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Nonlocal p-Kirchhoff equations with singular and critical nonlinearity terms

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Abstract. The objective of this work is to investigate a nonlocal problem involving singular and critical nonlinearities:

$$\begin{cases} ([u]^p_{s,p})^{\sigma-1}(-\Delta)^s_p u = \frac{\lambda}{u^\gamma} + u^{p^*_s-1} & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$, $0 < s < 1 < p < \infty$, $N > sp$, $1 < \sigma < p^*_s/p$, with $p^*_s = \frac{Np}{N-ps}$, $(-\Delta)^s_p$ is the nonlocal p-Laplace operator and $[u]_{s,p}$ is the Gagliardo p-seminorm. We combine some variational techniques with a truncation argument in order to show the existence and the multiplicity of positive solutions to the above problem.

Keywords: Kirchhoff problem, nonlocal operator, variational methods, singular nonlinearity, multiplicity results

1. Introduction

In this paper, we shall consider the following singular critical nonlocal problem:

$$\begin{cases} ([u]^p_{s,p})^{\sigma-1}(-\Delta)^s_p u = \frac{\lambda}{u^\gamma} + u^{p^*_s-1} & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $0 < s < 1 < p < \infty$, $N > sp$, $1 < \sigma < p_s^*/p$, $p_s^* = \frac{Np}{N-ps}$, $(-\Delta)_p^s$ is a nonlocal operator defined by

$$(-\Delta)_p^s u(x) := 2 \lim_{\epsilon\to 0} \int_{\Omega\setminus B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \mathrm{d}y, \quad x \in \Omega,$$

where $B_\epsilon(x) := \{y \in \Omega : |x-y| < \epsilon\}$, and $[u]_{s,p}$ is the Gagliardo p-seminorm given by

$$[u]_{s,p}^p := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy.$$

Problems of this type describe diffusion processes in heterogeneous or complex medium (anomalous diffusion) due to random displacements executed by jumpers that are able to walk to neighbouring nearby sites. These problems are also due to excursions to remote sites by way of Lévy flights, they can be used in modelling turbulence, chaotic dynamics, plasma physics and financial dynamics. For more details, see [1,6] and references therein.

For $p = 2$, problem (1.1) has been investigated by many authors in order to show the existence and the multiplicity of solutions. For further details, one can refer the reader to [8,9,11,17–22,29] and the references therein.

For $s = 1$, the local setting case has been extensively investigated in the recent past. The existence, the uniqueness, the multiplicity of weak solutions and regularity of solutions have been studied in [5,7,10,13–16,26,28,30,32] and the references therein.

Motivated by the previous results, and the work of Fiscella [11], who established the existence and the multiplicity of positive solutions using some variational methods combined with an appropriate truncation. The aim of this work is to extend the multiplicity results to a more general non-local problem. More precisely, we shall establish the following result.

Theorem 1.1. *Suppose that the parameters in problem (1.1) satisfy the following two conditions*

$$0 < 1-\gamma < 1 < p\sigma < p_s^* \quad \text{and} \quad 1 < \sigma < p_s^*/p.$$

Then there exists a parameter $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, problem (1.1) has at least two positive solutions.

2. Preliminaries

This section is devoted to basic definitions, notations, and function spaces that will be used in the forthcoming sections. For the other background material we refer the reader to [24,27]. We begin by defining the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u \text{ measurable}, |u|_{s,p} < \infty\},$$

with the Gagliardo norm

$$\|u\|_{s,p} := \left(\|u\|_p^p + |u|_{s,p}^p\right)^{\frac{1}{p}}.$$

Denote

$$\mathcal{Q} := \mathbb{R}^{2N} \setminus \big((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega) \big)$$

and define the space

$$X := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ Lebesgue measurable} : u\,|_{\Omega} \in L^p(\Omega) \text{ and } \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \in L^p(\mathcal{Q}) \right\}$$

with the norm

$$\|u\|_X := \|u\|_{L^p(\Omega)} + \left(\int_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p}.$$

Throughout this paper, we shall consider the space

$$X_0 := \big\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \big\},$$

with the norm

$$\|u\| := \left(\int_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p}$$

and the scalar product

$$\langle u, \varphi \rangle_{X_0} := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y.$$

We define a weak solution to problem (1.1) as follows:

Definition 2.1. We say that $u \in X_0$ is a weak solution of problem (1.1) if for all $\varphi \in X_0$, one has

$$\begin{aligned} &\big([u]_{s,p}^p\big)^{\sigma-1} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \lambda \int_{\Omega} \big(u^+\big)^{-\gamma} \varphi \, \mathrm{d}x + \int_{\Omega} \big(u^+\big)^{p_s^*-1} \varphi \, \mathrm{d}x. \end{aligned} \tag{2.1}$$

In order to find solutions of problem (1.1), we shall use the variational approach. More precisely, we shall find two distinct critical points of the energy functional $J_\lambda : X_0 \to (-\infty, \infty]$ defined by

$$J_\lambda(u) := \frac{1}{p\sigma} \|u\|^{p\sigma} - \frac{\lambda}{1-\gamma} \int_{\Omega} \big(u^+\big)^{1-\gamma} \, \mathrm{d}x - \frac{1}{p_s^*} \int_{\Omega} \big(u^+\big)^{p_s^*} \, \mathrm{d}x. \tag{2.2}$$

Now, we prove the following result.

Lemma 2.1. *There exist $\rho \in (0, 1]$, λ_1 and $\alpha > 0$ such that for every $\lambda \in (0, \lambda_1]$, we have*

$$J_\lambda(u) \geqslant \alpha \quad \text{for all } u \in X_0 \text{ with } \|u\| = \rho.$$

Moreover, the following holds

$$m_\lambda := \inf\{J_\lambda(u) : u \in \overline{B}_\rho\} < 0,$$

where $\overline{B}_\rho := \{u \in X_0 : \|u\| \leqslant \rho\}$.

