', E'_\lambda : X \to X^*$ *are sequentially weakly-strongly continuous. For any open set* $D \subset X\backslash\{0\}$ *with* $\overline{D} \subset X\backslash\{0\}$*, mappings* Φ' *and* $J'_\lambda : \overline{D} \to X^*$ *are bounded and of type* $(S+)$.

Proof. (i) Since function $\widehat{M}(t)$ is increasing and functional Λ is sequentially weakly lower semi-continuous, we can see that functional $\Phi : X := W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is sequentially weakly lower semi-continuous.

(ii) Noting that embedding $X \hookrightarrow L^{q(x)}(\Omega)$ is compact, we can see that Ψ, Ψ', E_λ, and E'_λ are sequentially weakly-strongly continuous. Now let $\overline{D} \subset X\backslash\{0\}$. It is clear that mappings Φ' and $J'_\lambda : \overline{D} \to X^* := (W^{-1,p'(x)}(\Omega))$ are bounded. To prove that $\Phi' : \overline{D} \to X^*$ is of type $(S+)$, assume that $(u_n) \subset \overline{D}$, $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty} \Phi'(u_n)(u_n - u) \leq 0$. Then there exist positive constants C_1 and C_2 such that $C_1 \leq \Lambda(u_n) \leq C_2$ and therefore there exist positive constants C_3 and C_4 such that $C_3 \leq M(\Lambda(u_n)) \leq C_4$. Noting that $\Phi'(u_n) = M(\Lambda(u_n))\Lambda'(u_n)$, it follows from $\limsup_{n\to+\infty} \Phi'(u_n)(u_n - u) \leq 0$ that $\limsup_{n\to+\infty} \Lambda'(u_n)(u_n - u) \leq 0$. Since Λ' is of type $(S+)$, we obtain $u_n \to u$ in X. This shows that mapping $\Phi' : \overline{D} \to X^*$ is of type $(S+)$. Moreover, since Ψ' and E'_λ are sequentially weakly-strongly continuous, mapping $J'_\lambda : \overline{D} \to X^*$ is of type $(S+)$.

$\square$

We can now complete the proof of Step 2. Since $J_\lambda(0)=0$ and $J_\lambda(u_n)\to c\neq 0$, there exists $\varepsilon>0$ small enough such that for sufficiently large n, $\|u_n\|>\varepsilon$. Setting $D=\{u\in W_0^{1,p(x)}(\Omega)\,/\,\|u_n\|>\varepsilon\}$, then $u_n\in D$ for n sufficiently large. Because (u_n) is bounded, we can consider a subsequence of (u_n), still denoted by (u_n), such that $u_n\in D$ and $u_n\rightharpoonup u$. The condition $J'_\lambda(u_n)\to 0$ implies $J'_\lambda(u_n)(u_n-u)\to 0$. Since $J'_\lambda:\overline{D}\to W_0^{1,p(x)}(\Omega)^*$ is of $(S+)$ type, we have $u_n\to u\in\overline{D}$. □

4. **Proof of Theorem 1.1.** In this section, the existence of nontrivial weak solutions for (1) is shown by applying the Mountain pass theorem and a variant of the Ekeland variational principle under suitable assumptions. To verify the conditions of the Mountain pass theorem (see e.g., [27]), we first need to prove two lemmas.

Lemma 4.1. *Suppose that conditions (M_1),(H_1) and (H_2) are satisfied. Then for any $w\in W_0^{1,p(x)}(\Omega)\backslash\{0\}$, $J_\lambda(sw)\to-\infty$ as $s\to+\infty$.*

Proof. Let $w\in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ be given. From (M_1) and for $t\geq 1$, we can easily obtain that $\widehat{M}(t)\leq\widehat{M}(1)t^\gamma$. Then

$$E(sw)=\widehat{M}\left(\int_\Omega\frac{1}{p(x)}|\nabla sw|^{p(x)}dx\right)\leq d_1s^{\gamma p^+},$$

for s large enough and d_1 a positive constant depending on w. By conditions (H_1) and (H_2), we have

$$\left[\int_\Omega F(x,sw)dx\right]^{r+1}\geq d_2s^{(r+1)\theta},$$

for s large enough and where d_2 is a positive constant depending on w. Finally, we have

