Mediterr. J. Math. (2022) 19:170
https://doi.org/10.1007/s00009-022-02076-5

Mediterranean Journal of Mathematics

Check for updates

Degenerate Fractional Kirchhoff-Type System with Magnetic Fields and Upper Critical Growth

Mingzhe Sun, Shaoyun Shi and Dušan D. Repovš

Abstract. This paper deals with the following degenerate fractional Kirchhoff-type system with magnetic fields and critical growth:

$$\begin{cases} -\mathfrak{M}(\|u\|_{s,A}^2)[(-\Delta)_A^s u + u] = G_u(|x|, |u|^2, |v|^2) \\ \quad + \left(\mathcal{I}_\mu * |u|^{p^*}\right)|u|^{p^*-2}u & \text{in } \mathbb{R}^N, \\ \mathfrak{M}(\|v\|_{s,A})[(-\Delta)_A^s v + v] = G_v(|x|, |u|^2, |v|^2) \\ \quad + \left(\mathcal{I}_\mu * |v|^{p^*}\right)|v|^{p^*-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$\|u\|_{s,A} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |u|^2\mathrm{d}x\right)^{1/2},$$

and $(-\Delta)_A^s$ and A are called magnetic operator and magnetic potential, respectively, $\mathfrak{M} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous Kirchhoff function, $\mathcal{I}_\mu(x) = |x|^{N-\mu}$ with $0 < \mu < N$, C^1-function G satisfies some suitable conditions, and $p^* = \frac{N+\mu}{N-2s}$. We prove the multiplicity results for this problem using the limit index theory. The novelty of our work is the appearance of convolution terms and critical nonlinearities. To overcome the difficulty caused by degenerate Kirchhoff function and critical nonlinearity, we introduce several analytical tools and the fractional version concentration-compactness principles which are useful tools for proving the compactness condition.

Mathematics Subject Classification. 35J20, 35J60, 35R11.

Keywords. Fractional Kirchhoff-type system, upper critical exponent, concentration-compactness principle, variational method, multiple solutions.

Birkhäuser

Published online: 22 June 2022

1. Introduction

This paper deals with the following degenerate fractional Kirchhoff-type system with magnetic fields and critical growth:

$$\begin{cases} -\mathfrak{M}(\|u\|_{s,A}^2)[(-\Delta)_A^s u + u] = G_u(|x|, |u|^2, |v|^2) \\ \quad + \left(\mathcal{I}_\mu * |u|^{p^*}\right) |u|^{p^*-2} u & \text{in } \mathbb{R}^N, \\ \mathfrak{M}(\|v\|_{s,A}^2)[(-\Delta)_A^s v + v] = G_v(|x|, |u|^2, |v|^2) \\ \quad + \left(\mathcal{I}_\mu * |v|^{p^*}\right) |v|^{p^*-2} v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where

$$\mathcal{I}_\mu(x) = |x|^{N-\mu}, \quad \text{with } 0 < \mu < N, \quad \|z\|_s = \left([z]_{s,A}^2 + \int_{\mathbb{R}^N} |z|^2 \mathrm{d}x\right)^{1/2},$$

$$[z]_{s,A} = \left(\iint_{\mathbb{R}^{2N}} \frac{|z(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} z(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\right)^{1/2},$$

$(-\Delta)_A^s$ and A are called magnetic operator and magnetic potential, respectively. According to the Hardy–Littlewood–Sobolev inequality (see (2.2)), the exponent $p^* = \frac{N+\mu}{N-2s}$ is called upper critical. The continuous Kirchhoff function $\mathfrak{M} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and C^1-function $G : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^+$ will satisfy the following assumptions throughout the paper:

($\mathcal{M}$) (M_1) $\inf_{t>0} \mathfrak{M}(t) = \mathfrak{m}^* > 0$.
(M_2) For all $t \in [0, +\infty)$, there exists $\sigma \in (1, p^*/2)$ such that $\sigma \mathscr{M}(t) \geq \mathfrak{M}(t)t$, where $\mathscr{M}(t) = \int_0^t \mathfrak{M}(s)ds$.
(M_3) For all $t \in (0, +\infty)$, there exists $\mathfrak{m}_1 > 0$ such that $\mathfrak{M}(t) \geq \mathfrak{m}_1 t^{\sigma-1}$, moreover $\mathfrak{M}(0) = 0$.