Proof. Let $\lambda > 0$. Then by virtue of the Hölder inequality and the Sobolev embedding theorem, we get for any $u \in X_0$

$$\begin{aligned}\int_\Omega u^{1-\gamma}\,dx &\leqslant |\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} \|u\|_{p_s^*}^{1-\gamma} \\ &\leqslant C\|u\|^{1-\gamma}.\end{aligned}$$

So from the Sobolev embedding, we obtain

$$\begin{aligned}J_\lambda(u) &= \frac{1}{p\sigma}\|u\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}\,\mathrm{d}x - \frac{1}{p_s^*}\int_\Omega u^{p_s^*}\,\mathrm{d}x \\ &\geqslant \frac{1}{p\sigma}\|u\|^{p\theta} - \frac{C\lambda}{1-\gamma}\|u\|^{1-\gamma} - \frac{c_1}{p_s^*}\|u\|^{p_s^*} \\ &= \|u\|^{1-\gamma}\left(\varphi(\|u\|) - \frac{C\lambda}{1-\gamma}\right)\end{aligned}$$

where $\varphi(t) = \frac{1}{p\sigma}t^{p\sigma-1+\gamma} - \frac{c_1}{p_s^*}t^{p_s^*-1+\gamma}$. Since $1-\gamma < 1 < p\sigma < p_s^*$, we find $\rho \in (0,1)$ sufficiently small and satisfying

$$\max_{0<t<1}\varphi(t) = \varphi(\rho). \tag{2.3}$$

Put

$$\lambda_1 := \frac{(1-\gamma)\varphi(\rho)}{2C}. \tag{2.4}$$

Thus, for all $u \in X_0$ with $\|u\| = \rho$ and all $\lambda \leqslant \lambda_1$, one has

$$J_\lambda(u) \geqslant \frac{C\rho^{1-\gamma}}{1-\gamma}(2\lambda_1 - \lambda) > \frac{C\rho^{1-\gamma}}{1-\gamma}\lambda_1 = \alpha > 0.$$

Moreover, since $1-\gamma < 1 < p\sigma < p_s^*$, it follows that for $u \in X_0$ with $u^+ \not\equiv 0$ and for $t \in (0,1)$ sufficiently small, one has

$$J_\lambda(tu) = \frac{t^{p\sigma}}{p\sigma}\|u\|^{p\sigma} - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}\,\mathrm{d}x - \frac{t^{p_s^*}}{p_s^*}\int_\Omega (u^+)u^{p_s^*}\,\mathrm{d}x$$
$$< 0. \qquad \square$$

Lemma 2.2. *For every $\lambda \in (0, \lambda_1]$, problem (1.1) has a positive solution $u_\lambda \in X_0$ with $J_\lambda(u_\lambda) < 0$.*

Proof. Let ρ and λ_1 be the constants given respectively by (2.3) and (2.4). Let $\{u_k\} \subset \overline{B}_\rho$ be a minimizing sequence for m_λ, i.e.

$$\lim_{k\to\infty} J_\lambda(u_k) = m_\lambda.$$

As $\{u_k\}$ is bounded, for any $1 \leqslant r < p_s^*$, one has

$$\begin{cases} u_k \rightharpoonup u_\lambda & \text{weakly in } X_0, \\ u_k \rightharpoonup u_\lambda & \text{weakly in } L^{p_s^*}(\Omega), \\ u_k \to u_\lambda & \text{strongly in } L^r(\Omega), \\ u_k \to u_\lambda & \text{a.e. in } \Omega. \end{cases} \tag{2.5}$$

By the Hölder inequality, we get for all integers k,

$$\left|\int_\Omega (u_k^+)^{1-\gamma}\,dx - \int_\Omega (u_\lambda^+)^{1-\gamma}\,dx\right| \leqslant \int_\Omega |u_k^+) - u_\lambda^+|^{1-\gamma}\,dx$$
$$\leqslant |\Omega|^{\frac{p-1+\gamma}{p}}\|u_k^+) - u_\lambda^+\|_p^{1-\gamma}. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$\lim_{k\to\infty}\int_\Omega (u_k^+)^{1-\gamma}\,dx = \int_\Omega (u_\lambda^+)^{1-\gamma}\,dx. \tag{2.7}$$

Put $\widetilde{u}_k := u_k - u_\lambda$. Then, by invoking the Brezis–Lieb Lemma [4], we obtain

$$\lim_{k\to\infty}\|u_k\|^p - \|\widetilde{u}_k\|^p = \|u_\lambda\|^p \quad \text{and} \quad \lim_{k\to\infty}\|u_k\|_{p_s^*}^{p_s^*} - \|\widetilde{u}_k\|_{p_s^*}^{p_s^*} = \|u_\lambda\|_{p_s^*}^{p_s^*}. \tag{2.8}$$

Since $\{u_k\} \subset \overline{B}_\rho$, it follows that (2.8) implies that for k large enough, $\widetilde{u}_k \in \overline{B}_\rho$. So, from Lemma 2.1, we deduce that for all $u \in X_0$ with $\|u\| = \rho$,

$$\frac{1}{p\sigma}\|u\|^{p\sigma} - \frac{1}{p_s^*}\int_\Omega u^{p_s^*}\,\mathrm{d}x \geqslant \alpha > 0,$$

that is, if $\rho \leqslant 1$ and k is large enough,

$$\frac{1}{p\sigma}\|\widetilde{u}_k\|^{p\sigma} - \frac{1}{p_s^*}\int_\Omega \widetilde{u}_k^{p_s^*}\,\mathrm{d}x > 0, \tag{2.9}$$

since $\{u_k\}$ is a minimizing sequence. Hence, by combining (2.7)–(2.9), we obtain for k large enough,

$$\begin{aligned}
m_\lambda &= J_\lambda(u_k) + \circ(1)\\
&= \frac{1}{p\sigma}\|\widetilde{u}_k + u_\lambda\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega \big((\widetilde{u}_k + u_\lambda)^+\big)^{1-\gamma}\,\mathrm{d}x - \frac{1}{p_s^*}\int_\Omega \big((\widetilde{u}_k + u_\lambda)^+\big)^{p_s^*}\,\mathrm{d}x + \circ(1)\\
&\geqslant \frac{1}{p\sigma}\|\widetilde{u}_k\|^{p\sigma} + \frac{1}{p\sigma}\|u_\lambda\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega \big(u_\lambda^+\big)^{1-\gamma}\,\mathrm{d}x\\
&\quad - \frac{1}{p_s^*}\int_\Omega \big(\widetilde{u}_k^+\big)^{p_s^*} - \frac{1}{p_s^*}\int_\Omega \big(u_\lambda^+\big)^{p_s^*}\,\mathrm{d}x + \circ(1)\\
&\geqslant J_\lambda(u_\lambda) + \frac{1}{p\sigma}\|\widetilde{u}_k\|^{p\sigma} - \frac{1}{p_s^*}\int_\Omega \big(\widetilde{u}_k^+\big)^{p_s^*} + \circ(1)\\
&\geqslant J_\lambda(u_\lambda) + \circ(1)\\
&\geqslant m_\lambda,
\end{aligned}$$

Hence, $J_\lambda(u_\lambda) = m_\lambda < 0$.