$$\left|\int_\Omega\frac{1}{p(x)}|sw|^{p(x)}dx\right|\leq\frac{1}{p^+}\left(\int_\Omega|w|^{p(x)}dx\right)s^{p+}=d_3s^{p^+},$$

for s large enough, where d_3 is a positive constant depending on w. Hence for any $w\in W_0^{1,p(x)}(\Omega)\backslash\{0\}$ and s large enough,

$$J_\lambda(sw)\leq\begin{cases}d_1s^{\gamma p^+}-d_2s^{(r+1)\theta}+\lambda d_3s^{p^+} & \text{if}\quad\lambda>0,\\ d_1s^{\gamma p^+}-d_2s^{(r+1)\theta}-\lambda d_3s^{p^+} & \text{if}\quad\lambda\leq 0.\end{cases}$$

Thus, since $p^+\leq\gamma p^+<(r+1)\theta$, we conclude that $J_\lambda(sw)\to-\infty$ as $s\to+\infty$. □

Lemma 4.2. *Suppose that conditions (M_1),(H_1) and (H_3) are satisfied. Then there exist positive numbers a,ρ,λ_0 such that $J_\lambda(u)\geq a>0$ if $\|u\|=\rho$ and $\lambda<\lambda_0$.*

Proof. Conditions (H_1) and (H_3) imply that

$$|F(x,t)|\leq\varepsilon|t|^{p(x)}+C_\varepsilon|t|^{q(x)},\quad\text{for all }(x,t)\in\Omega\times\mathbb{R}.$$

For $\|u\|$ small enough, we have

$$\begin{aligned}\int_\Omega F(x,u)dx&\leq\varepsilon\int_\Omega|u|^{p(x)}dx+C_\varepsilon\int_\Omega|u|^{q(x)}dx\leq\varepsilon\left(|u|_{p(x)}^{p^+}+|u|_{p(x)}^{p^-}\right)\\&\quad+C_\varepsilon\left(|u|_{q(x)}^{q^+}+|u|_{q(x)}^{q^-}\right)\\&\leq\varepsilon\left(C_1^{p^+}\|u\|^{p^+}+C_1^{p^-}\|u\|^{p^-}\right)+C_\varepsilon\left(C_2^{q^+}\|u\|^{q^+}+C_2^{q^-}\|u\|^{q^-}\right)\\&\leq\varepsilon\left(C_1^{p^+}+C_1^{p^-}\right)\|u\|^{p^-}+C_\varepsilon\left(C_2^{q^+}+C_2^{q^-}\right)\|u\|^{q^-}\end{aligned}$$

$$\leq \varepsilon\left(C_1^{p^+}+C_1^{p^-}\right)\|u\|^{p^-}+C_\varepsilon\left(C_2^{q^+}+C_2^{q^-}\right)\|u\|^{p^-}\leq C_3\|u\|^{p^-},$$

where

$$C_3=\varepsilon\left(C_1^{p^+}+C_1^{p^-}\right)+C_\varepsilon\left(C_2^{q^+}+C_2^{q^-}\right).$$

Therefore

$$\Psi(u)=\frac{1}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1}\leq\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^-}. \tag{11}$$

Moreover, condition (M_1) gives

$$\widehat{M}(t)\geq\widehat{M}(1)t^\gamma,\ \text{ for all } t\in[0,1]. \tag{12}$$

Thus, using (11) and (12), we obtain

$$\begin{aligned}
J_\lambda(u)&=\widehat{M}(\Lambda(u))-\lambda\int_\Omega\frac{1}{p(x)}|u|^{p(x)}dx-\Psi(u)\\
&\geq\begin{cases}\widehat{M}(1)\left(\Lambda(u)\right)^\gamma-\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^-}, & \text{if } \lambda\leq 0\\ \widehat{M}(1)\left(\Lambda(u)\right)^\gamma-\frac{\lambda C_1^{p^-}}{p^-}\|u\|^{p^-}-\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^-}, & \text{if } \lambda>0\end{cases}\\
&\geq\begin{cases}\frac{\widehat{M}(1)}{(p^+)^\gamma}\|u\|^{\gamma p^+}-\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^-}, & \text{if } \lambda\leq 0\\ \frac{\widehat{M}(1)}{(p^+)^\gamma}\|u\|^{\gamma p^+}-\frac{\lambda C_1^{p^-}}{p^-}\|u\|^{p^-}-\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^-}, & \text{if } \lambda>0\end{cases}\\
&=\begin{cases}\|u\|^{\gamma p^+}\left(\frac{\widehat{M}(1)}{(p^+)^\gamma}-\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^--\gamma p^+}\right), & \text{if } \lambda\leq 0\\ \|u\|^{\gamma p^+}\left(\frac{\widehat{M}(1)}{(p^+)^\gamma}-\frac{\lambda C_1^{p^-}}{p^-}\|u\|^{p^--\gamma p^+}-\frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^--\gamma p^+}\right), & \text{if } \lambda>0.\end{cases}
\end{aligned} \tag{13}$$

