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On the Schrödinger–Poisson system with (p, q)–Laplacian

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ABSTRACT

We study a class of Schrödinger–Poisson systems with (p, q)–Laplacian. Using fixed point theory, we obtain a new existence result for nontrivial solutions. The main novelty of the paper is the combination of a double phase operator and the nonlocal term. Our results generalize some known results.

1. Introduction

In this article, we shall study the following Schrödinger–Poisson system with (p, q)–Laplacian

$$\begin{cases} -\Delta_p u - \Delta_q u + (|u|^{p-2} + |u|^{q-2})u - \phi|u|^{q-2}u = h(x,u) + \lambda g(x) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^q & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where Δ_ς =div$(|\nabla u|^{p-2}\nabla u)$ is the ς-Laplacian ($\varsigma = p, q$), $\frac{3}{4} < p < q < 3$, λ is a positive parameter, the nonnegative function $g \in L^{\frac{3q}{4q-3}}(\mathbb{R}^3)$ is a perturbation term, and $g(x) \not\equiv 0$. Here, $h : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and it satisfies certain assumptions.

Our study of problem (1.1) was motivated by two main reasons. On the one hand, when $p = q = 2$ and $\lambda \equiv 0$, problem (1.1) becomes the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u - \phi u = h(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

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System (1.2) depicts how charged particles interact with the motion electromagnetic field. While the nonlocal term ϕu describes interactions with the electric field, the nonlinear term models interactions between the particles. By virtue of its strong physical background, system (1.2) has drawn wide attention in recent decades. For $p = q \neq 2$, system (1.1) was studied for the first time by Du et al. [1] and the existence of nontrivial solutions of the system was obtained by invoking the Mountain Pass Theorem. For the quasilinear Schrödinger–Poisson system, we refer to Du et al. [2]. Readers interested in learning more about the results on the Schrödinger–Poisson system using the variational methods, are referred to Ambrosetti–Ruiz [3], D'Aprile–Mugnai [4], Ruiz [5] and the references therein.

On the other hand, when $p \neq q$, problem (1.1) is driven by a differential operator with unbalanced growth. When problem (1.1) without the nonlocal term ϕu becomes a $p\&q$-Laplacian equation

$$-\Delta_p u - \Delta_q u + (|u|^{p-2} + |u|^{q-2})u - \phi|u|^{q-2}u = h(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

this problem has a rich physical background, since the double phase operator has been applied to describe steady-state solutions of reaction diffusion problems in biophysics, plasma physics, and chemical reaction analysis. Using the variational methods, some results for problem (1.3) can be found in Bartolo et al. [6], Figueiredo [7], Papageorgiou et al. [8], and the references therein.

Inspired by the above references, we prove in this paper the existence of nontrivial solutions for problem (1.1) by using fixed point theory. Although some authors have already used fixed point theory, see Carl–Heikkilä [9], de Souza [10], and Tao–Zhang [11,12], as far as we know, problem (1.1) has not been studied before. Because of the occurrence of a nonhomogeneous term, we can prove that a weak solution to problem (1.1) exists by the fixed point theory. The results in this paper can be regarded as an extension of results in Du et al. [1,2] and Tao–Zhang [11,12]. In some sense, our results are new, even in the $p = q$ case.

Our existence result, which is the main result of this paper, can be stated as follows.

Theorem 1.1. *Assume that $h(x,u) : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^+ := [0, +\infty)$ is a nondecreasing function in u, and $h(x,u) = 0$ when $u < 0$. Moreover, assume that it satisfies the following condition*

$$|h(x,t)| \le d_1(x)|t|^{\tau-1} + d_2(x)|t|^{q^*-1}, \text{ for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.4}$$

where $q \le \tau < q^* := \frac{3p}{3-p}$, $0 \le d_1 \in L^\eta(\mathbb{R}^3)$, $0 \le d_2 \in L^\infty(\mathbb{R}^3)$, and $\eta = \frac{6}{6-\tau}$. Then there exists $\lambda_0 > 0$ such that for every $0 < \lambda \leqslant \lambda_0$, problem (1.1) has a positive solution.