Now, let us prove that u_λ is a positive solution to problem (1.1). Our proof uses similar techniques as [12]. Consider $\phi \in X_0$ and $0 < \epsilon < 1$. Let $\Psi \in X_0$ be defined by $\Psi := (u_\lambda + \epsilon\phi)^+$ with $(u_\lambda + \epsilon\phi)^+ := \max\{u_\lambda + \epsilon\phi, 0\}$. Let $\Omega_\epsilon := \{u_\lambda + \epsilon\phi \leqslant 0\}$ and $\Omega^\epsilon := \{u_\lambda + \epsilon\phi < 0\}$. Put $\Theta_\epsilon := \Omega_\epsilon \times \Omega_\epsilon$. Since u_λ is a local minimizer for J_λ, replacing φ with Ψ in (2.1), one gets

$$\begin{aligned}
0 &\leqslant \big([u_\lambda]_{s,p}^p\big)^{\sigma-1}\\
&\quad\times \iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \lambda\int_\Omega \big(u_\lambda^+\big)^{-\gamma}\Psi\,\mathrm{d}x - \int_\Omega \big(u_\lambda^+\big)^{p_s^*-1}\Psi\,\mathrm{d}x\\
&= \big([u_\lambda]_{s,p}^p\big)^{\sigma-1}\\
&\quad\times \iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + \epsilon\phi)(x) - (u_\lambda + \epsilon\phi)(y))}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad - \int_{\{(x,y)\in\Omega^\epsilon\times\Omega^\epsilon\}} \big(\lambda\big(u_\lambda^+\big)^{-\gamma}(u_\lambda + \epsilon\phi) + \big(u_\lambda^+\big)^{p_s^*-1}(u_\lambda + \epsilon\phi)\big)\,\mathrm{d}x\\
&= \big([u_\lambda]_{s,p}^p\big)^{\sigma-1}\\
&\quad\times \iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + \epsilon\phi)(x) - (u_\lambda + \epsilon\phi)(y))}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\Omega} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}(u_\lambda + \epsilon\phi) + \left(u_\lambda^+\right)^{p_s^*-1}(u_\lambda + \epsilon\phi)\right) \mathrm{d}x \\
&= \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \\
&\quad \times \iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + \epsilon\phi)(x) - (u_\lambda + \epsilon\phi)(y))}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\{(x,y)\in\Theta_\epsilon\}} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}(u_\lambda + \epsilon\phi) + \left(u_\lambda^+\right)^{p_s^*-1}(u_\lambda + \epsilon\phi)\right) \mathrm{d}x \\
&= \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \|u_\lambda\|^p - \lambda \int_{\Omega} \left(u_\lambda^+\right) \mathrm{d}x - \lambda \int_{\Omega} \left(u_\lambda^+\right)^{p_s^*} \mathrm{d}x \\
&\quad - \int_{\Omega} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}\phi + \left(u_\lambda^+\right)^{p_s^*-1}\right)\phi\right) \mathrm{d}x \\
&\quad + \epsilon \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} \,dx\,dy \\
&\quad - \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \\
&\quad \times \int_{\{(x,y)\in\Theta_\epsilon\}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + \epsilon\phi)(x) - (u_\lambda + \epsilon\phi)(y))}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\{(x,y)\in\Theta_\epsilon\}} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}(u_\lambda + \epsilon\phi) + \left(u_\lambda^+\right)^{p_s^*-1}(u_\lambda + \epsilon\phi)\right) \mathrm{d}x \\
&= \epsilon \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} \,dx\,dy \\
&\quad - \epsilon \int_{\Omega} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}\phi + \left(u_\lambda^+\right)^{p_s^*-1}\phi\right) \mathrm{d}x \\
&\quad - \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \\
&\quad \times \int_{\{(x,y)\in\Theta_\epsilon\}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + \epsilon\phi)(x) - (u_\lambda + \epsilon\phi)(y))}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_{\{(x,y)\in\Theta_\epsilon\}} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}(u_\lambda + \epsilon\phi) + \left(u_\lambda^+\right)^{p_s^*-1}(u_\lambda + \epsilon\phi)\right) \mathrm{d}x \\
&\leqslant \epsilon \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}} \,dx\,dy \\
&\quad - \epsilon \int_{\Omega} \left(\lambda \left(u_\lambda^+\right)^{-\gamma}\phi + \left(u_\lambda^+\right)^{p_s^*-1}\phi\right) \mathrm{d}x \\
&\quad - \left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \\
&\quad \times \int_{\{(x,y)\in\Theta_\epsilon\}} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + \epsilon\phi)(x) - (u_\lambda + \epsilon\phi)(y))}{|x-y|^{N+ps}} \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

since the measure Ω_ϵ goes to zero as $\epsilon \to 0^+$. We deduce that,

$$\left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \int_{\{(x,y)\in\Theta_\epsilon\}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))((u_\lambda+\epsilon\phi)(x)-(u_\lambda+\epsilon\phi)(y))}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \to 0.$$

as $\epsilon \to 0^+$. We divide by ϵ and passing to the limit as $\epsilon \to 0^+$, one has

$$\left([u_\lambda]_{s,p}^p\right)^{\sigma-1} \int_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy - \int_\Omega \left(\lambda\left(u_\lambda^+\right)^{-\gamma}\phi + \left(u_\lambda^+\right)^{p_s^*-1}\phi\right)\mathrm{d}x \geqslant 0.$$

The equality holds if we change ϕ by $-\phi$. So we deduce that u_λ is a nonnegative solution of problem (1.1). □

3. A perturbed problem

Since J_λ is not Fréchet differentiable due to the singular term, we cannot apply the usual variational theory to the functional energy. Therefore, in order to establish the existence of a second solution, we introduce the following perturbed problem

$$\begin{cases} ([u]_{s,p}^p)^{\sigma-1}(-\Delta)_p^s u = \frac{\lambda}{(u^+ + \frac{1}{n})^\gamma} + (u^+)^{p_s^*-1} & \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

Associated to problem (3.1), we consider the functional $J_{n,\lambda} : X_0 \to \mathbb{R}$ defined by

$$J_{n,\lambda}(u) := \frac{1}{p\sigma}\|u\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega \left(\left(u^+ + \frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma}\right)\mathrm{d}x - \frac{1}{p_s^*}\int_\Omega \left(u^+\right)^{p_s^*}\mathrm{d}x.$$

It is clear that $J_{n,\lambda}$ is Fréchet differentiable, and for all $\varphi \in X_0$, we have

$$\left\langle J'_{n,\lambda}(u), \varphi\right\rangle = \|u\|^{p\sigma-2}\langle u, \varphi\rangle - \lambda\int_\Omega \frac{\varphi}{(u^+ + \frac{1}{n})^{1-\gamma}}\,\mathrm{d}x - \int_\Omega \left(u^+\right)^{p_s^*-1}\varphi\,\mathrm{d}x. \tag{3.2}$$

Lemma 3.1. *Let $\rho \in (0, 1]$, λ_1 and α be the constants given by Lemma 2.1. Then for any $\lambda \in (0, \lambda_1]$, one has*

$$J_{n,\lambda}(u) \geqslant \alpha, \quad \textit{for all } u \in X_0 \textit{ with } \|u\| \leqslant \rho.$$

Moreover, there exists $e \in X_0$, with $\|e\| > \rho$ and $J_{n,\lambda}(e) < 0$.

