Discrete and Continuous Dynamical Systems - Series S
Vol. 16, No. 6, June 2023, pp. 1401–1413
doi:10.3934/dcdss.2022130

ON NONLOCAL DIRICHLET PROBLEMS WITH OSCILLATING TERM

Boštjan Gabrovšek

Faculty of Mechanical Engineering
University of Ljubljana, Ljubljana, 1000, Slovenia

Giovanni Molica Bisci*

Dipartimento di Scienze Pure e Applicate (DiSPeA)
Università degli Studi di Urbino Carlo Bo, Urbino, 61029, Italy

Dušan D. Repovš

Faculty of Education and Faculty of Mathematics and Physics
University of Ljubljana, Ljubljana, 1000, Slovenia

Dedicated to the loving memory of Gaetana Restuccia

Abstract. In this paper, a class of nonlocal fractional Dirichlet problems is studied. By using a variational principle due to Ricceri (whose original version was given in J. Comput. Appl. Math. 113 (2000), 401–410), the existence of infinitely many weak solutions for these problems is established by requiring that the nonlinear term f has a suitable oscillating behaviour either at the origin or at infinity.

1. **Introduction.** In the present paper we deal with the following nonlocal fractional problem

$$\begin{cases} (-\Delta)_p^s u = \lambda\alpha(x)f(u) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth (Lipschitz) boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$, $s \in (0,1)$, $p > N/s$, $\lambda \in \mathbb{R}$, while $\alpha \in L^\infty(\Omega)$ with $\alpha_0 := \operatorname{essinf}_{x\in\Omega}\alpha(x) > 0$ and the reaction term $f : \mathbb{R} \to \mathbb{R}$ is a suitable continuous function. Finally, the leading operator $(-\Delta)_p^s$ in (1.1) is the degenerate fractional p-Laplacian, defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough, and $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s(x) := 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dx,$$

which for $p = 2$ reduces to the linear fractional Laplacian, up to a dimensional constant $C(N, s) > 0$; see, for instance, [7, 8, 11, 23].

Since elliptic problems involving the fractional p-Laplacian operator have been intensively studied in recent years by several authors, a bibliography list is always

2020 *Mathematics Subject Classification.* Primary: 47J30, 35R11; Secondary: 35S15, 35A15.

Key words and phrases. Variational methods, p-fractional Laplacian operator, infinitely many solutions.

* Corresponding author: Giovanni Molica Bisci.

far from being complete. To avoid this, we mention here only the papers [4, 10, 12, 13, 14] and [17, 18, 27, 28, 29], as well as the references therein.

Motivated by the wide interest in problem (1.1), and in order to treat it, we crucially use that, in our setting, the nonlocal fractional Sobolev space

$$X_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed by the norm

$$\|u\| := \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy \right)^{1/p},$$

is compactly embedded into the space $C^0(\bar{\Omega})$ of continuous functions up to the boundary $\partial\Omega$. This regularity result allow us to obtain essential analytical properties of the Euler–Lagrange functional associated to (1.1) in the low-dimensional case; see [3, 9, 15] for related topics.

Inspired by the results contained in [1, 3, 24, 25, 26] and invoking Lemma 2.1, we then study the number and the asymptotic behavior of the solutions of problem (1.1), when f oscillates near the origin or at the infinity. This analysis is carried out by exploiting variational and topological techniques; see Theorem 2.2 below and [31, Theorem 2.5].

More precisely, fixed $L \in \{0^+, \infty\}$, let

$$A_L := \liminf_{t \to L} \frac{\max_{|\zeta| \leqslant t} F(\zeta)}{t^p} \quad \text{and} \quad B_L := \limsup_{t \to L} \frac{F(t)}{t^p},$$

where

$$F(t) := \int_0^t f(\zeta) d\zeta, \text{ for every } t \in \mathbb{R}.$$

With the above notations, let us define

$$\lambda_1^L := \kappa_{p,N,s} \frac{\omega_N}{p\tau^{sp}\alpha_0} \frac{2^N}{B_L} \quad \text{and} \quad \lambda_2^L := \frac{1}{p\|\alpha\|_\infty |\Omega| K^p A_L},$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$,

$$\kappa_{p,N,s} := \frac{2^{p(3-s)-N}}{p} \left(1 - \frac{1}{2^N}\right)^2 + \frac{2^{2+ps-N}}{ps(N + p(1-s))} + \frac{2}{(N-ps)ps} \left(1 - \frac{1}{2^{N-ps}}\right),$$

$$\tau := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega), \quad \text{and} \quad K := \sup \left\{ \frac{\|u\|_\infty}{\|u\|} : u \in X_0^{s,p}(\Omega) \setminus \{0\} \right\}.$$

The main result reads as follows.

Theorem 1.1. *Assume that*

$$\inf_{t \geqslant 0} F(t) = 0 \quad \textit{and} \quad \liminf_{t \to L} \frac{\max_{|\zeta| \leqslant t} F(\zeta)}{t^p} < C \limsup_{t \to L} \frac{F(t)}{t^p},$$

where $C = C(p, N, s, \alpha, \tau, |\Omega|, K)$ is the geometric constant given by

$$C := \left(\frac{\tau^{sp}}{2^N \kappa_{p,N,s} K^p |\Omega| \omega_N} \right) \frac{\alpha_0}{\|\alpha\|_\infty}. \tag{1.2}$$

Then for every $\lambda \in (\lambda_1^L, \lambda_2^L)$, problem (1.1) admits a sequence $(u_{\lambda,j})_j$ of weak solutions in the fractional Sobolev space $X_0^{s,p}(\Omega)$.

