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On elliptic problems with Choquard term and singular nonlinearity

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Abstract. Using variational methods, we establish the existence of infinitely many solutions to an elliptic problem driven by a Choquard term and a singular nonlinearity. We further show that if the problem has a positive solution, then it is bounded a.e. in the domain Ω and is Hölder continuous.

Keywords: Choquard term, variational method, dual fountain theorem

1. Introduction

We shall study the following problem:

$$\begin{cases} -\Delta u + V(x)u = \mu(J_\alpha * |u|^p)|u|^{p-2}u + \lambda|u|^{-\beta-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$0 < \beta < 1, \qquad 0 < \alpha < N, \qquad \mu \in \mathbb{R}, \qquad \lambda > 0,$$
$$p \in [2_\alpha, 2_\alpha^*], \qquad 2_\alpha = \frac{N+\alpha}{N}, \qquad 2_\alpha^* = \frac{N+\alpha}{N-2},$$

and V is a continuous function (popularly called the *potential well*), satisfying the following conditions

(V_1) $V : \Omega \to \mathbb{R}$ is continuous and there exists a constant $V_0 > 0$ such that $V(x) > V_0$ for all $x \in \Omega$. Here, $N \geqslant 2$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain,

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(V_2) $V : \Omega \to \mathbb{R}$ satisfies the integrability condition

$$\int_{\Omega} \frac{1}{V(x)}\, dx < \infty.$$

The symbol $(*)$ in problem (1.1) denotes the convolution. We note that both 2_α and 2^*_α are less than $2^* = \frac{2N}{N-2}$, which is the Sobolev critical exponent. The Riesz potential can be described as

$$J_\alpha(x) = \frac{A_\alpha(N)}{|x|^{N-\alpha}}, \qquad A_\alpha(N) = \frac{\Gamma(N-\alpha/2)}{\Gamma(\alpha/2)\pi^{N/2}2^\alpha}.$$

There are many articles pertaining to elliptic nonlocal problems driven by the Choquard term with various assumptions on the potential function V. For example, the readers may refer to Moroz–Schaftingen [8], Carvalho et al. [1], Mukherjee–Sreenadh [9], and the references therein. The consideration of a upper critical Choquard term with singularity and Radon measure, albeit in a bounded domain, can be found in Panda et al. [10], who established the existence of positive solutions.

A systematic study of elliptic problems driven by singular nonlinearity began four decades ago with the seminal work by Lazer–McKenna [5]. This gave a new direction for research to the field of elliptic PDEs. We refer to some of the important works that answer the question of existence of solution to singularity driven elliptic problems in Giacomoni et al. [2], Saoudi et al. [14], and the references therein.

1.1. Significance of the considered problem

Problem (1.1) addresses a wider class of problems in the sense that we have considered $\mu \in \mathbb{R}$. This is an improvement of the existing works in which the authors required the coefficient of a Choquard term to be strictly positive. One can see that if $\lambda = 0$, $\mu = 1$, $V(x) = c$, $N = 3$ in problem (1.1), then we have

$$-\Delta u + cu - \left(\int_{\mathbb{R}^3} u^2(y)V(x-y)\,dy\right)u(x) = 0. \tag{1.2}$$

Here, V is a positive function. This problem was studied by Lions [7] for an unbounded domain $\mathbb{R}^3$. So the study of problem (1.1) is not very new and has been of interest for quite a long time.

Furthermore, when $V(x) \equiv 0$, $\mu = 0$, the problem reduces to the classical problem, similar to the one considered by Lazer–McKenna [5]. However, a combination of the Choquard term and a singular term with $\mu \in \mathbb{R}$ is a new idea. The main results of our paper are stated below.

Theorem 1.1. *For any $\lambda > 0$ and $\mu \in \mathbb{R}$, the functional $I_{\lambda,\mu}$ has a sequence of critical points (u_n) such that $I_{\lambda,\mu}(u_n) < 0$ and $I_{\lambda,\mu}(u_n) \to 0$, as $n \to \infty$.*

Our second main result establishes the boundedness of positive weak solutions of (1.1). Such a result, To the best of our knowledge, this was unknown for elliptic PDEs involving a Choquard term and a singular nonlinearity.

Theorem 1.2. *Let $0 < u \in X$ be a weak solution of problem (1.1). Then $u \in L^\infty(\Omega)$.*

Our third main result shows that such positive solutions are not only bounded but are also Hölder continuous regular.