Now, for each $\lambda>0$, we define a continuous function $h_\lambda:(0,\infty)\to\mathbb{R}$,

$$h_\lambda(t)=\frac{\lambda C_1^{p^-}}{p^-}t^{p^--\gamma p^+}+\frac{C_3^{r+1}}{r+1}t^{(r+1)p^--\gamma p^+}.$$

Since $1<p^-<\gamma p^+<(r+1)p^-$, it follows that $\lim_{t\to 0^+}h_\lambda(t)=\lim_{t\to+\infty}h_\lambda(t)=+\infty$. Thus we can find the infimum of $h_\lambda(t)$. Note that equating

$$h'_\lambda(t)=\frac{\lambda C_1^{p^-}(p^--\gamma p^+)}{p^-}t^{p^--\gamma p^+-1}+\frac{C_3^{r+1}((r+1)p^--\gamma p^+)}{r+1}t^{(r+1)p^--\gamma p^+-1}=0,$$

we get

$$t_0=t=C_4\lambda^{\frac{1}{rp^-}},\ \text{ where } C_4=\left(\frac{c_1^{p^-}(r+1)(\gamma p^+-p^-)}{C_3^{r+1}p^-((r+1)p^--\gamma p^+)}\right)^{\frac{1}{rp^-}}>0.$$

Clearly, $t_0>0$. It can also be checked that $h''_\lambda(t_0)>0$ and hence the infimum of $h_\lambda(t)$ is achieved at t_0. Now, observing that

$$h_\lambda(t_0)=\left(\frac{C_1^{p^-}C_4^{p^--\gamma p^+}}{p^-}+\frac{C_3^{r+1}C_4^{(r+1)p^--\gamma p^+}}{r+1}\right)\lambda^{\frac{(r+1)p^--\gamma p^+}{rp^-}}\to 0 \text{ as } \lambda\to 0^+,$$

we can infer from (13) that there exists $\lambda_0>0$ such that for all $\lambda<\lambda_0$ we can choose ρ small enough and $a>0$ such that $J_\lambda(u)\geq a>0$, for all $u\in X$ with $\|u\|=\rho$. □

Let $\lambda_0 > 0$ be a constant as given in Lemma 4.2. By Lemmas 3.2, 4.1, 4.2 and the Mountain pass theorem, we deduce that for all $\lambda \in (0, \lambda_0)$, J_λ has a critical point $u_1 \in X$ which is a weak solution for problem (1). Moreover, u_1 satisfies

$$J_\lambda(u_1) \geq a > 0, \tag{14}$$

which implies that u_1 is nontrivial.

We will show that there exists a second weak solution $u_2 \neq u_1$ by using the Ekeland variational principle. By Lemma 4.2, we have

$$\inf_{u \in \partial B(0,r)} (J_\lambda(u)) > 0,$$

and by Lemma 4.1, there exists $w \in X$ such that $J_\lambda(tw) < 0$ for $t > 0$ large enough. Moreover, as in the proof of Lemma 4.2, for $u \in B(0,r)$, we have

$$J_\lambda(u) \geq \begin{cases} \|u\|^{\gamma p^+}\left(\frac{\widehat{M}(1)}{(p^+)^\gamma} - \frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^- - \gamma p^+}\right), & \text{if } \lambda \leq 0 \\ \|u\|^{\gamma p^+}\left(\frac{\widehat{M}(1)}{(p^+)^\gamma} - \frac{\lambda C_1^{p^-}}{p^-}\|u\|^{p^- - \gamma p^+} - \frac{C_3^{r+1}}{r+1}\|u\|^{(r+1)p^- - \gamma p^+}\right), & \text{if } \lambda > 0. \end{cases}$$