Proof. Since $(u^+ + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma} \leqslant (u^+)^{1-\gamma}$, we have

$$J_{n,\lambda}(u) \geqslant J_\lambda(u).$$

Therefore, Lemma 2.1 implies that the first part of Lemma 3.1 has been proved.

Now, let $u \in X_0$ with $u^+ \not\equiv 0$. Then for any $t > 0$, we have

$$J_{n,\lambda}(tu) = \frac{t^{p\sigma}}{p}\|u\|^{p\sigma} - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega\left(\left(u^+ + \frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma}\right)\mathrm{d}x - \frac{t^{p_s^*}}{p_s^*}\int_\Omega (u^+)^{p_s^*}\,\mathrm{d}x.$$

Since $1-\gamma < 1 \leqslant p\sigma \leqslant p_s^*$, it follows that $J_{n,\lambda}(tu) \to -\infty$ as $t \to \infty$. Hence, the second part of Lemma 3.1 is proved. □

Now, put

$$C_\lambda := \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right) S^{\frac{N\sigma}{ps\sigma - N(\sigma-1)}} - \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)^{-\frac{1-\gamma}{p\sigma-1+\gamma}} \left[\lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\right]^{\frac{p\sigma}{p\sigma-1+\gamma}} \tag{3.3}$$

We show the following useful result.

Lemma 3.2. *The functional $J_{n,\lambda}$ satisfies the (PS) condition at any level $c \in \mathbb{R}$ such that $c < C_\lambda$ for any $\lambda > 0$.*

Proof. Let $\{u_k\} \subset X_0$ be a (PS) minimizing sequence for the functional $J_{n,\lambda}$ at level $c \in \mathbb{R}$, that is

$$J_{n,\lambda}(u_k) \to c \quad \text{and} \quad J'_{n,\lambda}(u_k) \to 0 \quad \text{as } k \to \infty. \tag{3.4}$$

Then by the Sobolev embedding and the Hölder inequality, there exist $\epsilon > 0$ and $C > 0$ satisfying

$$\begin{aligned} c + \epsilon\|u_k\| + \circ(1) &\geqslant J_{n,\lambda} - \frac{1}{p_s^*} \prec J'_{n,\lambda}(u_k), u_k \succ \\ &= \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|u_k\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega\left(\left(u_k^+ + \frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma}\right)\mathrm{d}x \\ &\quad + \frac{\lambda}{p_s^*}\int_\Omega\left(u_k^+ + \frac{1}{n}\right)^{-\gamma} u_k\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|u_k\|^{p\sigma} - \lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)\int_\Omega |u_k|^{1-\gamma}\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|u_k\|^{p\sigma} - \lambda C\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*+1-\gamma}{p_s^*}}\|u_k\|^{1-\gamma}. \end{aligned}$$

Since $1-\gamma < 1 < p\sigma < p_s^*$, it follows that $\{u_k\}$ is bounded. Moreover, $\{u_k^-\}$ is bounded in X_0. So from (3.4), we deduce that

$$\lim_{k\to\infty} \prec J'_{n,\lambda}(u_k), u_k \succ = \lim_{k\to\infty} \|u_k\|^{p(\sigma-1)} \prec u_k, -u_k^- \succ .$$

On the other hand, by an elementary inequality

$$(a-b)\big(a^- - b^-\big) \leqslant -\big(a^- - b^-\big)^2$$

we have

$$\begin{aligned} 0 &\leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant -\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u^-(x)-u^-(y))^2}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.5}$$

From (3.5), we have $\|u_k^-\| \to 0$ as k tends to infinity. Hence, for k large enough, we have

$$J_{n,\lambda}(u_k) = J_{n,\lambda}\big(u_k^+\big) + \circ(1) \quad \text{and} \quad J'_{n,\lambda}(u_k) = J'_{n,\lambda}\big(u_k^+\big) + \circ(1),$$

i.e., we can assume that $\{u_k\}$ is a sequence of nonnegative functions.

Now, since $\{u_k\}$ is bounded, up to a subsequence and using [2,31], there exist $\{u_k\} \subset X_0$, u in X_0, and nonnegative numbers l, μ such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } X_0, \\ u_k \rightharpoonup u & \text{weakly in } L^{p_s^*}(\Omega), \\ u_k \to u & \text{strongly in } L^q(\Omega) \text{ for } q \in [1, p_s^*), \\ u_k \to u & \text{a.e. in } \Omega, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \|u_k\| \to \mu, \\ \|u_k - u\|_{p_s^*} \to l. \end{cases} \tag{3.7}$$

Moreover, for a fixed $q \in [1, p_s^*)$, there is $h \in L^q(\Omega)$ such that

$$u \leqslant h \quad \text{a.e. in } \Omega.$$

It is easy to see that if $\mu = 0$, then $u_k \to 0$ in X_0. So let us assume that $\mu > 0$. It follows from the above assertion that

$$\left|\frac{u_k - u}{(u_k + \frac{1}{n})^\gamma}\right| \leqslant n^\gamma\big(h + |u|\big).$$

Therefore, the dominated convergence theorem implies that

$$\lim_{k\to\infty}\int_\Omega \frac{u_k-u}{(u_k+\frac{1}{n})^\gamma}\,dx=0. \tag{3.8}$$

Hence, the Bresis-Lieb Lemma [4] yields

$$\|u_k\|^p=\|u_k-u\|^p+\|u\|^p+\circ(1) \quad \text{and} \quad \|u_k\|_{p_s^*}^{p_s^*}=\|u_k-u\|_{p_s^*}^{p_s^*}+\|u\|_{p_s^*}^{p_s^*}+\circ(1). \tag{3.9}$$

Now, using (3.8) and (3.9), we can deduce that:

$$\begin{aligned}\circ(1)&=\prec J'_{n,\lambda}(u_k),u_k-u\succ\\ &=\|u_k\|^{p(\sigma-1)}\prec u_k,u_k-u\succ-\lambda\int_\Omega\frac{u_k-u}{(u_k+\frac{1}{n})^\gamma}\,dx-\int_\Omega u_k^{p_s^*-1}(u_k-u)\,dx\\ &=\mu^{p(\sigma-1)}\big(\|u_k\|^p-\|u\|^p\big)-\|u_k\|_{p_s^*}^{p_s^*}+\|u\|_{p_s^*}^{p_s^*}+\circ(1)\\ &=\mu^{p(\sigma-1)}\|u_k-u\|^p-\|u_k-u\|_{p_s^*}^{p_s^*}+\circ(1).\end{aligned}$$

Therefore,

$$\mu^{p(\sigma-1)}\lim_{k\to\infty}\|u_k-u\|^p=\lim_{k\to\infty}\|u_k-u\|_{p_s^*}^{p_s^*}=l. \tag{3.10}$$

Since $\mu>0$, if $l=0$, we obtain that $u_k\to u$ in X_0 and the proof is complete.