Theorem 1.3. *Let $u > 0$ be a weak solution of problem (1.1). Then $u \in C^{\theta}_{\mathrm{loc}}(\Omega)$ for some $\theta \in (0, 1)$.*

We complete this section with a description of the structure of our paper. In Section 2, the mathematical preliminaries are discussed. In Section 3, Theorem 1.1 is proved. In Section 4, Theorem 1.2 is proved. In Section 5, Theorem 1.3 is proved. We end the paper in Section 6, with an interesting application of the Choquard term.

2. Preliminaries

We shall seek for a solution in the space X which is defined as follows:

$$X = \left\{ v \in H_0^{1,2}(\Omega) : \int_\Omega V(x) v^2 \, dx < \infty \right\},$$

and endowed with the norm

$$\|v\| = \left(\int_\Omega \left(|\nabla v|^2 + V(x) v^2 \right) dx \right)^{1/2}.$$

We define the inner product on X as follows:

$$\langle \phi, \psi \rangle = \int_\Omega \left(\nabla\phi \cdot \nabla\psi + V(x)\phi\psi \right) dx, \quad \text{where } \phi, \psi \in X. \tag{2.1}$$

A consequence of the well known embedding result is that $X \hookrightarrow L^r(\Omega)$, for each $r \in [1, 2^*]$. Likewise, the embedding $X \hookrightarrow L^r(\Omega)$ is compact for $r \in [1, 2^*)$.

The following is the energy functional corresponding to problem (1.1):

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\mu}{2p} \int_\Omega \left(J_\alpha * \left(u^+(x)\right)^p \right) \left(u^+(x)\right)^p dx - \frac{\lambda}{1-\beta} \int_\Omega \left(u^+(x)\right)^{1-\beta} dx, \quad u \in X. \tag{2.2}$$

However, I is not a C^1 functional and hence the results from the premise of the variational methods cannot be used. However, we shall overcome this difficulty by defining a *cutoff* energy function as follows:

$$\mathcal{C}(s) = \begin{cases} 1, & \text{if } \ |s| \leqslant l, \\ \eta \text{ is decreasing}, & \text{if } \ l \leqslant s \leqslant 2l, \\ 0, & \text{if } \ |s| \geqslant 2l. \end{cases}$$

Clearly, $0 \leqslant \mathcal{C}(s) \leqslant 1$ for every $s \in \mathbb{R}$. Accordingly, we consider the following modified problem

$$(\mathcal{P}) : -\Delta u + V(x)u = \mu g(u) + \lambda |u|^{-\beta-1} u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Here,

$$g\big(u(x)\big)=\big\{\big(J_\alpha*|u(x)|^p\big)|u(x)|^{p-2}u\big\}\mathcal{C}\bigg(\frac{\|u\|^2}{2}\bigg),$$
$$G\big(u(x)\big)=\frac{1}{2p}\big\{\big(J_\alpha*|u(x)|^p\big)|u(x)|^{p}\big\}\mathcal{C}\bigg(\frac{\|u\|^2}{2}\bigg).$$

The corresponding energy functional is thus defined by

$$I_{\lambda,\mu}(u)=\frac{1}{2}\|u\|^2-\frac{\mu}{2p}\int_\Omega\big(J_\alpha*|u(x)|^p\big)|u(x)|^p\,dx-\lambda\int_\Omega G\big(u(x)\big)\,dx,\quad u\in X. \tag{2.3}$$

Corresponding to $I_{\lambda,\mu}$, we define a modified version of problem (1.1) as follows:

Definition 2.1. A function $u\in X$ is said to be a *weak solution* to $(\mathcal{P})$ if

$$\begin{aligned}&\langle u,\phi\rangle-\mu\int_\Omega\big(J_\alpha*|u(x)|^p\big)|u(x)|^{p-2}u\phi(x)\,dx\\&\quad-\frac{\mu}{2p}\mathcal{C}'\bigg(\frac{\|u\|^2}{2}\bigg)\langle u,\phi\rangle\int_\Omega\big(J_\alpha*|u(x)|^p\big)|u(x)|^p\phi(x)\,dx\\&\quad-\lambda\int_\Omega|u|^{-\beta-1}u\phi(x)\,dx=0\quad\text{for every }\phi\in X.\end{aligned} \tag{2.4}$$

We observe that if $\|u\|\leqslant\sqrt{2l}$ and u obeys the identity in (2.4), then u also is a weak solution to problem (1.1). Henceforth, we shall use the word solution instead of weak solution.