Remark 1.1. We point out that there are many functions that satisfy the assumptions of Theorem 1.1. For example, we can take $h(x,t) = \frac{1}{1+x^2}|t|^{q^*-1}$.

2. Preliminaries

In this section, we shall present some preliminary results, as well as some notations and useful results. To this end, let W be the subspace of $W^{1,p}(\mathbb{R}^3)$ and $W^{1,q}(\mathbb{R}^3)$, defined by $W = W^{1,p}(\mathbb{R}^3) \bigcap W^{1,q}(\mathbb{R}^3)$, with respect to the norm $\|u\| = \|u\|_{W^{1,p}(\mathbb{R}^3)} + \|u\|_{W^{1,q}(\mathbb{R}^3)}$. Since $W^{1,r}(\mathbb{R}^3)$, with $1 < r < \infty$, is a separable reflexive Banach space, we deduce that W is a separable reflexive Banach space. Moreover, we also know that the embeddings $W \hookrightarrow L^p(\mathbb{R}^3), L^q(\mathbb{R}^3)$ are continuous. On the other hand, according to Du et al. [2], for any given $u \in W^{1,q}(\mathbb{R}^3)$, there exists a unique

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^q}{|x-y|}\mathrm{d}y, \quad \phi_u \in D^{1,2}(\mathbb{R}^3),$$

satisfying $-\Delta\phi_u = |u|^q$.

We now summarize the properties of ϕ_u which will be used later.

Lemma 2.1 (*Du et al. [2]*). *Let $u \in W^{1,q}(\mathbb{R}^3)$. Then the following properties hold:*

(1) $\phi_u \geqslant 0$, *for all* $x \in \mathbb{R}^3$;

(2) *For any* $t \in \mathbb{R}^+, \phi_{tu} = t^q \phi_u$, *and* $\phi_{u_t} t^{kq-2} \phi_u(tx)$ *with* $u_t(x) = t^k u(tx)$;

(3) $\|\phi_u\|_{D^{1,2}} \leqslant C\|u\|^q$, *where* C *is independent of* u;

(4) *If* $u_n \rightharpoonup u$ *in* $W^{1,q}(\mathbb{R}^3)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$, *and* $\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{q-2}u_n\varphi \mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u|u|^{q-2}u\varphi \mathrm{d}x$, *for all* $\varphi \in W^{1,q}(\mathbb{R}^3)$.

Substituting $\phi = \phi_u$ into system (1.1), we can rewrite (1.1) as a single equation

$$-\Delta_p u - \Delta_q u + (|u|^{p-2}u + |u|^{q-2}u) - \phi_u|u|^{q-2}u = h(x,u) + \lambda g(x), \text{ for all } u \in W. \tag{2.1}$$

We define the energy functional I on W by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^3}\left(|\nabla u|^p + |u|^p\right)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^3}\left(|\nabla u|^q + |u|^q\right)\mathrm{d}x - \frac{1}{2q}\int_{\mathbb{R}^3}\phi_u|u|^q\mathrm{d}x - \int_{\mathbb{R}^3}\Big(H(x,u) + \lambda g(x)\Big)\mathrm{d}x,$$

where $H(x,t) = \int_0^t h(x,s)ds$. It is straightforward to show that $I \in C^1(W,\mathbb{R})$ and

$$\begin{aligned}\langle I'(u), \psi\rangle = &\int_{\mathbb{R}^3}\left(|\nabla u|^{p-2}\nabla u\nabla\psi + |u|^{p-2}u\psi\right)\mathrm{d}x + \int_{\mathbb{R}^3}\left(|\nabla u|^{q-2}\nabla u\nabla\psi + |u|^{q-2}u\psi\right)\mathrm{d}x\\ &- \int_{\mathbb{R}^3}\phi_u|u|^{q-2}u\psi\mathrm{d}x - \int_{\mathbb{R}^3}\Big(h(x,u) + \lambda g(x)\Big)\psi\mathrm{d}x.\end{aligned}$$

It is easy to verify that $(u, \phi_u) \in W \times D^{1,2}(\mathbb{R}^3)$ is a solution of system (1.1) if and only if $u \in W$ is a critical point of I.