Therefore

$$-\infty < \underline{c} = \inf_{u \in \overline{B(0,r)}} (J_\lambda(u)) < 0.$$

Let $\varepsilon > 0$, be such that

$$0 < \varepsilon < \inf_{u \in \partial B(0,r)} (J_\lambda(u)) - \inf_{u \in B(0,r)} (J_\lambda(u)).$$

We deduce from the above information that functional $J_\lambda : \overline{B(0,r)} \to \mathbb{R}$, is lower bounded and $J_\lambda \in C^1(\overline{B(0,r)}, \mathbb{R})$. Therefore, by using the Ekeland principle, we conclude that there exists $u_\varepsilon \in \overline{B(0,r)}$, such that

$$\begin{cases} \underline{c} \leq J_\lambda(u_\varepsilon) \leq \underline{c} + \varepsilon \\ \\ J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon||u - u_\varepsilon||, u \neq u_\varepsilon. \end{cases}$$

Since

$$J_\lambda(u_\varepsilon) \leq \inf_{u \in \overline{B(0,r)}} (J_\lambda(u)) + \varepsilon \leq \inf_{B(0,r)} (J_\lambda(u)) + \varepsilon < \inf_{\partial B(0,r)} (J_\lambda(u)),$$

we can deduce that $u_\varepsilon \in B(0,r)$. Now, we define

$$\Xi_\lambda : \overline{B(0,r)} \to \mathbb{R} \ \text{ by } \Xi_\lambda(u) = J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|.$$

It is clear that u_ε is a minimum of Ξ_λ. Therefore, for $t > 0$ large enough and for any $v \in B(0,1)$, we have

$$\frac{\Xi_\lambda(u_\varepsilon + tv) - \Xi_\lambda(u_\varepsilon)}{t} \geq 0, \text{ that is, } \frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

By letting t tend to infinity, we obtain

$$J'_\lambda(u_\varepsilon)(v) + \varepsilon\|v\| \geq 0.$$

This implies that $\|J'_\lambda(u_\varepsilon)\| \leq \varepsilon$. By the argument above, we deduce the existence of a sequence $(u_n) \subset B(0,r)$, such that

$$J_\lambda(u_n) \to \underline{c} < 0, \ \text{ and } \ J'_\lambda(u_n) \to 0. \tag{15}$$

Since $(u_n) \subset B(0,r)$, it follows that (u_n) is bounded in X. So, up to a subsequence, there exists $u_2 \in X$ such that (u_n) converges weakly to $u_2 \in X$. Hence, by the proof of Lemma 3.2, we deduce that $u_n \to u$ strongly in X.

Since $J_\lambda \in C^1(X, \mathbb{R})$, we have $J'_\lambda(u_n) \to J'_\lambda(u_2)$, as $n \to \infty$. Hence, from (15), we conclude that

$$J'_\lambda(u_2) = 0, \ \|u_2\| < r, \text{ and } J_\lambda(u_2) < 0. \tag{16}$$

This implies that u_2 is a nontrivial solution for problem (1). Finally, by combining (14) and (16), we obtain $J_\lambda(u_2) < 0 < J_\lambda(u_1)$. The proof of Theorem 1.1 is now complete. □

5. **Proof of Theorem 1.2.** In this section, we will show that problem (1) has infinitely many pairs $(u_j, -u_j)$ of critical points with $I(u_j) \to \infty$ as $j \to \infty$ by using the Symmetric mountain pass theorem [26]. We first need the following lemma:

Lemma 5.1. *Suppose that conditions (H_1) and (H_2) are satisfied. Then for any finite-dimensional subspace $\widetilde{X} \subset X$, $J_\lambda(u) \to -\infty$, $\|u\| \to +\infty$, $u \in \widetilde{X}$.*