Moreover, $\lim_{j\to\infty} \|u_{\lambda,j}\| = \infty$ if $L = \infty$, and $\lim_{j\to\infty} \|u_{\lambda,j}\| = \lim_{j\to\infty} \|u_{\lambda,j}\|_\infty = 0$ if $L = 0^+$.

A special and a meaningful case of Theorem 1.1 is the following.

Corollary 1. *Assume that f is nonnegative with $f(0) = 0$. Furthermore, suppose that*

$$\liminf_{t\to L} \frac{F(t)}{t^p} = 0 \quad \textit{and} \quad \limsup_{t\to L} \frac{F(t)}{t^p} = \infty. \tag{1.3}$$

Then for every $\lambda > 0$ problem (1.1) admits a sequence $(u_{\lambda,j})_j$ of nonnegative weak solutions in the fractional Sobolev space $X_0^{s,p}(\Omega)$.

Moreover, $\lim_{j\to\infty} \|u_{\lambda,j}\| = \infty$ if $L = \infty$, and $\lim_{j\to\infty} \|u_{\lambda,j}\| = \lim_{j\to\infty} \|u_{\lambda,j}\|_\infty = 0$ if $L = 0^+$.

We notice that the existence of sequences of weak solutions for fractional nonlocal equations, without any symmetry hypothesis on the nonlinear term f, has been investigated in [2, Theorems 5 and 6]. However, in the low-dimensional case treated here, it can be easily seen that Theorem 1.1 is more general than the results proved in the aforementioned paper. We refer to the monograph [23] as a general reference for nonlocal problems and variational methods used in this manuscript.

2. Fractional framework.

This section is devoted to the notations used throughout the paper. In order to give the weak formulation of problem (1.1), we need to work in a special functional space. Indeed, one of the difficulties in treating problem (1.1) is related to encoding the Dirichlet boundary condition in the variational formulation. In this respect, the standard fractional Sobolev spaces are not sufficient in order to study this problem. We overcome this difficulty by working in a new functional space, whose definition is recalled here.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth (Lipschitz) boundary, fix $s \in (0,1)$ and take $p > N/s$. Let

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy < \infty \right\}$$

be the fractional space endowed with the norm

$$\|u\|_{s,p} := \left(\int_{\mathbb{R}^N} |u(x)|^p dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \right)^{1/p}, u \in W^{s,p}(\mathbb{R}^N).$$

We work on the closed linear subspace defined by

$$X_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

and equivalently renormed by setting

$$\|u\| := \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \right)^{1/p}, \; u \in X_0^{s,p}(\Omega),$$

namely, the Poincaré inequality holds in $X_0^{s,p}(\Omega)$. The following Rellich-type result will be crucial for our purposes.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and let $p > 1$, $s \in (0,1)$ such that $sp > N$. Then the embedding*

$$X_0^{s,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$$

is compact.

Proof. Since $p > N/s$, by [5, Theorem 4.47] it follows that $X_0^{s,p}(\Omega) \subset W^{s,p}(\mathbb{R}^N)$ is continuously embedded into $C^0(\bar{\Omega})$; see also [8, Theorem 8.2]. Now, in order to show that this embedding is also compact, let B be a bounded subset of $X_0^{s,p}(\Omega)$ and let us prove that B is relatively compact in $C^0(\bar{\Omega})$. By virtue of the Arzelà–Ascoli theorem, the conclusion will be achieved by proving that B is equibounded and equicontinuous on $C^0(\bar{\Omega})$. To this end, since $X_0^{s,p}(\Omega)$ by [5, Theorem 4.47], is continuously embedded in $C^0(\bar{\Omega})$, there exists a constant $c_1 > 0$ such that

$$\|u\|_\infty \leqslant c_1\|u\|, \quad \text{for every } u \in B.$$

Hence, the set B is equibounded in $C^0(\bar{\Omega})$. Moreover, arguing as in the proof [8, Theorem 8.2], for every $u \in B$, the following Morrey-type inequality holds

$$|u(x) - u(y)| \leqslant c_2\|u\|_{s,p}|x - y|^{s-N/p}, \quad \text{for every } \ x, y \in \mathbb{R}^N \tag{2.1}$$

for some constant $c_2 > 0$. Indeed, by formula (8.8) in [8], it follows that

$$|u(x) - u(y)| \leqslant c[u]_{p,ps}|x - y|^{s-N/p}, \quad \text{for every } \ x, y \in \mathbb{R}^N,$$

where

$$[u]_{p,sp} := \left(\sup_{x_0\in\Omega\, \rho>0} \rho^{-sp} \int_{B_\rho(x_0)\cap\Omega} |u(x) - \langle u\rangle_{B_\rho(x_0)\cap\Omega}|^p dx\right)^{1/p},$$

with

$$\langle u\rangle_{B_\rho(x_0)\cap\Omega} := \frac{1}{|B_\rho(x_0) \cap \Omega|} \int_{B_\rho(x_0)\cap\Omega} u(x)dx.$$

Consequently, (2.1) has been proved by [8, formula 8.4]. Finally,

$$\|u\|_{s,p} \leqslant c_3 \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dxdy\right)^{1/p}, \text{ for every } u \in B. \tag{2.2}$$

In conclusion, by combining (2.2) with (2.1) the equicontinuity of B easily follows. This completes the proof of Lemma 2.1. □

Note that, since the embedding $X_0^{s,p}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ is continuous, it follows that

$$K := \sup\left\{\frac{\|u\|_\infty}{\|u\|} : u \in X_0^{s,p}(\Omega) \setminus \{0\}\right\} < \infty.$$

It remains an open problem to determine an explicit upper bound for the constant K.