We also refer to an important inequality called the Hardy–Littlewood–Sobolev (HLS) inequality (see [8, Lemma 2.1]) which is usually used in the literature to estimate the Choquard term .

In the course of the proof of the main result, we shall need the best Sobolev constant which is defined by

$$S:=\inf_{u\in X\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2\,dx}{(\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}\,dx\,dy)^{1/2^*_\alpha}}.$$

For all other preliminary information we refer the reader to the comprehensive monograph by Papageorgiou et al. [11].

3. Proof of Theorem 1.1

The first step involved in variational methods to verify whether a functional has a critical point or not is to see if the functional exhibits a mountain pass geometry or not. The following lemma verifies this.

Lemma 3.1. *There exists $r>0$ such that* $\inf_{\|u\|=r}I_{\lambda,\mu}(u)>0$.

Proof. We begin by the following estimates

$$I_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 - \frac{\mu}{2p}\int_\Omega \big(J_\alpha * |u(x)|^p\big)|u(x)|^p\,dx - \lambda\int_\Omega G\big(u(x)\big)\,dx \tag{3.1}$$

$$\geqslant \frac{1}{2}\|u\|^2 - \frac{\mu}{2p}\int_\Omega \big(J_\alpha * |u(x)|^p\big)|u(x)|^p\,dx - \lambda\int_\Omega G\big(u(x)\big)\,dx \tag{3.2}$$

$$\geqslant \frac{1}{2}\|u\|^2 - \frac{C\mu}{2p}\|u\|^{2p} - C'\lambda\|u\|^{1-\beta}, \tag{3.3}$$

where the Sobolev embedding result has been used along with the Hardy–Littlewood–Sobolev inequality. Therefore, for a sufficiently small choice of positive numbers $\lambda, r := \|u\|$, we have $I_\lambda(u) > 0$ for any $\|u\| = r$. $\square$

We shall employ the *dual fountain theorem* due to Bartsch–Willem (see [15, Theorem 3.18]) to show that there are infinitely many critical points of the functional $I_{\lambda,\mu}$.

Proof of Theorem 1.1. Let (e_n) be a sequence of orthonormal basis of X and let $X_n := \{ae_n : a \in \mathbb{R}\}$. We shall use the following subspaces in the proof

$$Y_n := \bigoplus_{0\leqslant k\leqslant j} X_k, \qquad Z_k := \overline{\bigoplus_{j\leqslant k<\infty} X_k}.$$

We further define

$$B_n := \sup_{u\in Z_n, \|u\|=1}\int_\Omega |u|^{1-\beta}\,dx.$$

We notice that for a small enough $l > 0$, $0 < R < l/2 < 1$, one has

$$|\mu|\frac{C}{2p}\|u\|^{2p} \leqslant \frac{1}{4}\|u\|^2. \tag{3.4}$$

We further observe that $0 < B_{n+1} \leqslant B_n$ for each n, hence $B_n \to B$ as $n \to \infty$. By the definition of B_n we have for every $n \geqslant 1$, $u_n \in Z_n$, $\|u_n\| = 1$, such that

$$\int_\Omega |u_n|^{1-\beta}\,dx > \frac{B_n}{2}.$$

Also, since $\|u_n\| = 1$, we have $u_n \rightharpoonup 0$ as $n \to \infty$. By the embedding theorem, we can conclude that $B_n \to 0$ as $n \to \infty$.

Thus by (3.4), for all $u \in Z_n$, $\|u\| \leqslant R$, we have

$$I_{\lambda,\mu}(u) \geqslant \frac{1}{2}\|u\|^2 - C\frac{|\mu|}{2p}\|u\|^{2p} - B_n^{1-\beta}\frac{\lambda}{1-\beta}\|u\|^{1-\beta}$$

$$\geqslant \frac{1}{4}\|u\|^2 - B_n^{1-\beta}\frac{\lambda}{1-\beta}\|u\|^{1-\beta}. \tag{3.5}$$

We define $\rho_n := (4\lambda B_n^{1-\beta})^{\frac{1}{1+\beta}} \to 0$ as $n \to \infty$. Thus there exists n_0 such that $\|u\| \leqslant R$ for any $n \geqslant n_0$. Hence, for any $n \geqslant n_0$, $u \in Z_n$, $\|u\| = \rho_n$, we have $I_{\lambda,\mu}(u) \geqslant 0$.