Proof. Arguing indirectly, assume that there exists a sequence $(u_n) \subset \widetilde{X}$ such that

$$\|u_n\| \to +\infty, \ n \to +\infty \text{ and } J_\lambda(u_n) \geq B, \text{ for all } n \in \mathbb{N}, \tag{17}$$

where $B \in \mathbb{R}$ is a fixed constant not depending on $n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then it is obvious that $\|v_n\| = 1$. Since $\dim \widetilde{X} < +\infty$, there exists $v \in \widetilde{X} \backslash \{0\}$ such that up to a subsequence,

$$\|v_n - v\| \to 0, \text{ and } v_n(x) \to v(x) \quad \text{a.e. } x \in \Omega, \text{ as } n \to +\infty.$$

If $v(x) \neq 0$, then $|u_n(x)| \to +\infty$ as $n \to +\infty$. Clearly, condition (H_2) implies condition

$$\lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^{\frac{\gamma p^+}{r+1}}} = +\infty, \text{ uniformly a.e. } x \in \Omega. \tag{18}$$

By virtue of (18),

$$\lim_{n \to +\infty} \frac{F(x, u_n(x))}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} = \lim_{n \to +\infty} \frac{F(x, u_n(x))}{|u_n|^{\frac{\gamma p^+}{r+1}}} |v_n|^{\frac{\gamma p^+}{r+1}} = +\infty, x \in \Omega_0 = \{x \in \Omega : v(x) \neq 0\}.$$

Moreover, we can find $t_0 > 0$, such that

$$\frac{F(x,t)}{|t|^{\frac{\gamma p^+}{r+1}}} \geq c > 0, \text{ for all } x \in \Omega \text{ and } |t| > t_0. \tag{19}$$

On the other hand, condition (H_1) implies that there exists a positive constant C_1 such that

$$|F(x,t)| \leq C_1, \text{ for all } (x,t) \in \Omega \times [-t_0, t_0]. \tag{20}$$

Then, by (19) and (20), we deduce that there exists a constant $C_2 \in \mathbb{R}$ such that $F(x,t) \geq C_2$, for all $(x,t) \in \Omega \times \mathbb{R}$. From this, we conclude that

$$\frac{F(x, u_n(x)) - C_2}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} \geq 0, \text{ for all } x \in \Omega \text{ and } n \in \mathbb{N},$$

which implies that

$$\frac{F(x, u_n(x))}{|u_n(x)|^{\frac{\gamma p^+}{r+1}}} |v_n(x)|^{\frac{\gamma p^+}{r+1}} - \frac{C_2}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} \geq 0, \text{ for all } x \in \Omega \text{ and } n \in \mathbb{N}. \tag{21}$$

Therefore using (17) and (21), we have

$$0 \leq \lim_{n \to +\infty} J_\lambda(u_n(x))$$

$$\leq \begin{cases} \dfrac{\widehat{M}(1)}{(p^+)^\gamma} - \lim\limits_{n\to+\infty} \left[\dfrac{\lambda \int_\Omega |u_n(x)|^{p(x)}dx}{p^-\|u_n\|^{\frac{\gamma p^+}{r+1}}} + \dfrac{\left(\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} dx \right)^{r+1}}{r+1} \right], \text{if } \lambda \leq 0 \\ \dfrac{\widehat{M}(1)}{(p^+)^\gamma} - \lim\limits_{n\to+\infty} \dfrac{1}{r+1} \left(\displaystyle\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} dx \right)^{r+1}, \text{if } \lambda > 0 \end{cases}$$

$$\leq \begin{cases} \dfrac{\widehat{M}(1)}{(p^+)^\gamma} - \lim\limits_{n\to+\infty} \left[\dfrac{\lambda C_3 \|u_n\|^{p^+}}{p^-\|u_n\|^{\frac{\gamma p^+}{r+1}}} + \dfrac{\left(\int_\Omega \frac{F(x,u_n(x))-C_2}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} dx \right)^{r+1}}{r+1} \right], \text{if } \lambda \leq 0 \\ \dfrac{\widehat{M}(1)}{(p^+)^\gamma} - \lim\limits_{n\to+\infty} \dfrac{1}{r+1} \left(\displaystyle\int_\Omega \frac{F(x,u_n(x))-C_2}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} dx \right)^{r+1}, \text{if } \lambda > 0 \end{cases}$$

$$\leq \frac{\widehat{M}(1)}{(p^+)^\gamma} - \lim_{n\to+\infty} \frac{1}{r+1} \left(\int_\Omega \frac{F(x,u_n(x))-C_2}{\|u_n\|^{\frac{\gamma p^+}{r+1}}} dx \right)^{r+1}$$