Now, we introduce the necessary fixed-point theorem due to Carl–Heikkilä [9], which plays a crucial role in proving our conclusions. For this, let $\mathcal{E}$ be a real Banach space. A nonempty subset $\mathcal{E}_+ \neq \{0\}$ of $\mathcal{E}$ is called an order cone if it satisfies the following conditions: (a) $\mathcal{E}_+$ is closed and convex; (b) if $v \in \mathcal{E}_+$ and $\delta \geqslant 0$, then $\delta v \in \mathcal{E}_+$; (c) if $v \in \mathcal{E}_+$ and $-v \in \mathcal{E}_+$, then $v = 0$. An order cone $\mathcal{E}_+$ induces a partial order in W in the following way: $x \preceq y$ and only if $y - x \in \mathcal{E}_+$, and $(W, \preceq)$ is called an ordered Banach space. If $\inf\{x,y\}$ and $\sup\{x,y\}$ exist for all $x, y \in W$ with respect to $\preceq$, then $(W, \|\cdot\|)$ is called a lattice. In addition, if $\|x^{\pm}\| \leqslant \|x\|$ for each $x \in W$, where $x^+ := \sup\{0,x\}$ and $x^- := -\inf\{0,x\}$, then $(W, \|\cdot\|)$ is a Banach semilattice. We also note that the dual space W' of W has the following partial order:

$$\varphi_1, \varphi_2 \in W', \varphi_1 \lhd \varphi_2 \Leftrightarrow \langle\varphi_1, v\rangle \leqslant \langle\varphi_2, w\rangle, \text{ for all } w \in \mathcal{E}_+.$$

Next, we give the definition of fixed point property, which, according to Carl–Heikkilä [9], is the following one: P is said to have a fixed point property if each increasing mapping $G : P \to P$ has a fixed point.

Proposition 2.1 (*Carl–Heikkilä [9, Corollary 2.2]*). *Let W be a reflexive Banach semilattice. Then every closed ball in W has the fixed point property.*

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first prove some key lemmas. To begin, we define the functional $\mathcal{B} : W \to W'$ by $\langle\mathcal{B}u, v\rangle = \int_{\mathbb{R}^3}(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv)\mathrm{d}x + \int_{\mathbb{R}^3}(|\nabla u|^{q-2}\nabla u\nabla v + |u|^{q-2}uv)\mathrm{d}x$. Clearly, $\mathcal{B}u$ is linear for all $u \in W$. This means that the Hölder inequality holds

$$|\langle\mathcal{B}u, v\rangle| \leqslant C_1\|u\|^{p-1}\|v\| + C_2\|u\|^{q-1}\|v\|, \quad \text{for some } C_1, C_2 > 0.$$

Therefore $\mathcal{B}u \in W'$ and $\mathcal{B}$ are well-defined. In addition, we have the following property of $\mathcal{B}$.

Lemma 3.1. *The operator $\mathcal{B}: W \to W'$ is continuous and invertible.*

Proof. Let $\{u_k\}$ in W be such that $u_k \to u$ in W. Using the Hölder inequality for $v \in W$ with $\|v\| \leqslant 1$, we have

$$\|\mathcal{B}u_k - \mathcal{B}u\|_{W'} = \sup_{v\in W, \|v\|\leqslant 1} |\langle \mathcal{B}u_k - \mathcal{B}u, v\rangle| \leqslant \|u_k - u\|^p_{W^{1,p}(\mathbb{R}^3)} + \|u_k - u\|^q_{W^{1,q}(\mathbb{R}^3)} \to 0.$$