By (3.5) we have

$$I_{\lambda,\mu}(u) \geqslant -\lambda B_n^{1-\beta}\|u_n\|^{1-\beta}/(1-\beta) \geqslant -\lambda B_n^{1-\beta}\rho_n^{1-\beta}/(1-\beta) \tag{3.6}$$

for any $n \geqslant n_0$, $u \in Z_n$, $\|u\| \leqslant \rho_n$. Also, since for any $n \geqslant n_0$ the ρ_n's are sufficiently small, and $\|u\| < \rho_n$, we have $I_{\lambda,\mu}(u) < 0$. Since $B_n \to 0$, $\rho_n \to 0$ as $n \to \infty$, it follows that

$$D_n := \inf_{u\in Z_n, \|u\|\leqslant \rho_n} I_{\lambda,\mu}(u) \to 0$$

as $n \to \infty$. In fact, we name D_{n_0} to be the first member of the sequence (D_n) after which eventually all the D_n's are negative.

Now, since any two norms are topologically equivalent in a finite-dimensional space Y_n, there exists a sufficiently small $r_n > 0$, for $\lambda > 0$, such that

$$\max_{u\in Y_n, \|u\|=r_n} I_{\lambda,\mu}(u) < 0. \tag{3.7}$$

We now show that the functional $I_{\lambda,\mu}$ obeys the Palais–Smale condition. We shall first devise a mechanism to select a suitable sequence in such a way that the functional $I_{\lambda,\mu}$ is still well defined. Let $(u_n) \subset X$ be an eventually zero sequence, then it converges to 0. We shall discard it as the functional may not be defined and moreover such sequences anyway converge to zero.

Furthermore, if $(u_n) \subset X$ is a sequence with infinitely (or finitely) many terms being equal to zero, then we choose a subsequence with only nonzero terms of (u_n). Therefore, without loss of generality, let (u_n) be a sequence such that $u_n \neq 0$ for every $n \in \mathbb{N}$ and also which does not converge to zero in X as $n \to \infty$.

We divide the proof into three cases:

Case 1: Suppose $p \in (2_\alpha, 2_\alpha^*)$. Consider

$$\begin{aligned} c + \sigma\|u_n\| + o(1) &= I_{\lambda,\mu}(u_n) - \frac{1}{2p}\langle I'_{\lambda,\mu}(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2 - \lambda\left(\frac{1}{1-\beta} - \frac{1}{2p}\right)\int_\Omega |u_n|^{1-\beta}\,dx \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2 - \lambda B_0^{1-\beta}\left(\frac{1}{1-\beta} - \frac{1}{2p}\right)\|u_n\|^{1-\beta}. \end{aligned} \tag{3.8}$$

This implies that (u_n) is bounded and hence there exists a subsequence, still denoted as (u_n), such that $u_n \rightharpoonup u$ in X. By the weak convergence $u_n \rightharpoonup u$ and the embedding theorem, we have $\langle I'_{\lambda,\mu}u, u\rangle = 0$. Furthermore, we consider the following

$$\begin{aligned} o(1) = \langle I'_{\lambda,\mu}u_n, u_n - u\rangle &= \|u_n\|^2 - \|u\|^2 - \mu\int_\Omega \left(J_\alpha * |u_n|^p\right)|u_n|^p\,dx + \mu\int_\Omega \left(J_\alpha * |u|^p\right)|u|^p\,dx \\ &\quad - \lambda\int_\Omega |u_n|^{1-\beta}\,dx + \lambda\int_\Omega |u_n|^{-\beta-1}u_n u\,dx = \|u_n\|^2 - \|u\|^2. \end{aligned}$$

We have again used a combination of the embedding theorems and the Egorov's theorem above. Thus $u_n \to u$ in X and hence the (PS) condition is satisfied by the functional $I_{\lambda,\mu}$.