$$\leq \frac{\widehat{M}(1)}{(p^+)^\gamma} - \lim_{n\to+\infty} \frac{1}{r+1} \left(\int_\Omega \frac{F(x,u_n(x))-C_2}{|u_n(x)|^{\frac{\gamma p^+}{r+1}}} |v_n(x)|^{\frac{\gamma p^+}{r+1}} dx \right)^{r+1} \to -\infty,$$

which is a contradiction. The proof of Lemma 5.1 is thus complete. □

Proof of Theorem 1.2. Clearly, by condition (H_4), J_λ is an even functional. Since $J_\lambda(0) = 0$, thanks to Lemmas 3.2, 4.2, 5.1 and the Symmetric mountain pass theorem [26], we deduce the existence of an unbounded sequence of weak solutions to problem (1). □

6. **Proof of Theorem 1.3.** In this part, we will prove Theorem 1.3 by using Clarke's theorem [11] which will be stated below. To this end, let us begin by defining the notion of genus and its basic properties.

Let Σ be the class of closed subset A of $X \setminus 0$ such that $A = -A$, i.e. symmetric with respect to the origin. Recall that for $A \in \Sigma$, the genus $\gamma(A)$ is defined as the least integer k such that there exists an odd function $f \in C(X, \mathbb{R}^k \setminus 0)$. Moreover, if such function does not exist then $\gamma(A) = \infty$ and by convenience, $\gamma(\emptyset) = 0$.

It's well known that in general, the computation of the genus is a difficult task. Often, it suffices to use some estimates which can be given by comparison with sets whose genus is known as for example the sphere. We shall use the definition of the genus from [21].

Consider now

$$\Sigma_k = \{A \in \Sigma, \gamma(A) \geq k\},\ k \in \mathbb{N}, \quad \text{and} \quad c_k := \inf_{A\in\Sigma_k} \sup_{u\in A} I(u).$$

We have

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_{k+1} \leq \cdots .$$

Moreover, in order to prove Theorem 1.3, we use the following theorem due to Clarke [11].

Theorem 6.1 ([11]). *Let $J \in C^1(X,\mathbb{R})$ be a functional satisfying the following conditions*

- *(i) J satisfies the (PS) condition.*
- *(ii) J is bounded from below and even.*
- *(iii) There exists a compact set $K \in \mathcal{A}$ such that $\gamma(A) = k$ and $\sup_{x\in K} J(x) < J(0)$.*

Then J possesses at least k pairs of distinct critical points, and their corresponding critical values $c_k < 0$ such that $\lim_{k\to\infty} c_k = 0$ are less than $J(0)$.

In order to get the infinity of solutions, we shall use Theorem 6.1. Since X is a separable reflexive Banach space, there exist $(e_n) \subset X$ and $(e_n^\star) \subset X^\star$ such that

$$\langle e_n^\star, e_m\rangle = \delta_{nm} = \begin{cases} 1 \text{ if } n = m \\ 0 \text{ if } n \neq m, \end{cases}$$

$$X = \overline{\text{span}\{e_n,\ n = 1, 2, \cdots, \}},\ X^\star = \overline{\text{span}\{e_n^\star,\ n = 1, 2, \cdots, \}}.$$

For each $k \in \mathbb{N}$, consider the subspace

$$X_k = \text{span}\{e_1, \cdots, e_k\} \subset X = W_0^{1,p(x)}(\Omega), \quad \text{spanned by } e_1, \cdots, e_k.$$

It is well-known that

$$X_k \hookrightarrow L^{\delta(x)}(\Omega), \quad \text{continuously for } 1 < \delta(x) < p^\star.$$

Moreover, the norms in X and $L^{\delta(x)}(\Omega)$ are equivalent in X_k. Furthermore, by using condition (H_2), we have

$$|F(x,u)| \geq C_1|u|^\theta - C_2,$$

hence we get

$$J_\lambda(u) \leq \frac{\widehat{M}(1)}{p^{-\gamma}}\left[\int_\Omega |\nabla u|^{p(x)}dx\right]^\gamma - \frac{\lambda}{p^+}\int_\Omega |u|^{p(x)}dx - \frac{C_3}{r+1}\left[\int_\Omega |u|^{(r+1)\theta}dx\right] + \frac{C_4|\Omega|^{r+1}}{r+1}.$$