Remark 1. We note that a more precise version of Lemma 2.1 can be proved by using [5, Lemma 2.85]. More precisely, if $p > N/s$ and $C_b^{0,\mu}(\bar{\Omega})$ denotes the space of Hölder continuous functions of order μ on $\bar{\Omega}$, the embedding $X_0^{s,p}(\Omega) \hookrightarrow C_b^{0,\mu}(\bar{\Omega})$ is compact provided that $\mu < s - N/p$. The above regularity argument, with a slight modification, seems to work also in anisotropic fractional Sobolev spaces; see [30] for related topics.

For further details on the fractional Sobolev spaces we refer to [8, 23] and to the references therein.

Let us fix $\lambda \in \mathbb{R}$. We recall that a *weak solution* for problem (1.1), is a function $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy \\ \qquad\qquad\qquad = \lambda \displaystyle\int_\Omega \alpha(x)f(u(x))\varphi(x)dx, \text{ for every } \varphi \in X_0^{s,p}(\Omega) \\ u \in X_0^{s,p}(\Omega)\,. \end{cases} \tag{2.3}$$

Let $\mathcal{J}_\lambda : X_0^{s,p}(\Omega) \to \mathbb{R}$ be defined as follows

$$\mathcal{J}_\lambda(u) := \frac{1}{p}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy - \lambda\int_\Omega \alpha(x)F(u(x))dx, \tag{2.4}$$

where, as usual, we set

$$F(t) := \int_0^t f(\zeta)\,d\zeta, \text{ for every } t \in \mathbb{R}.$$

Since $f \in C^0(\mathbb{R},\mathbb{R})$ and $\alpha \in L^\infty(\Omega)$, the functional $\mathcal{J}_\lambda \in C^1(X_0^{s,p}(\Omega))$ and its derivative at $u \in X_0^{s,p}(\Omega)$ is given by

$$\langle \mathcal{J}_\lambda'(u), \varphi\rangle = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy$$

$$-\lambda\int_\Omega \alpha(x)f(u(x))\varphi(x)dx, \text{ for every } \varphi \in X_0^{s,p}(\Omega).$$

Thus the weak solutions of problem (1.1) are exactly the critical points of the energy functional $\mathcal{J}_\lambda$.

Therefore, the proof of the main result reduces to finding critical points of the functional by using suitable abstract approaches.

In this direction, we rephrase [31, Theorem 2.1] in a slightly different version; see, for instance, [3, Theorem 2.1].

Theorem 2.2. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive, and Ψ is sequentially weakly upper semicontinuous. Set $J_\lambda := \Phi - \lambda\Psi$. Moreover, for every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\displaystyle\sup_{v\in\Phi^{-1}((-\infty,r))}\Psi(v) - \Psi(u)}{r-\Phi(u)},$$

$$\gamma := \liminf_{r\to\infty}\varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+}\varphi(r).$$

Then one has

(a) *If $\gamma < \infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternatives exist:*
(a_1) *either J_λ possesses a global minimum,*
(a_2) *or there is a sequence $(u_j)_j$ of critical points (local minima) of J_λ such that*

$$\lim_{j\to\infty}\Phi(u_j) = \infty.$$

(b) *If $\delta < \infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternatives exist:*
(b_1) *either there is a global minimum of Φ which is a local minimum of J_λ,*

(b_2) *or there is a sequence* $(u_j)_j$ *of pairwise distinct critical points* (*local minima*) *of* J_λ *which weakly converges to a global minimum of* Φ, *with*

$$\lim_{j\to\infty} \Phi(u_j) = \inf_{u\in X} \Phi(u).$$

Following the seminal work of Ricceri [31], an impressive number of publications appeared, most of them dedicated to the study of suitable extensions of his variational principle as well as of its consequences; see, for instance, the books [16, 21, 23] and the references therein. Recent applications of [31, Theorem 2.1] can be found in [6, 19, 22]. See also [20] for related topics.

3. **A proof of Theorem 1.1.** Let $X := X_0^{s,p}(\Omega)$ be endowed with the norm

$$\|u\| := \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy\right)^{1/p}, \ u\in X.$$

Moreover, let $\Phi : X\to\mathbb{R}$ and $\Psi : X\to\mathbb{R}$ be defined as follows

$$\Phi(u) := \frac{1}{p}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \quad \text{and} \quad \Psi(u) := \int_\Omega \alpha(x)F(u(x))dx,$$

so that if we set $J_\lambda := \Phi - \lambda\Psi$ as in Theorem 2.2, then we get $\mathcal{J}_\lambda = J_\lambda$.