Case 2: Suppose $p = 2^*_\alpha$. We shall show that $I_{\lambda,\mu}$ satisfies the Palais–Smale $(PS)_c$ condition for energy level

$$c < c^* := \frac{1}{2}\left(\frac{\alpha+2}{N+\alpha}\right)S^{\frac{N-2}{\alpha+2}} - \frac{1}{2}\left(\frac{N+\alpha}{\alpha+2}\right)^{\frac{1+\beta}{1-\beta}}\left(\lambda C\left(\frac{1}{1-\beta} - \frac{1}{2\cdot 2^*_\alpha}\right)\right)^{\frac{2}{1+\beta}} \quad \text{for any } \lambda > 0.$$

We again consider $(u_n) \subset X$ such that $I_{\lambda,\mu}(u_n) \to c < c^*$, $I'_{\lambda,\mu}(u_n) \to 0$. Suppose $\lim_{n\to\infty}\|u_n - u\|^2 = l^2 > 0$. Then

$$\begin{aligned} c + \sigma\|u_n\| + o(1) &= I_{\lambda,\mu}(u_n) - \frac{1}{2\cdot 2^*_\alpha}\langle I'_{\lambda,\mu}(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_\alpha}\right)\|u_n\|^2 - \lambda\left(\frac{1}{1-\beta} - \frac{1}{2\cdot 2^*_\alpha}\right)\int_\Omega \left(u_n^+\right)^{1-\beta}\,dx \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_\alpha}\right)\|u_n\|^2 - \lambda B_0^{1-\beta}\left(\frac{1}{1-\beta} - \frac{1}{2\cdot 2^*_\alpha}\right)\|u_n\|^{1-\beta}. \end{aligned} \tag{3.9}$$

It follows from (3.9) that the sequence (u_n) is bounded in X. Therefore by the reflexivity of X, there exists a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u$ as $n \to \infty$. It is straightforward to show that u is a weak solution to the modified problem $(\mathcal{P})$ and hence is a weak solution to (1.1), i.e. $\langle I'_{\lambda,\mu}(u), u\rangle = 0$.

Furthermore, invoking the weak convergence of (u_n), the Brezis–Lieb lemma, the embedding theorems, and the Egorov theorem, we can make the following observation

$$\begin{aligned} o(1) &= \langle I_{\lambda,\mu}(u_n), u_n\rangle \\ &= \|u_n\|^2 - \mu\int_\Omega \left(J_\alpha * |u_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha}\,dx - \lambda\int_\Omega |u_n|^{1-\beta}\,dx \\ &= \|u_n - u\|^2 + \|u\|^2 - \mu\int_\Omega \left(J_\alpha * |u_n - u|^{2^*_\alpha}\right)|u_n - u|^{2^*_\alpha}\,dx + \mu\int_\Omega \left(J_\alpha * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha}\,dx \\ &\quad - \lambda\int_\Omega |u|^{1-\beta}\,dx \\ &= \langle I_{\lambda,\mu}(u), u\rangle + \|u_n - u\|^2 - \mu\int_\Omega \left(J_\alpha * |u_n - u|^{2^*_\alpha}\right)|u_n - u|^{2^*_\alpha}\,dx \\ &= \|u_n - u\|^2 - \mu\int_\Omega \left(J_\alpha * |u_n - u|^{2^*_\alpha}\right)|u_n - u|^{2^*_\alpha}\,dx. \end{aligned} \tag{3.10}$$

Let

$$\lim_{n\to\infty}\|u_n - u\|^2 = \lim_{n\to\infty}\int_\Omega \left(J_\alpha * |u_n - u|^{2^*_\alpha}\right)|u_n - u|^{2^*_\alpha}\,dx = l.$$

From the definition of the best Sobolev constant given in Section 2, we have

$$S \leqslant \frac{l}{l^{1/2_\alpha^*}}.$$

Hence, by an application of the Young inequality in (3.9), we get

$$\begin{aligned} c + o(1) &\geqslant \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right)\left(l^2 + \|u_n\|^2\right) - \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right)\|u_n\|^2 \\ &\quad - \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right)^{-\frac{1-\beta}{1+\beta}} \left(\lambda C\left(\frac{1}{1-\beta} - \frac{1}{2 \cdot 2_\alpha^*}\right)\right)^{\frac{2}{1+\beta}} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right) S^{2 \cdot \frac{N+\alpha}{\alpha+2}} - \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right)^{-\frac{1-\beta}{1+\beta}} \left(\lambda C\left(\frac{1}{1-\beta} - \frac{1}{2 \cdot 2_\alpha^*}\right)\right)^{\frac{2}{1+\beta}}. \end{aligned} \tag{3.11}$$