If $\|u\|$ is small enough, then we have

$$\int_\Omega |\nabla u|^{p(x)}dx \leq \|u\|^{p^-} \quad \text{and} \quad -|u|_{p(x)}^{p^+} \geq -\int_\Omega |u|^{p(x)}dx.$$

By using the equivalence of the norms in X_k, we deduce that

$$-C(k)\|u\|^{p^+} \geq -\int_\Omega |u|^{p(x)}dx,$$

where $C(k)$ is a positive constant. Consequently, we get

$$J_\lambda(u) \leq \frac{\widehat{M}(1)}{p^{-\gamma}}\|u\|^{\gamma p^-} - \frac{\lambda C(k)}{p^+}\|u\|^{p^+} - \tilde{C}(k)\|u\|^{(r+1)\theta} + C_5.$$

Hence, we have

$$J_\lambda(u) \leq \|u\|^{(r+1)\theta}\left[\frac{\widehat{M}(1)\|u\|^{\gamma p^- -(r+1)\theta}}{p^{-\gamma}} - \frac{\lambda C(k)\|u\|^{p^+-(r+1)\theta}}{p^+} + \frac{C_5}{\|u\|^{(r+1)\theta}} - \tilde{C}(k)\right].$$

Let R be a positive constant such that

$$\frac{\widehat{M}(1)}{p^{-\gamma}}\|u\|^{\gamma p^- -(r+1)\theta} - \frac{\lambda C(k)}{p^+}\|u\|^{p^+-(r+1)\theta} + C_5\|u\|^{-(r+1)\theta} \leq \tilde{C}(k).$$

Let $0 < r_0 < R$ and consider the set $K = \{u \in X_k;\ \|u\| = r_0\}$. Then

$$\begin{aligned} J_\lambda(u) &\leq r_0^{(r+1)\theta}\left[\frac{\widehat{M}(1)}{p^{-\gamma}} r_0^{\gamma p^- -(r+1)\theta} - \frac{\lambda C(k)}{p^+} r_0^{p^+-(r+1)\theta} + C_5 r_0^{-(r+1)\theta} - \tilde{C}(k)\right] \\ &\leq R^{(r+1)\theta}\left[\frac{\widehat{M}(1)R^{\gamma p^- -(r+1)\theta}}{p^{-\gamma}} - \frac{\lambda C(k)|g|_\infty R^{p^+-(r+1)\theta}}{p^+} + \frac{C_5}{R^{(r+1)\theta}} - \tilde{C}(k)\right] \\ &< 0 = J_\lambda(0), \end{aligned}$$

which implies that $\sup_K J_\lambda(u) < 0 = J_\lambda(0)$. Since X_k and $\mathbb{R}^k$ are isomorphic and K and S^{k-1} are homomorphic, it follows that $\gamma(K) = k$. The Clarke theorem 6.1 shows that problem (1) admits at least k pairs of distinct critical points, and their corresponding critical values $c_k < 0$ such that $\lim_{k\to\infty} c_k = 0$ are less than $J_\lambda(0)$. If k is chosen arbitrary then problem (1) possesses infinitely many critical points.

Lemma 6.2. *For each $n \in \mathbb{N}$, there exists $\varepsilon > 0$ such that*

$$\gamma(A_\lambda^{-\varepsilon}) \geq n,\ \textit{where } A_\lambda^{-\varepsilon} = \{u \in X;\ J_\lambda(u) \leq -\varepsilon\}.$$

Proof. Consider X_n be a subspace of X of dimension n and any $u \in X_n$ such that $\|u\| = 1$ and $0 < t < R$. Then we have

$$\begin{aligned} J_\lambda(tu) &\leq \frac{\widehat{M}(1)t^{\gamma p^-}}{p^{-\gamma}}\|u\|^{\gamma p^-} - \frac{\lambda C(k)t^{p^+}}{p^+}\|u\|^{p^+} - \tilde{C}(k)t^{(r+1)\theta}\|u\|^{(r+1)\theta} + C_5 \\ &\leq \frac{\widehat{M}(1)t^{\gamma p^-}}{p^{-\gamma}} - \frac{\lambda C(k)t^{p^+}}{p^+} - \tilde{C}(k)t^{(r+1)\theta} + C_5. \end{aligned}$$