By standard arguments, one shows that Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous and that its Gâteaux derivative is the functional $\Phi'(u)\in X^*$ given by

$$\Phi'(u)(\varphi) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy, \ \varphi\in X.$$

Moreover, Ψ is continuously Gâteaux differentiable, its Gâteaux derivative is given by

$$\Psi'(u)(\varphi) = \int_\Omega \alpha(x)f(u(x))\varphi(x)dx, \ \varphi\in X.$$

Now, thanks to Lemma 2.1, Ψ is a sequentially weakly continuous functional. Indeed, for every sequence $(u_j)_j$ in X such that $u_j \rightharpoonup u_0$ for some $u_0\in X$, we shall prove that

$$\lim_{j\to\infty} \Psi(u_j) = \Psi(u_0). \tag{3.1}$$

Since the embedding $X \hookrightarrow C^0(\bar\Omega)$ is compact by Lemma 2.1, there exists $c>0$ such that $u_j \to u_0$ in $C^0(\bar\Omega)$ and

$$\|u_j\|_\infty \leqslant c, \ \text{for every } j\in\mathbb{N}. \tag{3.2}$$

On the other hand,

$$\lim_{j\to\infty} \alpha(x)F(u_j(x)) = \alpha(x)F(u_0(x)),$$

and, by inequality (3.2)

$$|\alpha(x)F(u_j(x))| \leqslant \alpha(x)\max_{|t|\leqslant c}|F(t)|, \ \text{for a.e. } x\in\bar\Omega \text{ and every } j\in\mathbb{N}.$$

Hence, since $\alpha\in L^\infty(\Omega)$, by using the Lebesgue dominated convergence theorem it follows that (3.1) holds.

Now, let $(c_j)_j$ be a positive real sequence such that

$$\lim_{j\to\infty} c_j = L \quad \text{and} \quad \lim_{j\to\infty} \frac{\max_{|t|\leqslant c_j} F(t)}{c_j^p} = A_L.$$

Set

$$r_j := \frac{c_j^p}{K^p p} > 0, \text{ for every } j \in \mathbb{N}.$$

The continuous embedding $X \hookrightarrow C^0(\bar{\Omega})$ yields

$$\|v\|_\infty \leqslant c_j, \text{ for every } v \in \Phi^{-1}((-\infty, r_j)) \text{ and } j \in \mathbb{N}.$$

Thus

$$\|v\|_\infty \leqslant K\|v\| = K(pr_j)^{1/p} = c_j, \text{ for every } j \in \mathbb{N}.$$

Then

$$\begin{aligned}\varphi(r_j) &= \inf_{u\in\Phi^{-1}((-\infty,r_j))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r_j))} \Psi(v) - \Psi(u)}{r_j - \Phi(u)} \\ &\leqslant \frac{\sup_{v\in\Phi^{-1}((-\infty,r_j))} \Psi(v)}{r_j} \leqslant p\|\alpha\|_\infty |\Omega| K^p \frac{\max_{|t|\leqslant c_j} F(t)}{c_j^p}, \text{ for every } j \in \mathbb{N},\end{aligned}$$

taking into account that $\Phi(0) = \Psi(0) = 0$.

Hence, if $\lambda < \lambda_2^L := \dfrac{1}{p\|\alpha\|_\infty |\Omega| K^p A_L}$ it follows that

$$\beta_L \leqslant \liminf_{j\to\infty} \varphi(r_j) \leqslant p\|\alpha\|_\infty |\Omega| K^p A_L < \frac{1}{\lambda} < \infty,$$

where

$$\beta_L := \begin{cases} \gamma := \liminf\limits_{r\to\infty} \varphi(r) & \text{if } L = \infty \\ \delta := \liminf\limits_{r\to 0^+} \varphi(r) & \text{if } L = 0^+. \end{cases}$$

Let us denote by $B_r(x_0)$ the N-dimensional open ball centered at $x_0 \in \mathbb{R}^N$ and of radius $r > 0$. As Ω is open, we can certainly choose a point $x_0 \in \Omega$ and a number $\tau > 0$ so that $B_\tau(x_0) \subseteq \Omega$.

If we set $\tau := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$, the point x_0 is the Chebyshev center of Ω. Hence let us fix such x_0 and τ and define the function θ to be

$$\theta(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B_\tau(x_0) \\ 1 & \text{if } x \in B_{\tau/2}(x_0) \\ 2\dfrac{\tau - |x - x_0|}{\tau} & \text{if } x \in B_\tau(x_0) \setminus B_{\tau/2}(x_0) \end{cases} \tag{3.3}$$

for every $x \in \mathbb{R}^N$, where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. Since $\theta \equiv 0$ outside the compact ball $\overline{B}_\tau(x_0)$, we can easily deduce that $\theta \in X$.

We now state and prove our main lemma.