This is a contradiction the energy level, below which the $(PS)_c$ condition holds. Hence,

$$\lim_{n \to \infty} \|u_n - u\|^2 = 0.$$

Case 3: The obeying of $(PS)_c$ condition by the functional $I_{\lambda,\mu}$ for the case of $p = 2_\alpha$ follows in a similar manner as in *Case 2*. In fact, it holds for

$$c < c^{**} := \frac{1}{2}\left(\frac{\alpha}{N+\alpha}\right) S^{2\frac{N+\alpha}{\alpha}} - \frac{1}{2}\left(\frac{N+\alpha}{\alpha}\right)^{\frac{1+\beta}{1-\beta}} \left(\lambda C\left(\frac{1}{1-\beta} - \frac{1}{2p}\right)\right)^{\frac{2}{1+\beta}} \quad \text{for any } \lambda > 0.$$

Therefore for the critical cases of $p = 2_\alpha, 2_\alpha^*$, the (PS) condition holds for $c < \min\{c^*, c^{**}\}$ for any $\lambda > 0$.

Since all the hypothesis of the dual fountain theorem are satisfied, we conclude the existence of a sequence of negative critical values that converge to 0. Hence $I_{\lambda,\mu}$ has infinite number of solutions and consequently (1.1) has infinitely many solutions. □

4. Proof of Theorem 1.2

We shall prove that all nonnegative solutions of (1.1) are bounded. The sketch of the proof is as follows (it runs along the lines of the argument in [9]).

Proof of Theorem 1.2. Let $u \in X$ be a nonnegative solution with $|\{x \in \Omega : u(x) = 0\}| = 0$. We shall prove the boundedness for the case of $p = 2_{\alpha^*}$. The cases when $p \in (2_\alpha, 2_\alpha^*)$ and $p = 2_\alpha$ follow the same argument. We define $w(x) := \frac{u(x)}{d}$ which, for a suitable $d > 0$, obeys

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \phi \, dx + \int_\Omega V(x) u \phi \, dx \\ &\quad \leqslant |\mu| \int_\Omega \left(J_\alpha * \left(v(x)\right)^{2_\alpha^*}\right)\left(v(y)\right)^{2_\alpha^*-2} v(y)\phi(x)\, dx + \lambda \int_\Omega (v)^{-\beta} \phi \, dx \end{aligned} \tag{4.1}$$

for every nonnegative $\phi \in C_c^\infty(\Omega)$. We further define the following:

$$D_n := 1 - 3^{-n}, \qquad w_n := w - D_n, \qquad v_n := w_n^+, \qquad V_n := \|v_n\|_{2^*}. \tag{4.2}$$

Thus $0 \leqslant |w| + D_n \leqslant |w| + 1$. Of course, $|w| + 1 \in L^2(\Omega) \subset L^{2^*}(\Omega)$ since Ω is bounded. Moreover, $\lim_{n\to\infty} v_n = (w-1)^+$. By the dominated convergence theorem we get

$$\lim_{n\to\infty} V_n = \left(\int_\Omega \left[(w-1)^+\right]^{2^*dx}\right)^{1/2^*}. \tag{4.3}$$

From the definitions above we make the following observations:

$$D_n < D_{n+1}, \quad \text{which implies } v_{n+1} \leqslant v_n \text{ a.e. in } \Omega. \tag{4.4}$$

Furthermore, we define

$$B_n := \frac{D_{n+1}}{D_{n+1} - D_n} = (1/2)\left(3^{n+1} - 1\right) < \left(3^{n+1} - 1\right) \quad \text{for all } n \in \mathbb{N}.$$

We make a similar observation as in [9] that

$$w < B_n v_n \quad \text{on } \{v_{n+1} > 0\}. \tag{4.5}$$

Employing (4.1), (4.5) in tandem with Hölder's inequality, we obtain

$$\begin{aligned}
\int_\Omega |\nabla v_{n+1}|^2\,dx &\leqslant \int_\Omega \nabla v_{n+1} \cdot \nabla v_{n+1}^+ \leqslant \int_\Omega \nabla w \cdot \nabla v_{n+1}^+\,dx \\
&\leqslant B_n^{2_\alpha^*-1}|\mu| \int_{\{v_{n+1}>0\}} \int_\Omega \frac{(v(y))^{2_\alpha^*}|v_n(x)|^{2_\alpha^*-1} v_{n+1}(x)}{|x-y|^{N-\alpha}}\,dy\,dx \\
&\quad + \lambda B_n \int_{\{v_{n+1}>0\}} |v_n|^2\,dx \\
&\leqslant B_n^{2_\alpha^*-1}|\mu| \int_{\{v_{n+1}>0\}} \int_\Omega \frac{(v(y))^{2_\alpha^*}|v_n(x)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}\,dy\,dx \\
&\quad + \lambda B_n \|v_n\|_{2^*}^{1-\beta} \left|\{v_{n+1} > 0\}\right|^{\frac{N+2+\beta(N-2)}{2N}}.
\end{aligned} \tag{4.6}$$