This means that the operator $\mathcal{B}$ is continuous. Considering $p, q \geqslant 2$ and $\langle \mathcal{B}u, u\rangle = \|u\|^p + \|u\|^q$ for all $u \in W$, we have $\lim_{\|u\|\to\infty} \frac{\langle \mathcal{B}u,u\rangle}{\|u\|} = \infty$. It well-known that

$$(|a|^{s-2}a - |b|^{s-2}b)(a-b) \geqslant C_p|a-b|^{s-2}, \text{ for all } s \geq 2, a, b \in \mathbb{R},$$

and we have $\langle \mathcal{B}u_1 - \mathcal{B}u_2, u_1 - u_2\rangle > 0$, for all $u_1, u_2 \in W,\ u_1 \neq u_2$. Therefore, by the Minty–Browder Theorem (see [13, Theorem 5.16]), we obtain that the operator W is reversible. Hence the proof of Lemma 3.1 is complete. □

Similar to the proof of Lemma 3.2 in [11], we can show that $\mathcal{B}^{-1} : (W', \lhd) \to (W, \preceq)$ is increasing.

Next, inspired by [14], let the operator $\mathcal{T} : W \to W'$ be defined by

$$\langle \mathcal{T}u, v\rangle = \int_{\mathbb{R}^3} \Big(\phi_{u^+}|u^+|^{q-2}u^+ + h(x, u^+) + \lambda g(x)\Big) v \mathrm{d}x, \text{ for all } u, v \in W,$$

where $u^+ := \max\{u, 0\}$ and $u^- := -\min\{u, 0\}$. By the Hölder inequality, the Sobolev Embedding Theorem, and the Hardy–Littlewood–Sobolev inequality, there exist some positive constants C^*, C^{**} and C^{***} such that

$$|\langle \mathcal{T}u, v\rangle| \leq \left(C^*\|u^+\|^{2q-1} + C^{**}\|d_1\|_\eta\|u^+\|^{\tau-1} + C^{***}\|d_2\|_\infty\|u^+\|^{q^*-1} + \lambda\|g\|_{\frac{3q}{4q-3}}\right)\|v\|. \tag{3.1}$$

Let $\mathcal{G} := \mathcal{B}^{-1} \circ \mathcal{T}$. Then we have the following result.

Lemma 3.2. *Under the hypotheses of Theorem 1.1, for any $0 < \lambda \leqslant \lambda_0$, there exists $R > 0$, such that $\mathcal{G}(\mathbb{B}_W[0, R]) \subset \mathbb{B}_W[0, R]$, where $\mathbb{B}_W[0, R] = \{u \in W : \|u\| \leqslant R\}$.*

Proof. Let $u \in W$, $v = (\mathcal{B}^{-1} \circ \mathcal{T})u = \mathcal{G}u$. We note that $\langle \mathcal{B}v, v\rangle = \|v\|^p_{W^{1,p}(\mathbb{R}^3)} + \|v\|^q_{W^{1,q}(\mathbb{R}^3)}$. We consider 3 possible cases:

Case 1. $\|v\|_{W^{1,q}(\mathbb{R}^3)} \geq 1$. Then $\|v\|^q_{W^{1,q}(\mathbb{R}^3)} \geq \|v\|^p_{W^{1,q}(\mathbb{R}^3)}$, hence

$$\langle \mathcal{B}v, v\rangle \geqslant \|v\|^p_{W^{1,p}(\mathbb{R}^3)} + \|v\|^p_{W^{1,q}(\mathbb{R}^3)} \geqslant 2^{1-p}(\|v\|_{W^{1,q}(\mathbb{R}^3)} + \|v\|_{W^{1,p}(\mathbb{R}^3)})^p = 2^{1-p}\|v\|^p. \tag{3.2}$$

Case 2. $\|v\|_{W^{1,q}(\mathbb{R}^3)} < 1$ and $\|v\|_{W^{1,p}(\mathbb{R}^3)} \geq 1$. Then $\|v\|_{W^{1,p}(\mathbb{R}^3)} \geq 1 > \|v\|_{W^{1,q}(\mathbb{R}^3)}$. Since $\|v\| = \|v\|_{W^{1,q}(\mathbb{R}^3)} + \|v\|_{W^{1,p}(\mathbb{R}^3)}$, we get $2\|v\|_{W^{1,q}(\mathbb{R}^3)} \leq \|v\| \leq 2\|v\|_{W^{1,p}(\mathbb{R}^3)}$, therefore

$$\langle \mathcal{B}v, v\rangle = \|v\|^p_{W^{1,p}(\mathbb{R}^3)} + \|v\|^q_{W^{1,q}(\mathbb{R}^3)} \geqslant \|v\|^p_{W^{1,p}(\mathbb{R}^3)} \geqslant \frac{1}{2^p}\|v\|^p. \tag{3.3}$$