Lemma 3.1. *Let $\theta \in X$ be the function defined in (3.3). Then*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\theta(x) - \theta(y)|^p}{|x - y|^{N+ps}}\, dxdy \leqslant \kappa_{p,N,s}\omega_N^2 \tau^{N-sp}, \tag{3.4}$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$, and

$$\kappa_{p,N,s} := \frac{2^{p(3-s)-N}}{p}\left(1 - \frac{1}{2^N}\right)^2 + \frac{2^{2+ps-N}}{ps(N + p(1 - s))} + \frac{2}{(N - ps)ps}\left(1 - \frac{1}{2^{N-ps}}\right).$$

Proof. A direct and straightforward computation ensures that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\theta(x)-\theta(y)|^p}{|x-y|^{N+ps}}\,dxdy = \sum_{j=1}^{4} J_j, \tag{3.5}$$

where we set

$$J_1 := \int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \frac{|\theta(x)-\theta(y)|^p}{|x-y|^{N+ps}}\,dxdy,$$

$$J_2 := 2\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \int_{\mathbb{R}^N\setminus B_{\tau/2(x_0)}} \frac{|\theta(x)-\theta(y)|^p}{|x-y|^{N+ps}}\,dxdy,$$

$$J_3 := 2\int_{B_{\tau/2}(x_0)} \int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \frac{|\theta(x)-\theta(y)|^p}{|x-y|^{N+ps}}\,dxdy,$$

and

$$J_4 := 2\int_{B_{\tau/2}(x_0)} \int_{\mathbb{R}^N\setminus B_\tau(x_0)} \frac{|\theta(x)-\theta(y)|^p}{|x-y|^{N+ps}}\,dxdy.$$

On the other hand, by virtue of

$$|y-x_0|-|x-x_0| \leqslant |x-y|,$$

and

$$|x-y|\leqslant |x-x_0|+|y-x_0| \leqslant 2\tau, \text{ for all } (x,y)\in (B_\tau(x_0)\setminus B_{\tau/2}(x_0))\times(B_\tau(x_0)\setminus B_{\tau/2}(x_0)),$$

one has

$$\begin{aligned}
J_1 &= \frac{2^p}{\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \frac{||y-x_0|-|x-x_0||^p}{|x-y|^{N+ps}}\,dxdy\\
&\leqslant \frac{2^p}{\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \frac{|x-y|^p}{|x-y|^{N+ps}}\,dxdy\\
&= 2^{p(3-s)-N}\left(1-\frac{1}{2^N}\right)^2 \frac{\tau^{N-ps}}{p}\omega_N^2.
\end{aligned}$$

Furthermore, since for every $y\in B_\tau(x_0)\setminus B_{\tau/2}(x_0)$,

$$\int_{\mathbb{R}^N\setminus B_\tau(x_0)} \frac{|\tau-|y-x_0||^p}{|x-y|^{N+ps}}\,dx = \omega_N\int_{\tau-|y-x_0|}^{\infty} \frac{|\tau-|y-x_0||^p}{\varrho^{ps+1}}d\varrho,$$

it follows that

$$\begin{aligned}
J_2 &= \frac{2^{p+1}}{\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \int_{\mathbb{R}^N\setminus B_\tau(x_0)} \frac{|\tau-|y-x_0||^p}{|x-y|^{N+ps}}\,dxdy\\
&= \frac{2^{p+1}\omega_N}{\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \int_{\tau-|y-x_0|}^{\infty} \frac{|\tau-|y-x_0||^p}{\varrho^{ps+1}}d\varrho dy\\
&= \frac{2^{p+1}\omega_N}{ps\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} |\tau-|x-y||^{p(1-s)}dy\\
&= \frac{2^{p+1}\omega_N^2}{ps\tau^p}\int_0^{\tau/2} z^{N+(1-s)p-1}dz = \frac{2^{1+ps-N}}{s(N+p(1-s))}\frac{\tau^{N-ps}}{p}\omega_N^2,
\end{aligned}$$

as well as

$$\begin{aligned}
J_3 &= \frac{2^{p+1}}{\tau^p}\int_{B_{\tau/2}(x_0)}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} \frac{||x-x_0|-\tau/2|^p}{|x-y|^{N+ps}}\,dxdy\\
&= \frac{2^{p+1}}{\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)}\int_{B_{\tau/2}(x_0)} \frac{||x-x_0|-\tau/2|^p}{|x-y|^{N+ps}}\,dydx\\
&= \frac{2^{p+1}\omega_N}{\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2(x_0)}} ||x-x_0|-\tau/2|^p\int_{|x-x_0|-\tau/2}^{|x-x_0|+\tau/2}\frac{d\varrho}{\varrho^{ps+1}}\,dx\\
&\leqslant \frac{2^{p+1}\omega_N}{ps\tau^p}\int_{B_\tau(x_0)\setminus B_{\tau/2}(x_0)} ||x-x_0|-\tau/2|^{p(1-s)}dx\\
&= \frac{2^{p+1}\omega_N}{ps\tau^p}\int_0^{\tau/2} z^{p(1-s)+N-1}dz = \frac{2^{1+ps-N}}{s(N+p(1-s))}\frac{\tau^{N-ps}}{p}\omega_N^2,
\end{aligned}$$

observing that

$$\int_{B_{\tau/2}(x_0)}\frac{dy}{|x-y|^{N+ps}} = \omega_N\int_{|x-x_0|-\tau/2}^{|x-x_0|+\tau/2}\frac{d\varrho}{\varrho^{ps+1}},\ \text{for every } x\in B_\tau(x_0)\setminus B_{\tau/2}(x_0).$$