Now, let us prove that $l=0$. Proceeding by contradiction, suppose that $l>0$. Then from (3.10) and the Sobolev embedding, we get

$$S\mu^{p(\sigma-1)}l^p\leqslant l^{p_s^*}, \tag{3.11}$$

that is,

$$l^{p_s^*-p}\geqslant S\mu^{p(\sigma-1)}. \tag{3.12}$$

On the other hand, by combining (3.9) and (3.10), we obtain

$$\mu^{p(\sigma-1)}\big(\mu^p-\|u\|^p\big)=l^{p_s^*},$$

that is,

$$l=\mu^{\frac{p(\sigma-1)}{p_s^*}}\big(\mu^p-\|u\|^p\big)^{\frac{N-ps}{Np}}.$$

So using (3.12), we get

$$l^{p_s^*-p}=\mu^{\frac{p(p_s^*-p)(\sigma-1)}{p_s^*}}\big(\mu^p-\|u\|^p\big)^{\frac{(N-ps)(p_s^*-p)}{Np}}\geqslant S\mu^{p(\sigma-1)}l^p.$$

We deduce that

$$\mu^{\frac{p_s^2}{N}} \geqslant \left(\mu^p - \|u\|^p\right)^{\frac{(N-ps)(p_s^*-p)}{Np}} \geqslant S\left(\mu^{p(\sigma-1)}\right)^{\frac{N-ps}{N}}.$$

Since $1 < \sigma < \frac{p_s^*}{p}$, it follows that $ps\sigma - N(\sigma - 1) > 0$. So

$$\mu^p \geqslant S^{\frac{N}{ps\sigma - N(\sigma-1)}}. \tag{3.13}$$

Now, the fact that $(u^+ + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma} \leqslant (u^+)^{1-\gamma}$ implies that for all integers k and n we have

$$J_{n,\lambda}(u_k) - \frac{1}{p_s^*} \prec J'_{n,\lambda}(u_k), u_k \succ \geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|u_k\|^{p\sigma} - \lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)\int_\Omega u_k^{1-\gamma}\,dx.$$

So from (3.9), (3.13), the Hölder inequality and the Young inequality, if k tends to infinity, we get

$$\begin{aligned}
c &\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\left(\mu^{p\sigma} + \|u\|^{p\sigma}\right) - \lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\|u\|^{1-\gamma} \\
&\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\left(\mu^{p\sigma} + \|u\|^{p\sigma}\right) - \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|u\|^{p\sigma} \\
&\quad - \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)^{-\frac{1-\gamma}{p\sigma-1+\gamma}}\left[\lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\right]^{\frac{p\sigma}{p\sigma-1+\gamma}} \\
&\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)S^{\frac{N\sigma}{ps\sigma - N(\sigma-1)}} - \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)^{-\frac{1-\gamma}{p\sigma-1+\gamma}}\left[\lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\right]^{\frac{p\sigma}{p\sigma-1+\gamma}} \\
&= C_\lambda,
\end{aligned}$$

which is a contradiction. □

4. Existence of an upper bound

Under some suitable condition, we shall prove that $J_{n,\lambda}$ is bounded from above. To this end, we can assume without loss of generality, that $0 \in \Omega$ and we fix $r > 0$ such that $B_{4r} \subset \Omega$ where $B_{4r} := \{x \in \mathbb{R}^N : |x| < 4r\}$. Let $\varepsilon > 0$ and ψ_ϵ be the function defined by

$$\psi_\epsilon := \frac{\phi U_\epsilon}{\|\phi U_\epsilon\|_{p_s^*}^p}, \tag{4.1}$$

where U_ϵ is the family of functions (for more details see [25]) and $\phi \in C^\infty(\mathbb{R}^N, [0, 1])$ is satisfying

$$\phi = \begin{cases} 1 & \text{in } B_r, \\ 0 & \text{in } \mathbb{R}^N \backslash B_{2r} \end{cases}$$

Lemma 4.1. *There exist* $\lambda_2 > 0$ *and* $\psi \in E$ *satisfying*

$$\sup_{t>0} J_{n,\lambda}(t\psi) < C_{\lambda,}$$

for all $\lambda \in (0, \lambda_1)$.

Proof. Let $\epsilon > 0$ and let u_ϵ and ψ_ϵ be as above. Since

$$0 < 1 - \gamma < p\sigma < p_s^*,$$

it is easy to see that

$$J_{n,\lambda}(t\psi_\epsilon) \longrightarrow -\infty \quad \text{as } t \to \infty.$$

Thus, there exists $t_\epsilon > 0$ satisfying

$$J_{n,\lambda}(t_\epsilon\psi_\epsilon) = \max_{t\geqslant 0} J_{n,\lambda}(t\psi_\epsilon).$$

From Lemma 2.1, we get $J_{n,\lambda} \geqslant \alpha > 0$. So since the functional $J_{n,\lambda}$ is continuous, we deduce the existence of two values $t_0, t^* > 0$ satisfying

$$t_0 < t_\epsilon < t_1, \quad \text{and} \quad J_{n,\lambda}(t_0\psi_\epsilon) = J_{n,\lambda}(t_1\psi_\epsilon) = 0.$$

On the other hand, since $\|u_\epsilon\|_{p_s^*}$ is independent from ϵ, it follows from [23] that

$$\|\psi_\epsilon\|^p \leqslant \frac{\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy}{\|\phi u_\epsilon\|_{p_s^*}^p} = S + O\big(\epsilon^{\frac{N-ps}{p-1}}\big).$$

In fact, for any $a > 0$, $b \in [0, 1]$, $p \geqslant 1$,

$$(a + b)^p \leqslant a^p + p(a + 1)^{p-1}b.$$

We obtain for ϵ small enough,

$$\|\psi_\epsilon\|^{p\sigma} \leqslant \big(S + O\big(\epsilon^{\frac{N-ps}{p-1}}\big)\big)^\sigma \leqslant S^\sigma + O\big(\epsilon^{\frac{N-ps}{p-1}}\big).$$

Hence, for any $\epsilon > 0$ sufficiently small, and using the fact that $t_0 < t_\epsilon < t_1$ and $\|\psi_\epsilon\|^{p^*\sigma} = 1$, we obtain

$$J_{n,\lambda}(t_\epsilon\psi_\epsilon) \leqslant \left(\frac{t_\epsilon^{p\sigma}}{\sigma p}S^\sigma - \frac{t_\epsilon^{p_s^*}}{p_s^*}\right) - \frac{\lambda}{1-\gamma}\int_\Omega \left(\left(t_0\psi_\epsilon + \frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma}\right) \mathrm{d}x$$
$$+ O\big(\epsilon^{\frac{N-ps}{p-1}}\big). \tag{4.2}$$