Case 3. $\|v\|_{W^{1,q}(\mathbb{R}^3)} < 1$ and $\|v\|_{W^{1,p}(\mathbb{R}^3)} < 1$. Then $\|v\|^q_{W^{1,p}(\mathbb{R}^3)} \leqslant \|v\|^p_{W^{1,p}(\mathbb{R}^3)}$, therefore

$$\langle \mathcal{B}v, v\rangle \geqslant \|v\|^q_{W^{1,p}(\mathbb{R}^3)} + \|v\|^q_{W^{1,q}(\mathbb{R}^3)} \geqslant 2^{1-q}(\|v\|_{W^{1,q}(\mathbb{R}^3)} + \|v\|_{W^{1,p}(\mathbb{R}^3)})^q = 2^{1-q}\|v\|^q. \tag{3.4}$$

From (3.2), (3.3) and (3.4), we have

$$\langle \mathcal{B}v, v\rangle \geqslant 2^{-p}\|v\|^p \quad \text{or} \quad \langle \mathcal{B}v, v\rangle \geqslant 2^{1-q}\|v\|^q. \tag{3.5}$$

On the other hand, we have

$$\|\mathcal{G}u\|^p_{W^{1,p}(\mathbb{R}^3)} + \|\mathcal{G}u\|^q_{W^{1,q}(\mathbb{R}^3)} = \langle \mathcal{T}u, \mathcal{G}u\rangle \leq \|\mathcal{T}u\|_{W'}\|\mathcal{G}u\|. \tag{3.6}$$

If $\|u\| \leqslant R$, then by (3.1), (3.5) and (3.6), one has

$$\begin{aligned} 2^{-p}\|\mathcal{G}u\|^{p-1} &\leqslant \|\mathcal{T}u\|_{W'} \leqslant C^*\|u\|^{2q-1} + C^{**}\|d_1\|_\eta\|u\|^{\tau-1} + C^{***}\|d_2\|_\infty\|u\|^{q^*-1} + \lambda\|g\|_{\frac{3q}{4q-3}} \\ &\leqslant C^*R^{2q-1} + C^{**}\|d_1\|_\eta R^{\tau-1} + C^{***}\|d_2\|_\infty R^{q^*-1} + \lambda\|g\|_{\frac{3q}{4q-3}} \end{aligned}$$

and

$$2^{1-q}\|\mathcal{G}u\|^{q-1} \leqslant C^*R^{2q-1} + C^{**}\|d_1\|_\eta R^{\tau-1} + C^{***}\|d_2\|_\infty R^{q^*-1} + \lambda\|g\|_{\frac{3q}{4q-3}}.$$

From this, we obtain

$$\frac{\|\mathcal{G}u\|^{p-1}}{R^{p-1}} \leqslant 2^pC^*R^{2q-p} + 2^pC^{**}\|d_1\|_\eta R^{\tau-p} + 2^pC^{***}\|d_2\|_\infty\|d_2\|_\infty R^{q^*-p} + 2^p\lambda\frac{\|g\|_{\frac{3q}{4q-3}}}{R^{p-1}}. \tag{3.7}$$

Similarly, we can also get

$$\frac{\|\mathcal{G}u\|^{q-1}}{R^{q-1}} \leqslant 2^{q-1}C^*R^{q} + 2^{q-1}C^{**}\|d_1\|_\eta R^{\tau-q} + 2^{q-1}C^{***}\|d_2\|_\infty R^{q^*-q} + 2^{q-1}\lambda\frac{\|g\|_{\frac{3q}{4q-3}}}{R^{q-1}}. \tag{3.8}$$