Finally,

$$\begin{aligned}
J_4 &= 2\int_{B_{\tau/2}(x_0)}\int_{\mathbb{R}^N\setminus B_\tau(x_0)}\frac{1}{|x-y|^{N+ps}}\,dxdy\\
&= 2\omega_N\int_{B_{\tau/2}(x_0)}\int_{\tau-|y-x_0|}^{\infty}\frac{1}{\varrho^{1+ps}}d\varrho dy = \frac{2}{ps}\omega_N\int_{B_{\tau/2}(x_0)}\frac{1}{(\tau-|y-x_0|)^{ps}}dy\\
&= \frac{2\omega_N^2}{ps}\int_{\tau/2}^{\tau} z^{N-ps-1}dz = \frac{2}{(N-ps)s}\left(1-\frac{1}{2^{N-ps}}\right)\frac{\tau^{N-ps}}{p}\omega_N^2,
\end{aligned}$$

due to the fact that

$$\int_{\mathbb{R}^N\setminus B_\tau(x_0)}\frac{1}{|x-y|^{N+ps}}\,dx = \omega_N\int_{\tau-|y-x_0|}^{\infty}\frac{1}{\varrho^{1+ps}}d\varrho,\ \text{for every } y\in B_{\tau/2}(x_0).$$

The conclusion follows by (3.5) and the above estimates. $\square$

If $L=\infty$, then we claim that the functional J_λ is unbounded from below. For our goal, let $(\zeta_j)_j$ be a real sequence such that

$$\lim_{j\to\infty}\zeta_j=\infty$$

and

$$\lim_{j\to\infty}\frac{F(\zeta_j)}{\zeta_j^p}=B_\infty. \tag{3.6}$$

For each $j\in\mathbb{N}$, let $w_j=\zeta_j\theta\in X$. Then, by Lemma 3.1,

$$\Phi(w_j)\leqslant \kappa_{p,N,s}\omega_N^2\frac{\tau^{N-ps}}{p}\zeta_j^p,\ \text{for every } j\in\mathbb{N}.$$

Moreover, since $\inf\limits_{\zeta\geqslant 0}F(\zeta)=0$ and $\alpha_0:=\operatorname{essinf}_{x\in\Omega}\alpha(x)>0$, we have

$$\int_\Omega \alpha(x)F(w_j(x))dx\geqslant \alpha_0\int_{B_{\tau/2}(x_0)}F(w_j(x))dx\geqslant \alpha_0\frac{\tau^N}{2^N}\omega_N F(\zeta_j)>0,\ \text{for every } j\in\mathbb{N},$$

since $\theta\equiv 1$ on $B_{\tau/2}(x_0)$. Then, on account of (3.4), it follows that

$$J_\lambda(w_j)\leqslant \kappa_{p,N,s}\omega_N^2\frac{\tau^{N-ps}}{p}\zeta_j^p-\lambda\alpha_0\frac{\tau^N}{2^N}\omega_N F(\zeta_j),\ \text{for every } j\in\mathbb{N}.$$

If $B_\infty < \infty$, since $\lambda > \lambda_1^\infty$, we can fix

$$\varepsilon \in \left(\kappa_{p,N,s} \frac{\omega_N}{\lambda p \tau^{ps} \alpha_0} \frac{2^N}{B_\infty}, 1\right).$$

By using (3.6), we get ν_ε such that

$$F(\zeta_j) > \varepsilon B_\infty \zeta_j^p, \quad \text{for all } j > \nu_\varepsilon.$$

Then one has for every $j > \nu_\varepsilon$,

$$J_\lambda(w_j) \leqslant \kappa_{p,N,s} \omega_N^2 \frac{\tau^{N-ps}}{p} \zeta_j^p - \lambda \alpha_0 \frac{\tau^N}{2^N} \omega_N F(\zeta_j) \leqslant \left(\kappa_{p,N,s} \frac{\omega_N}{p \tau^{ps}} - \lambda \varepsilon B_\infty \frac{\alpha_0}{2^N}\right) \tau^N \omega_N \zeta_j^p.$$

Consequently, since

$$\lim_{j\to\infty} \zeta_j = +\infty \ \text{ and } \varepsilon > \kappa_{p,N,s} \frac{\omega_N}{\lambda p \tau^{ps} \alpha_0} \frac{2^N}{B_\infty},$$

it follows that,

$$\lim_{j\to\infty} J_\lambda(w_j) = -\infty.$$

If $B_\infty = \infty$, let us fix

$$M > 2^N \kappa_{p,N,s} \frac{\omega_N}{\lambda p \tau^{ps} \alpha_0}.$$

By using again (3.6), we get ν_M such that

$$F(\zeta_j) > M \zeta_j^p, \quad \text{for all } j > \nu_M.$$

Now we have for every $j > \nu_M$,

$$J_\lambda(w_j) \leqslant \kappa_{p,N,s} \omega_N^2 \frac{\tau^{N-ps}}{p} \zeta_j^p - \lambda \alpha_0 \frac{\tau^N}{2^N} \omega_N F(\zeta_j) = \left(\kappa_{p,N,s} \frac{\omega_N}{p \tau^{ps}} - \lambda M \frac{\alpha_0}{2^N}\right) \tau^N \omega_N \zeta_j^p.$$

Bearing in mind the choice of M again, we get

$$\lim_{j\to\infty} J_\lambda(w_j) = -\infty.$$

Therefore, thanks to Theorem 2.2 - Part (a), the functional J_λ admits an unbounded sequence $(u_{\lambda,j})_j \subset X$ of critical points.