If

$$J_\lambda(tu) \to -\infty,\ \gamma p^- \leq \gamma p^+ < (r+1)\theta,\ 0 < t < R,$$

then there exist $t_0 > 0$ and $\varepsilon > 0$ such that

$$J_\lambda(t_0 u) < -\varepsilon,\ u \in X_n,\ \|u\| = 1.$$

Consider now the sphere

$$S_{t_0,n} = \{u \in X_n,\ \|u\| = t_0\}.$$

Then $S_{t_0,n} \subset A_\lambda^{-\varepsilon}$ and by the properties of the genus, $\gamma(A_\lambda^{-\varepsilon}) \geq \gamma S_{t_0,n} = n$. □

Lemma 6.3. *Let*

$$\Sigma = \{A \subset X \setminus \{0\} \mid\ A \textit{ is closed and } A = -A\},\quad \Sigma_k = \{A \in \Sigma \mid\ \gamma(A) \geq k\}.$$

Then $c_k = \inf_{A\in\Sigma_k}\sup_{u\in A} J_\lambda(u)$ is a negative critical value of J_λ and if $c = c_k = \cdots = c_{k+r}$, then

$$\gamma(K_c) \geq r+1,\ \textit{where } K_c = \{u \in X;\ J_\lambda(u) = c;\ J'_\lambda(u) = 0\}.$$

Proof. First, we claim that $-\infty < c_k < \infty$. By the previous lemma, we know that for each $k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $\gamma(A_\lambda^{-\varepsilon}) \geq k$ or $A_\lambda^{-\varepsilon} \in \Sigma_k$. Then:

1. either $c_k \leq \sup_{u\in A_\lambda^{-\varepsilon}} J_\lambda(u) \leq -\varepsilon(k) < 0$, for all K,
2. or J_λ is bounded from below, hence $c_k > -\infty$, for all $k \in \mathbb{N}$.

Since $c < 0$, J_λ satisfies the (PS) condition at level c, K_c is compact and symmetric, it follows that $\gamma(K_c)$ is well-defined.

Let us now assume that

$$c = c_k = c_{k+1} = \cdots = c_{k+r}\ \text{ and } \gamma(K_c) < r+1.$$

By the properties of the genus, there exists a neighborhood K of K_c such that $\gamma(K) = \gamma(K_c) < r + 1$. Moreover, based on the Deformation lemma [24], there exists an odd homomorphism

$$\widehat{\eta} : X \to X \text{ such that } \widehat{\eta}(A_\lambda^{c+\beta} \setminus K) \subset A_\lambda^{c-\beta}, \text{ where } 0 < \beta < -c.$$

Functional J_λ satisfies the (PS) in A_λ^0. Furthermore, by definition, we have

$$c = c_{k+r} = \inf_{a\in\Sigma_{k+r}} \sup_{u\in A} J_\lambda(u).$$

Then there exists $A \in \Sigma_{k+r}$ such that $\sup_{u\in A} J_\lambda(u) < c + \beta$, which means that

$$A \subset A_\lambda^{c+\beta} \text{ and } \widehat{\eta}(A \setminus K) \subset \widehat{\eta}(A_\lambda^{c+\beta} \setminus K) \subset A_\lambda^{c-\beta}.$$

Hence, we conclude that

$$\gamma(\widehat{\eta}(\overline{A \setminus k})) \geq \gamma(\overline{A \setminus K}) \geq \gamma(A) - \gamma(K) \geq (k + r) - r = k,$$

i.e.,

$$\widehat{\eta}(\overline{A \setminus K}) \in \Sigma_k, \quad \text{hence} \sup_{u\in\widehat{\eta}(\overline{A\setminus K})} J_\lambda \geq c_k = c,$$

which yields a contradiction. Therefore, if $c = c_k = \cdots = c_{k+r}$, then $\gamma(K_c) \geq r + 1$. □

Remark 3. We note that if c_k is a critical value, then $\gamma(K_{c_k}) \geq 1$ and K_{c_k} is nonempty for all $k \in \mathbb{N}$. In addition, if the points c_k are not all distinct, then $\gamma(K_c) > 1$, K_c is an infinite subspace, and problem (1) admits infinitely many critical points with negative energy.

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