We estimate the first integral on the right hand side by dividing it into a sum of two integrals as follows:

$$\begin{aligned}
&\int_{\{v_{n+1}>0\}} \int_\Omega \frac{(v(y))^{2_\alpha^*}|v_n(x)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}\,dy\,dx \\
&\quad \leqslant \left(\int_{\{v_{n+1}>0\}} \int_{\{v(y)\leqslant D_{n+1}\}} + \int_{\{v_{n+1}>0\}} \int_{\{v(y)>D_{n+1}\}}\right) \frac{(v(y))^{2_\alpha^*}|v_n(x)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}\,dy\,dx := \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{4.7}$$

The HLS in equality and the Hölder inequality yields

$$\begin{aligned} \mathcal{I}_1 &\leqslant B_n^{2_\alpha^*} C(N,\alpha)\|v_n\|_{2^*}^{2\cdot 2_\alpha^*}, \\ \mathcal{I}_2 &\leqslant C B_{n+1}^{2_\alpha^*}\big|\{v_{n+1}>0\}\big|^{(N-\alpha)/2N}\|v_n\|_{2^*}^{2_\alpha^*}. \end{aligned} \tag{4.8}$$

The embedding theorem in combination with (4.6) and (4.8) yields the following:

$$\begin{aligned} S\|v_{n+1}\|_{2^*}^2 \leqslant B_n^{2_\alpha^*-1}\big(&\mu B_n^{2_\alpha^*} C(N,\alpha)\|v_n\|_{2^*}^{2\cdot 2_\alpha^*} + \mu C D_{n+1}^{2_\alpha^*}\big|\{v_{n+1}>0\}\big|^{(N-\alpha)/2N}\|v_n\|_{2^*}^{2_\alpha^*} \\ &+ \lambda B_n 3^{n+1}\|v_n\|_{2^*}^{1-\beta}\big|\{v_{n+1}>0\}\big|^{\frac{N+2+\beta(N-2)}{2N}}\big), \end{aligned} \tag{4.9}$$

where

$$S = \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{2/2^*}}.$$

Again, by following the proof in [9], we get that

$$\{v_{n+1}>0\} \subset \{v_n > 1/3^{n+2}\}. \tag{4.10}$$

Therefore by (4.10) we get

$$V_n^{2^*} = \|v_n\|_{2^*} \geqslant \int_{\{v_n>3^{-n-2}\}} v_n^{2^*}\,dx \geqslant 3^{-n-2}\big|\{v_{n+1}>0\}\big| \tag{4.11}$$

and

$$\begin{aligned} S\|v_{n+1}\|_{2^*}^2 \leqslant B_n^{2_\alpha^*-1}\big(&\mu B_n^{2_\alpha^*} C(N,\alpha)\|v_n\|_{2^*}^{2\cdot 2_\alpha^*} + \mu C D_{n+1}^{2_\alpha^*} 3^{(n+2)(N-\alpha)/2N}\|v_n\|_{2^*}^{2^*} \\ &+ \lambda B_n 3^{n+2}\|v_n\|_{2^*}^{2^*} 2^{(n+1)\frac{N+2+\beta(N-2)}{2N}}\big) \leqslant 3^{(2_\alpha^*-1)(n+2)}\mathcal{C}\big(\|v_n\|_{2^*}^{2\cdot 2_\alpha^*} + \|v_n\|_{2^*}^{2^*}\big), \end{aligned} \tag{4.12}$$

where

$$\mathcal{C} := \max\big\{3^{2_\alpha^*(n+2)}C(N,\alpha), 3^{(n+2)(N-\alpha)/2N}C + 3^{(n+2)(3N+2+\beta(N-2))/2N}\big\}.$$