If $L = 0^+$, then an argument similar to the one above shows that $u_0 \equiv 0$ is not a local minimum point for the functional J_λ. By Theorem 2.2 - Part (b), the functional J_λ admits a sequence $(u_{\lambda,j})_j \subset X$ of pairwise distinct critical points (local minima) such that

$$\lim_{j\to\infty} \|u_{\lambda,j}\| = 0.$$

Finally, by Lemma 2.1 one also have $\lim_{j\to\infty} \|u_{\lambda,j}\|_\infty = 0$ as claimed. This completes the proof of Theorem 1.1. $\square$

4. **Final comments and remarks.** Let us give some comments concerning Corollary 1. To this end, assume that $f:\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function with $f(0)=0$. It is easily seen that Corollary 1 can be obtained by applying Theorem 1.1 to the nonlocal problem

$$\left\{\begin{array}{ll}(-\Delta)_p^s u=\lambda\alpha(x)f^+(u) & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{array}\right. \tag{4.1}$$

where $f^+:\mathbb{R}\to\mathbb{R}$, defined by

$$f^+(t):=\begin{cases}f(t) & \text{if } t>0\\ 0 & \text{if } t\leqslant 0,\end{cases}$$

is continuous for every $t\in\mathbb{R}$.

Indeed, since f is nonnegative, one has

$$\max_{|\zeta|\leqslant t}\int_0^\zeta f^+(x)dx=\max_{|\zeta|\leqslant t}\int_0^\zeta f(x)dx=F(t),\ \text{for every } t\in[0,+\infty),$$

so that the assumptions of Corollary 1 actually give $A_L=0$ and $B_L=\infty$. Consequently, the main conclusions hold with $\lambda_1^L=0$ and $\lambda_2^L=\infty$.

Finally, to conclude the proof, we just need to prove that the solutions are nonnegative. To this end, let

$$\xi^\pm:=\max\{0,\pm\xi\},\quad \text{for every } \xi\in\mathbb{R},$$

and let $u\in X_0^{s,p}(\Omega)$ be a weak solution of (4.1). Then $u^\pm\in X_0^{s,p}(\Omega)$ and

$$\int_\Omega \alpha(x)f^+(u(x))u^-(x)dx=0. \tag{4.2}$$

Furthermore, notice that

$$|\xi^--\eta^-|^p\leqslant|\xi-\eta|^{p-2}(\xi-\eta)(\eta^--\xi^-),\ \text{for every } \xi,\eta\in\mathbb{R}. \tag{4.3}$$

Then, by virtue of (4.2) and (4.3), and by testing J'_λ with $-u^-\in X_0^{s,p}(\Omega)$, we obtain

$$\begin{aligned}0=\langle J'_\lambda(u),-u^-\rangle&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}}dxdy\\ &\geqslant\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u^-(x)-u^-(y)|^{p-2}}{|x-y|^{N+ps}}dxdy.\end{aligned}$$

This implies that u^- is constant on $\mathbb{R}^N$ and since u^- vanishes outside Ω, it follows that $u^-=0$ on the entire space $\mathbb{R}^N$. Thus, $u\geqslant 0$ a.e. in Ω as claimed. This completes the proof of Corollary 1. ☐

We conclude the paper by an application of Corollary 1.

Example 4.1. Let us consider the following nonlocal fractional problem

$$\left\{\begin{array}{ll}(-\Delta)_p^s u=\lambda f(u) & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega,\end{array}\right. \tag{4.4}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth (Lipschitz) boundary $\partial\Omega$, $s\in(0,1)$, $p>N/s$, and $f:\mathbb{R}\to\mathbb{R}$ is the function defined by

$$f(t):=\begin{cases}((k+1)!^p-k!^p)\dfrac{g_k(t)}{\displaystyle\int_{a_k}^{b_k}g_k(\zeta)d\zeta} & \text{if } t\in\bigcup_{k\geqslant1}[a_k,b_k]\\ 0 & \text{otherwise,}\end{cases}$$

where

$$a_k := \frac{2k!(k+2)!-1}{4(k+1)!} \quad \text{and} \quad b_k := \frac{2k!(k+2)!+1}{4(k+1)!},$$

and $g_k : [a_k, b_k] \to \mathbb{R}$ is the continuous function given for every $k \geqslant 1$, by

$$g_k(t) := \sqrt{\frac{1}{16(k+1)!} - \left(t - \frac{k!(k+2)}{2}\right)^2}, \quad t \in [a_k, b_k].$$

By virtue of Corollary 1, for every

$$\lambda > \kappa_{p,N,s} \frac{\omega_N}{p\tau^{ps}} 2^{N-p},$$

problem (4.4) admits a sequence $(u_{\lambda,j})_j \subset X_0^{s,p}(\Omega)$ of (nonnegative) weak solutions such that

$$\lim_{j \to \infty} \|u_{\lambda,j}\| = \infty.$$

Acknowledgments. The manuscript was realized under the auspices of the Italian MIUR project *Variational methods, with applications to problems in mathematical physics and geometry* (2015KB9WPT 009) and the Slovenian Research Agency program P1-0292 and grants N1-0114 and N1-0083.

REFERENCES

[1] C. Alves and G. Molica Bisci, A compact embedding result for anisotropic Sobolev spaces associated to a strip-like domain and some applications, *J. Math. Anal. Appl.*, **501** (2021), Paper No. 123490, 24 pp.