Since

$$\max_{t>0}\left(\frac{t^{p\sigma}}{\sigma p}S^{\sigma}-\frac{t^{p_s^*}}{p_s^*}\right)=\left(\frac{1}{\sigma p}-\frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}}, \tag{4.3}$$

it follows by (4.2) and (4.3) that

$$J_{n,\lambda}(t_\epsilon\psi_\epsilon)\leqslant\left(\frac{1}{\sigma p}-\frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}}-\frac{\lambda}{1-\gamma}\int_\Omega\left(\left(t_0\psi_\epsilon+\frac{1}{n}\right)^{1-\gamma}-\left(\frac{1}{n}\right)^{1-\gamma}\right)\mathrm{d}x$$
$$+O\left(\epsilon^{\frac{N-ps}{p-1}}\right). \tag{4.4}$$

In addition, for any $a>0$, $b>0$ large enough,

$$(a+b)^\epsilon-a^\epsilon\geqslant\epsilon b^{\frac{\epsilon}{p}}a^{\frac{\epsilon(p-1)}{p}}.$$

We can now deduce that for all $q>0$ small enough, we can establish the existence of $c_1>0$ satisfying

$$\int_\Omega\left(\left(t_0\psi_\epsilon+\frac{1}{n}\right)^{1-\gamma}-\left(\frac{1}{n}\right)^{1-\gamma}\right)\mathrm{d}x$$
$$\geqslant c_1(1-\gamma)\epsilon^{\frac{(N-ps)(1-\gamma)}{p_s^*}}\int_{x\in\Omega:|x|\leqslant\epsilon^q}\left(\frac{1}{\left(\frac{1}{|x|^{p'}+\epsilon^{p'}}\right)^{\frac{N-ps}{p}}}\right)^{\frac{p(1-\gamma)}{p_s^*}}dx$$
$$\geqslant c_1(1-\gamma)\epsilon^{\frac{(N-ps)(1-\gamma)-p(p-1)q(N-ps)(1-\gamma)+p_s^*qN}{p_s^*}}.$$

Combining this with (4.4), we get

$$J_{n,\lambda}(t_\epsilon\psi_\epsilon)\leqslant\left(\frac{1}{\sigma p}-\frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}}-\lambda c_1\epsilon^{\frac{(N-ps)(1-\gamma)-p(p-1)q(N-ps)(1-\gamma)+p_s^*qN}{p_s^*}}+O\left(\epsilon^{\frac{N-ps}{p-1}}\right)$$
$$\leqslant\left(\frac{1}{\sigma p}-\frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}}-\lambda c_1\epsilon^{\frac{(N-ps)(1-\gamma)-p(p-1)q(N-ps)(1-\gamma)+p_s^*qN}{p_s^*}}+c_2\epsilon^{\frac{N-ps}{p-1}}, \tag{4.5}$$

for some positive constant c_2.

Now, let $\widetilde{\lambda}>0$ be such that $C_\lambda>0$ for all $\lambda\in(0,\widetilde{\lambda})$, where C_λ is given by (3.3) and let us set

$$\beta:=1+\frac{p(p-1)\sigma((N-ps)(1-\gamma)-p(p-1)q(N-ps)(1-\gamma)+p_s^*qN)}{p_s^*(p\sigma-1+\gamma)(N-ps)}$$
$$-\frac{p\sigma}{p\sigma-1+\gamma},$$
$$\theta:=\left(\frac{1}{p\sigma}-\frac{1}{p_s^*}\right)^{-\frac{1-\gamma}{p\sigma-1+\gamma}}\left[\left(\frac{1}{1-\gamma}+\frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}S^{-\frac{1-\gamma}{p}}\right]^{\frac{p\sigma}{p\sigma-1+\gamma}},$$

and

$$\lambda_2 := \min\left\{\widetilde{\lambda}, r^{\frac{(p\sigma-1+\gamma)(N-ps)}{pq\sigma}}, \left(\frac{c_2+\theta}{c_1}\right)^{\frac{1}{\beta}}\right\},$$

where $r > 0$ is such that $B_{4r} \subset \Omega$ and $q > 0$ is such that $\beta < 0$.

Now, for $\lambda \in (0, \lambda_1)$, if we choose

$$\epsilon := \lambda^{\frac{p(p-1)\sigma}{(ps-1+\gamma)(N-ps)}}$$

in (4.5). Then using the fact that $c_1\lambda^\beta > c_1\lambda_1^\beta \geqslant c_2 + \theta$, we obtain

$$\begin{aligned} J_{n,\lambda}(t_\epsilon\psi_\epsilon) &\leqslant \left(\frac{1}{\sigma p} - \frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}} - \lambda c_1\lambda^{\beta+\frac{p\sigma}{ps-1+\gamma}-1} + c_2\lambda^{\frac{p\sigma}{ps-1+\gamma}} \\ &= \left(\frac{1}{\sigma p} - \frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}} + \lambda^{\frac{p\sigma}{p\sigma-1+\gamma}}\left(c_2 - c_1\lambda^\beta\right) \\ &< \left(\frac{1}{\sigma p} - \frac{1}{p_s^*}\right)S^{\frac{p_s^*\sigma}{p_s^*-p\sigma}} - \theta\lambda^{\frac{p\sigma}{p\sigma-1+\gamma}} = C_\lambda. \end{aligned}$$

□

Set

$$\lambda_0 := \min(\lambda_1, \lambda_2).$$

Then we have the following important result.

Lemma 4.2. *Problem (3.1) has a nonnegative solution $v_n \in X_0$ satisfying*

$$\alpha < J_{n,\lambda}(v_n) < C_\lambda,$$

for all $\lambda \in (0, \lambda_0)$, where α is from Lemma 2.1.