We now take $R > 0$ sufficiently small so that

$$2^pC^*R^{2q-p} + 2^pC^{**}\|d_1\|_\eta R^{\tau-p} + 2^pC^{***}\|d_2\|_\infty\|d_2\|_\infty R^{q^*-p} \leqslant \frac{1}{2}$$

and

$$2^{q-1}C^*R^{q} + 2^{q-1}C^{**}\|d_1\|_\eta R^{\tau-q} + 2^{q-1}C^{***}\|d_2\|_\infty R^{q^*-q} < \frac{1}{2}.$$

Let

$$\lambda_0 := \min\left\{\frac{R^{p-1}}{2^{p+1}\|g\|_{\frac{3q}{4q-3}}}, \frac{R^{q-1}}{2^{q}\|g\|_{\frac{3q}{4q-3}}}\right\}.$$

Then for all $0 < \lambda \leqslant \lambda_0$, we can derive from (3.7) and (3.8) that $\|\mathcal{G}u\| \leqslant R$. This completes the proof of Lemma 3.2. □

Proof of Theorem 1.1. It suffices to show that $\mathcal{G} : (W, \preceq) \to (W, \preceq)$ is an increasing operator, since by Proposition 2.1 and Lemma 3.2, we can then obtain the existence of the weak solutions. So let us show that the operator $\mathcal{T} : (W, \preceq) \to (W', \lhd)$ is increasing. In fact, take $u_1, u_2 \in W$ such that $u_1 \leqslant u_2$ almost everywhere on $\mathbb{R}^3$. Due to the assumptions on h in Theorem 1.1 and the definition of operator $\mathcal{T}$, we get

$$\begin{aligned} \langle \mathcal{T}u_1, v\rangle &= \int_{\mathbb{R}^3}\Big(\phi_{u_1^+}|u_1^+|^{q-2}u_1^+ + h(x,u_1^+) + \lambda g(x)\Big)v\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^3}\Big(\phi_{u_2^+}|u_2^+|^{q-2}u_2^+ + h(x,u_2^+) + \lambda g(x)\Big)v\mathrm{d}x = \langle \mathcal{T}u_2, v\rangle, \text{ for all } v \in \mathcal{E}_+. \end{aligned} \tag{3.9}$$

Therefore, the operator $\mathcal{G} : (W, \preceq) \to (W, \preceq)$ is indeed increasing. By Proposition 2.1 and Lemma 3.2, the operator $\mathcal{G}$ has a fixed point, that is, there exists $u_0 \in B_W[0, R]$ such that $\mathcal{G}u_0 = u_0$. Since $\mathcal{G} = \mathcal{B}^{-1} \circ \mathcal{T}$, we have $\langle \mathcal{B}u_0, v\rangle = \langle \mathcal{T}u_0, v\rangle$, for all $v \in W$. That is,

$$\begin{aligned} &\int_{\mathbb{R}^3}(|\nabla u_0|^{p-2}\nabla u_0\nabla v + |u_0|^{p-2}u_0v)\mathrm{d}x + \int_{\mathbb{R}^3}(|\nabla u_0|^{q-2}\nabla u_0\nabla v + |u_0|^{q-2}u_0v)\mathrm{d}x \\ &= \int_{\mathbb{R}^3}\phi_{u_0^+}|u_0^+|^{q-2}u_0^+v\mathrm{d}x - \int_{\mathbb{R}^3}\Big(h(x,u_0^+) + \lambda g(x)\Big)v\mathrm{d}x. \end{aligned} \tag{3.10}$$

Letting $v = u_0^-$ in (3.10), we can get $\|u_0^-\|^p_{W^{1,p}(\mathbb{R}^3)} + \|u_0^-\|^q_{W^{1,q}(\mathbb{R}^3)} = 0$, which means that $u_0^- = 0$ and is a nontrivial nonnegative weak solution of problem (1.1). According to the well-known Strong Maximum Principle, u_0 is a positive solution to problem (1.1). This completes the proof of Theorem 1.1. □

Data availability

No data was used for the research described in the article.

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