[2] V. Ambrosio, L. D'Onofrio, and G. Molica Bisci, On nonlocal fractional Laplacian problems with oscillating potentials, *Rocky Mountain J. Math.*, **48** (2018), 1399–1436.

[3] G. Bonanno and G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p–Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A*, **140** (2010), 737–752.

[4] L. M. Del Pezzo and A. Quaas, A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian, *J. Differential Equations*, **263** (2017), 765–778.

[5] F. Demengel and G. Demengel, *Functional Spaces for the Theory of Elliptic Partial Differential Equations*, Springer, Berlin, 2012.

[6] G. Devillanova, G. Molica Bisci and R. Servadei, A flower-shape geometry and nonlinear problems on strip-like domains, *J. Geom. Anal.*, **31** (2021), 8105–8143.

[7] A. Di Castro, T. Kuusi and G. Palatucci, Local behavior of fractional p-minimizers, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **267** (2015), 1807–1836.

[8] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.

[9] F. Faraci, A. Iannizzotto and A. Kristály, Low–dimensional compact embeddings of symmetric Sobolev spaces with applications, *Proc. Roy. Soc. Edinburgh Sect. A*, **141** (2011), 383–395.

[10] G. M. Figueiredo, G. Molica Bisci and R. Servadei, The effect of the domain topology on the number of solutions of fractional Laplace problems, *Calc. Var. Partial Differential Equations*, **57** (2018), Paper No. 103, 24 pp.

[11] G. Franzina and G. Palatucci, Fractional p-eigenvalues, *Riv. Mat. Univ. Parma*, **5** (2014), 373–386.

[12] A. Iannizzotto, S. Liu, K. Perera and M. Squassina, Existence results for fractional p-Laplacian problems via Morse theory, *Adv. Calc. Var.*, **9** (2016), 101–125.

[13] A. Iannizzotto, S. J. N. Mosconi and M. Squassina, Fine boundary regularity for the degenerate fractional p-Laplacian, *J. Funct. Anal.*, **279** (2020), 108659, 54 pp.

[14] A. Iannizzotto, S. Mosconi and M. Squassina, Sobolev versus Hölder minimizers for the degenerate fractional p-Laplacian, *Nonlinear Anal.*, **191** (2020), 111635, 14 pp.

[15] A. Kristály, Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$, *J. Differential Equations*, **220** (2006), 511–530.

[16] A. Kristály, V. D. Rădulescu and C. G. Varga, *Variational Principles in Mathematical Physics, Geometry, and Economics. Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, **136**, Cambridge University Press, Cambridge, 2010.

[17] E. Lindgren and P. Lindqvist, Fractional eigenvalues, *Calc. Var. Partial Differential Equations*, **49** (2014), 795–826.

[18] J. Mawhin and G. Molica Bisci, A Brezis–Nirenberg type result for a non–local fractional operator, *J. Lond. Math. Soc.*, **95** (2017), 73–93.

[19] G. Molica Bisci, A group–theoretical approach for nonlinear Schrödinger equations, *Adv. Calc. Var.*, **13** (2020), 403–423.

[20] G. Molica Bisci and P. F. Pizzimenti, Sequences of weak solutions for non-local elliptic problems with Dirichlet boundary condition, *Proc. Edinb. Math. Soc.*, **57** (2014), 779–809.

[21] G. Molica Bisci and P. Pucci, *Nonlinear Problems with Lack of Compactness*, De Gruyter Series in Nonlinear Analysis and Applications, 2021.

[22] G. Molica Bisci and V. D. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations, *Calc. Var. Partial Differential Equations*, **54** (2015), 2985–3008.

[23] G. Molica Bisci, V. D. Rădulescu and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, No. **162**, Cambridge University Press, Cambridge, 2016.

[24] F. Obersnel and P. Omari, Positive solutions of elliptic problems with locally oscillating nonlinearities, *J. Math. Anal. Appl.*, **323** (2006), 913–929.

[25] P. Omari and F. Zanolin, Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential, *Commun. Partial Differential Equations*, **21** (1996), 721–733.

[26] P. Omari and F. Zanolin, An elliptic problem with arbitrarily small positive solutions, *Proceedings of the Conference on Nonlinear Differential Equations (Coral Gables, FL, 1999)*, 301–308, Electron. J. Differ. Equ. Conf., 5, Southwest Texas State Univ., San Marcos, TX, 2000.

[27] R. Pei, C. Ma and J. Zhang, Existence results for asymmetric fractional p-Laplacian problem, *Math. Nachr.*, **290** (2017), 2673–2683.

[28] P. Piersanti and P. Pucci, Existence theorems for fractional p-Laplacian problems *Anal. Appl. (Singap.)*, **15** (2017), 607–640.

[29] P. Pucci and S. Saldi, Multiple solutions for an eigenvalue problem involving non-local elliptic p-Laplacian operators, in *Geometric Methods in PDE's, Springer INdAM Series*, Vol. 11, G. Citti, M. Manfredini, D. Morbidelli, S. Polidoro, F. Uguzzoni, Eds., Springer, Cham, 2015, 159–176.

[30] V. D. Rădulescu and D. D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[31] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.*, **113** (2000), 401–410.

Received April 2021; revised May 2022; early access June 2022.

E-mail address: bostjan.gabrovsek@fs.uni-lj.si
E-mail address: giovanni.molicabisci@uniurb.it
E-mail address: dusan.repovs@guest.arnes.si