Proof. Let $\lambda \in (0, \lambda_0)$. By Lemma 2.1, $J_{n,\lambda}$ satisfies the Mountain Pass geometry. So we can define the Mountain Pass level

$$c_{n,\lambda} := \inf_{g\in\Gamma}\max_{t\in[0,1]} J_{n,\lambda}\big(g(t)\big),$$

where

$$\Gamma := \left\{g \in C\big([0,1], E\big) : g(0) = 0, J_{n,\lambda}\big(g(1)\big) < 0\right\}.$$

Moreover,

$$0 < \alpha < c_{n,\lambda} \leqslant \sup_{t\geqslant} J_{n,\lambda}(t\psi) < C_{n,\lambda}.$$

Hence, by Lemma 3.2, $J_{n,\lambda}$ satisfies the (PS) condition at the level $c_{n,\lambda}$, i.e., there exists a non-regular point v_n for $J_{n,\lambda}$ at level $c_{n,\lambda}$. Moreover, $J_{n,\lambda}(v_n) = c_{n,\lambda} > \alpha > 0$. We can therefore deduce that v_n is a nontrivial critical point of the functional energy $J_{n,\lambda}$ and also a solution to problem (3.1). If we now replace φ by v_n^- in (3.2) and use (3.5), we get $\|v_n\| = 0$, that is, v_n is nonnegative. This leads to the positivity of v_n by the maximum principle [3]. □

5. Proof of Theorem 1.1

In order to complete the proof of our main result it now remains to obtain a second positive solution to problem (1.1) as a limit of the some subsequence of $|v_n|$. To this end, let $\lambda \in (0, \lambda_0)$ and $|v_n|$ be a family of the positive function given by Lemma 4.2. By Lemma 4.2, the Hölder inequality and since $(v_n + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma} \leqslant v_n^{1-\gamma}$, we see that

$$\begin{aligned}
C_\lambda &> J_{n,\lambda} - \frac{1}{p_s^*}\langle J'_{n,\lambda}(v_n), v_n\rangle \\
&= \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|v_n\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega\left(\left(v_n + \frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma}\right)\mathrm{d}x \\
&\quad + \frac{\lambda}{p_s^*}\int_\Omega\left(v_n + \frac{1}{n}\right)^{-\gamma} v_n\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|v_n\|^{p\sigma} - \frac{\lambda}{1-\gamma}\int_\Omega v_n^{1-\gamma}\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\|v_n\|^{p\sigma} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\|v_n\|^{1-\gamma}.
\end{aligned}$$

Since $0 < 1-\gamma < 1 < p\sigma$, v_n is bounded in X_0. So, there is $v_\lambda \in X_0$ satisfying

$$\begin{cases}
v_n \rightharpoonup v_\lambda & \text{weakly in } X_0, \\
v_n \rightharpoonup v_\lambda & \text{weakly in } L^{p_s^*}(\Omega), \\
v_n \to v_\lambda & \text{strongly in } L^r(\Omega), \text{ for any } r \in [1, p_s^*) \\
v_n \to v_\lambda & \text{a.e. in } \Omega.
\end{cases}$$

We shall now prove that $v_n \to v_\lambda$ strongly in X_0, i.e. $\|v_n - v_\lambda\| \to 0$ as $n \to \infty$.

First, we observe that if $\|v_n\| \to 0$, then $v_n \to v_\lambda$ strongly in X_0, so we assume that $\|v_n\| \to \eta > 0$. Since

$$0 \leqslant \frac{v_n}{(v_n + \frac{1}{n})^\gamma} \leqslant v_n^{1-\gamma} \quad \text{a.e. in } \Omega,$$

it follows by the Vitali theorem that

$$\lim_{n\to\infty}\int_\Omega \frac{v_n}{(v_n + \frac{1}{n})^\gamma}\,\mathrm{d}x = \int_\Omega v_\lambda^{1-\gamma}\,\mathrm{d}x.$$

Now, replace both u and φ by v_n in (3.2) to get

$$\eta^{p\sigma} - \lambda \int_\Omega v_\lambda^{1-\gamma}\,dx + \|v_n\|_{p_s^*}^{p_s^*} \to 0. \tag{5.1}$$

On the other hand, by a simple calculation in (3.1) we get

$$\|v_n\|^{p\sigma}(-\Delta)_p^s v_n \geqslant \min\left(1, \frac{\lambda}{p^\gamma}\right) \quad \text{in } \Omega,$$

since v_n is bounded in X_0. Now, by the strong maximum principle [3], there exist $\widetilde{\Omega} \subset \Omega$ and $\widetilde{c} > 0$ such that

$$v_n \geqslant \widetilde{c} > 0, \quad \text{a.e. in } \Omega, \tag{5.2}$$

for any integer n. Let $\varphi \in C_0^\infty(\Omega)$ satisfy $\operatorname{supp}(\varphi) = \widetilde{\Omega} \subset \Omega$. Then by (5.2),

$$0 \leqslant \left|\frac{\varphi}{(v_n + \frac{1}{n})^\gamma}\right| \leqslant \frac{|\varphi|}{\widetilde{c}}, \quad \text{a.e. in } \Omega.$$

Then the dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_\Omega \frac{\varphi}{(v_n + \frac{1}{n})^\gamma}\,dx = \int_\Omega v_\lambda^{-\gamma}\varphi\,dx.$$

Thus, by replacing u with v_λ in (3.2) and by letting n to infinity, we obtain

$$\eta^{p(\sigma-1)}\langle v_\lambda, \varphi\rangle - \lambda \int_\Omega v_\lambda^{-\gamma}\varphi\,dx + \int_\Omega v_\lambda^{p_s^*-1}\varphi\,dx = 0. \tag{5.3}$$

Now, if we replace φ by v_λ in (5.3) and invoke (3.2), we obtain

$$= \eta^{p(\sigma-1)}\left(\eta^p - \|v_\lambda\|^p\right) \lim_{n\to\infty}\left(\|v_n\|_{p_s^*}^{p_s^*} - \|v_\lambda\|_{p_s^*}^{p_s^*}\right).$$

Therefore, by the Brezis–Lieb Lemma [4], we obtain

$$\eta^{p(\sigma-1)} \lim_{n\to\infty}\left(\|v_n - v_\lambda\|^p\right) = l^{p_s^*}. \tag{5.4}$$

Now, let us prove that $l = 0$, by contradiction, i.e. we assume that $l > 0$. As in Lemma 3.2 we can prove that

$$l^{p_s^*-p} \geqslant S\mu^{p(\sigma-1)}.$$

Therefore, by Lemma 4.2 combined with Young inequality and Hölder inequality, we deduce

$$\begin{aligned} C_\lambda &> J_{n,\lambda}(v_n) - \frac{1}{p_s^*}\langle J'_{n,\lambda}(v_n), v_n\rangle \\ &\geqslant \left(\frac{1}{p\sigma} - \frac{1}{p_s^*}\right)\left(\eta^{p\sigma} + \|v_\lambda\|^{p\sigma}\right) - \lambda\left(\frac{1}{1-\gamma} + \frac{1}{p_s^*}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}} \|v_\lambda\|^{1-\gamma} \\ &\geqslant C_\lambda. \end{aligned}$$

Clearly, this is a contradiction, so $l = 0$ and $v_n \to v_\lambda$ strongly in X_0. In addition, one can easily see that v_λ is a solution of problem (1.1). Therefore by Lemma 4.2, $J_\lambda(v_\lambda) \geqslant \alpha > 0$ so v_λ is nontrivial. We can now proceed as in the proof of Lemma 4.2 and deduce that v_λ is a positive solution of problem (1.1). In conclusion, since $J_\lambda(u_\lambda) < 0 < J_\lambda(v_\lambda)$, this completes the proof.