Thus by the definition of V_n in (4.2), we have

$$V_{n+1} = \mathcal{D}^{n+2}\big(V_n^{2_\alpha^*} + V_n^{2^*/2}\big), \tag{4.13}$$

where $\mathcal{D} = (1+\sqrt{\mathcal{C}^{1/n+2}}) > 1$. We choose $\epsilon > 0$ sufficiently small so that

$$\epsilon^{2/N-2} < \frac{1}{(3^{2_\alpha^*}\mathcal{D})^{N-2/2}}. \tag{4.14}$$

On fixing $\gamma \in (\epsilon^{2/N-2}, \frac{1}{3^{2^*_\alpha}\mathcal{D}^{N-2/2}})$, we conclude that $\gamma \in (0,1)$ since $N/N-2, \mathcal{D} > 1$. Also

$$\epsilon^{2/N-2} \leqslant \gamma, \qquad 3^{2^*_\alpha}\mathcal{D}\gamma^{N/N-2} \leqslant 1. \tag{4.15}$$

It can be proved by induction that

$$V_{n+1} \leqslant 2\epsilon\gamma^{n+1}. \tag{4.16}$$

From the fact that $\gamma \in (0,1)$ and by (4.16) we see that $\lim_{n\to\infty} V_n = 0$. Thus by (4.3), we have $(w-1)^+ = 0$ a.e. in Ω. This implies that $0 \leqslant w \leqslant 1$ a.e. in Ω. Hence, $0 \leqslant u \leqslant d$ a.e. in Ω. Therefore $\|u\|_\infty \leqslant c$. □

5. Proof of Theorem 1.3

Finally, we prove a regularity result for the positive solutions of problem (1.1).

Proof of Theorem 1.3. Let $\tilde{\Omega} \Subset \Omega$. Thus for any $\phi \in C_c^\infty(\Omega)$, we have, by the boundedness of u, the following

$$\begin{aligned}\int_\Omega \big(\nabla u\cdot\nabla\phi + V(x)u\phi\big)\,dx &= \mu\int_{\tilde{\Omega}} \big(J_\alpha * \big(u(x)\big)^{2^*_\alpha}\big)\big(u(x)\big)^{2^*_\alpha-2}u(x)\phi(x)\,dx + \lambda\int_{\tilde{\Omega}} u^{-\beta}\phi\,dx\\ &\leqslant |\mu|\int_{\tilde{\Omega}} \big(J_\alpha * \big(u(x)\big)^{2^*_\alpha}\big)\big(u(x)\big)^{2^*_\alpha-2}u(x)\phi(x)\,dx + \lambda\int_{\tilde{\Omega}} u^{-\beta}\phi\,dx\\ &\leqslant C\int_{\tilde{\Omega}}\phi\,dx.\end{aligned}$$

Here we have used the fact that $u \geqslant C_K > 0$ a.e. for any compact $K \subset \Omega$. For if not, then the integral with the singular term ceases to exists. Therefore, $(-\Delta)u + V(x)u$ is weakly bounded in $\tilde{\Omega}$.

By Iannizzotto et al. [4, Theorem 4.4] and the covering argument applied to the inequality in of [4, Corollary 5.5] we produce a $\theta \in (0,1)$ such that $u \in C^\theta(\tilde{\Omega})$, for any $\tilde{\Omega} \Subset \Omega$. Hence $u \in C^\theta_{\text{loc}}(\tilde{\Omega})$. □

We end this section with the following remark.

Remark 5.1. We observe that if $u > 0$ is a solution to (1.1), then $-u < 0$ is also a solution which is bounded and Hölder continuous.

6. Application of the Choquard term

In his seminal work, Lieb [6] pointed out that Choquard used this term to approximate the Hartree–Fock theory of component plasma. For the case

$$N = 3, \qquad \alpha = 2, \qquad p = 2, \qquad \lambda = 0, \qquad V(x) \equiv 0,$$

problem (1.1) was used by Peker [12] to study the quantum theory of a polaron at rest. The Choquard term also found its application in multiparticle system courtesy Gross [3] and in quantum mechanics Penrose [13].

For each solution of problem (1.1), we define $\phi(t,x) = e^{it}u(x)$ which is called the wave function, where $i = \sqrt{-1}$. Here, ϕ is a solitary wave of the focusing time-dependent Hartree equation

$$-i\phi_t - \Delta\phi + W(x)\phi - \big(J_\alpha * |\phi|^p\big)|\phi|^{p-2}\phi - \lambda|u|^{q-2}u = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

where $W(x) = V(x) - 1$, $x \in \mathbb{R}^N$. Thus problem (1.1) can be interpreted as the stationary nonlinear Hartree equation.

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