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References

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, 2nd edn, Camb. Stud. Adv. Math., Vol. 116, Cambridge University Press, Cambridge, 2009.
[2] B. Barrios, I. De Bonis, M. Medina and I. Peral, Semilinear problems for the fractional Laplacian with a singular nonlinearity, *Open Math.* **13** (2015), 390–407.
[3] L. Brasco and G. Franzina, Convexity properties of Dirichlet integrals and Picone-type inequalities, *Kodai Math. J.* **37** (2014), 769–799. doi:10.2996/kmj/1414674621.
[4] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88**(3) (1983), 486–490. doi:10.1090/S0002-9939-1983-0699419-3.
[5] M.M. Coclite and G. Palmieri, On a singular nonlinear Dirichlet problem, *Comm. Partial Differential Equations* **14** (1989), 1315–1327. doi:10.1080/03605308908820656.
[6] A. Cotsiolis and N. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.* **295** (2004), 225–236. doi:10.1016/j.jmaa.2004.03.034.
[7] M.G. Crandall, P.H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Partial Differential Equations* **2** (1977), 193–222. doi:10.1080/03605307708820029.
[8] A. Daoues, A. Hammami and K. Saoudi, Multiple positive solutions for a nonlocal PDE with critical Sobolev–Hardy and singular nonlinearities via perturbation method, *Fractional Calculus and Applied Analysis* **23**(3) (2020), 837–860. doi:10.1515/fca-2020-0042.
[9] A. Daoues, A. Hammami and K. Saoudi, Multiplicity results of a nonlocal problem involving concave-convex nonlinearities, *Mathematical Notes* **109**(2) (2021), 192–207. doi:10.1134/S0001434621010235.
[10] R. Dhanya, J. Giacomoni, S. Prashanth and K. Saoudi, Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in $\mathbb{R}^2$, *Advances in Differential Equations* **17**(3–4) (2012), 369–400.
[11] A. Fiscella, A fractional Kirchhoff problem involving a singular term and a critical nonlinearity, *journal Advances in Nonlinear Analysis* **8**(1) (2019), 645–660. doi:10.1515/anona-2017-0075.
[12] A. Ghanmi and K. Saoudi, A multiplicity results for a singular problem involving the fractional p-Laplacian operator, *Complex Variables and Elliptic Equations* **61**(9) (2016), 1199–1216. doi:10.1080/17476933.2016.1154548.
[13] M. Ghergu and V. Rădulescu, Singular elliptic problems with lack of compactness, *Ann. Mat. Pura Appl. (4)* **185**(1) (2006), 63–79. doi:10.1007/s10231-004-0128-2.
[14] M. Ghergu and V. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 37, The Clarendon Press, Oxford University Press, Oxford, 2008.

[15] J. Giacomoni and K. Saoudi, Multiplicity of positive solutions for a singular and critical problem, *Nonlinear Anal.* **71**(9) (2009), 4060–4077. doi:10.1016/j.na.2009.02.087.
[16] W. He, D. Qin and Q. Wu, Existence, multiplicity and nonexistence results for Kirchhoff type equations, *Adv. Nonlinear Anal.* **10**(1) (2021), 616–635. doi:10.1515/anona-2020-0154.
[17] M. Kratou, K. Saoudi and A.S.K. Alshehri, Multiple solutions of a nonlocal system with singular nonlinearities, *International Journal of Mathematics* **32**(10) (2021), 2150072. doi:10.1142/S0129167X21500725.
[18] C.Y. Lei, J.F. Liao and C.L. Tang, Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents, *J. Math. Anal. Appl.* **421** (2015), 521–538. doi:10.1016/j.jmaa.2014.07.031.
[19] J.F. Liao, X.F. Ke, C.Y. Lei and C.L. Tang, A uniqueness result for Kirchhoff type problems with singularity, *Appl. Math. Lett.* **59** (2016), 24–30. doi:10.1016/j.aml.2016.03.001.
[20] J.F. Liao, P. Zhang, J. Liu and C.L. Tang, Existence and multiplicity of positive solutions for a class of Kirchhoff type problems with singularity, *J. Math. Anal. Appl.* **430** (2015), 1124–1148. doi:10.1016/j.jmaa.2015.05.038.
[21] R.Q. Liu, C.L. Tang, J.F. Liao and X.P. Wu, Positive solutions of Kirchhoff type problem with singular and critical nonlinearities in dimension four, *Commun. Pure Appl. Anal.* **15** (2016), 1841–1856. doi:10.3934/cpaa.2016006.
[22] X. Liu and Y. Sun, Multiple positive solutions for Kirchhoff type of problems with singularity, *Commun. Pure Appl. Anal.* **12** (2013), 721–733. doi:10.3934/cpaa.2013.12.721.
[23] S. Mosconi, K. Perera, M. Squassina and Y. Yang, The Brezis–Nirenberg problem for the fractional p-Laplacian, *Calc. Var.* **55** (2016), 105. doi:10.1007/s00526-016-1035-2.
[24] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.
[25] K. Perera, M. Squassina and Y. Yang, Bifurcation and multiplicity results for critical fractional p-Laplacian problems, *Math. Nachr.* **289**(2–3) (2016), 332–342. doi:10.1002/mana.201400259.
[26] V.D. Rădulescu, Combined effects in nonlinear singular elliptic problems with convection, *Rev. Roumaine Math. Pures Appl.* **53**(5–6) (2008), 543–553.
[27] V.D. Rădulescu and D.D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, Chapman and Hall /CRC, Taylor & Francis Group, Boca Raton, FL, 2015.
[28] K. Saoudi, Existence and non-existence for a singular problem with variables potentials, *Electronic Journal of Differential equations* **2017**(291) (2017), 1.
[29] K. Saoudi, A fractional Kirchhoff system with singular nonlinearities, *Analysis and Mathematical Physics* **9** (2019), 1463–1480. doi:10.1007/s13324-018-0251-7.
[30] K. Saoudi and M. Kratou, Existence of multiple solutions for a singular and quasilinear equation, *Complex Var. Elliptic Equ.* **60**(7) (2015), 893–925. doi:10.1080/17476933.2014.981169.
[31] R. Servadei and E. Valdinoci, Mountain pass solutions for nonlocal elliptic operators, *J. Math. Anal. Appl.* **389** (2012), 887–898. doi:10.1016/j.jmaa.2011.12.032.
[32] Z. Shen and J. Yu, Multiple solutions for weighted Kirchhoff equations involving critical Hardy–Sobolev exponent, *Adv. Nonlinear Anal.* **10**(1) (2021), 673–683. doi:10.1515/anona-2020-0152.