($\mathcal{G}$) (G_1) For all $(r, \xi, \eta) \in [0, +\infty) \times \mathbb{R}^2$, there exist $C > 0$ and $2 < \tau < 2^* := \frac{2N}{N-2}$ such that

$$|G_\xi(r, \xi, \eta)| + |G_\eta(r, \xi, \eta)| \leq C\left(|\xi|^{\frac{\tau-1}{2}} + |\eta|^{\frac{\tau-1}{2}}\right).$$

(G_2) For all $(r, \xi, \eta) \in [0, +\infty) \times \mathbb{R}^2$, there exists $2\sigma < \theta < 2p^*$ such that

$$0 < \theta G(r, \xi, \eta) \leq \xi G_\xi(r, \xi, \eta) + \eta G_\eta(r, \xi, \eta),$$

where σ is defined by (M_2).
(G_3) $\xi G_\xi(r, \xi, \eta) \geq 0$ for all $(r, \xi, \eta) \in [0, +\infty) \times \mathbb{R}^2$.
(G_4) $G(r, \xi, \eta) = G(r, -\xi, -\eta)$ for all $r \geq 0$ and $\xi, \eta \in \mathbb{R}$.

Remark 1.1. A typical function which satisfies conditions (M_1)-(M_3) is given by $\mathfrak{M}(t) = a + b\,t^{\sigma-1}$ for $t \in \mathbb{R}_0^+$, where $a \in \mathbb{R}_0^+$, $b \in \mathbb{R}_0^+$, and $a + b > 0$. In particular, when $\mathfrak{M}(t) \geq d > 0$ for some d and all $t \geq 0$, this case is said to be non-degenerate, while it is called degenerate if $\mathfrak{M}(0) = 0$ and $\mathfrak{M}(t) > 0$ for $t > 0$. However, in proving the compactness condition, the two cases of degenerate and non-degenerate are completely different, and it is more complicated in the degenerate case. In this paper, we mainly deal with the degenerate fractional Kirchhoff-type system with magnetic fields. Therefore we need to develop new techniques to conquer some difficulties induced by the degeneration.

The fractional magnetic operator $(-\Delta)^s_A$ was recently introduced by d'Avenia and Squassina [4], which up to normalization constants, can be defined on smooth functions u as follows

$$(-\Delta)^s_A u(x) := 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \quad x\in\mathbb{R}^N.$$

The equation with fractional magnetic operator often arises as a model for various physical phenomena, in particular in the study of the infinitesimal generators of Lévy stable diffusion processes [6]. A vast literature on nonlocal operators and on their applications exists, we refer the interested reader to [1, 7, 17, 32, 46]. To further research this type of question by variational methods, many scholars have established the basic properties of fractional Sobolev spaces - for this the reader is referred to [6, 33, 35].

First, we make a quick overview of the literature on the magnetic Schrödinger equation. To begin, we note that there are works concerning the magnetic Schrödinger equation

$$-(\nabla u - \mathrm{i}A)^2u + V(x)u = f(x,|u|)u, \tag{1.2}$$

which have appeared in recent years, where the magnetic operator in (1.2) is given by

$$-(\nabla u - \mathrm{i}A)^2u = -\Delta u + 2iA(x)\cdot\nabla u + |A(x)|^2u + iu\,\mathrm{div}A(x).$$

As stated in Squassina and Volzone [41], up to correcting the operator by the factor $(1-s)$, it follows that $(-\Delta)^s_A u$ converges to $-(\nabla u - \mathrm{i}A)^2u$ as $s\to 1$. Thus, up to normalization, the nonlocal case can be seen as an approximation of the local one. Ji and Rădulescu [13] obtained the multiplicity and concentration properties of solutions for a class of nonlinear magnetic Schrödinger equation using variational methods, penalization techniques, and the Ljusternik–Schnirelmann theory. For more interesting results, we refer to [14, 28, 44, 48]. Recently, many researchers have paid attention to the equations with fractional magnetic operator. In particular, Mingqi et al. [30] proved some existence results for Schrödinger–Kirchhoff type equation involving the fractional p–Laplacian and the magnetic operator

$$M([u]^2_{s,A})(-\Delta)^s_A u + V(x)u = f(x,|u|)u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where f satisfies the subcritical growth condition. For the critical growth case, Wang and Xiang [42] have obtained the existence of two solutions and infinitely many solutions to fractional Schrödinger–Choquard–Kirchhoff type equations with external magnetic operator. Subsequently, Liang et al. [23] investigated the existence and multiplicity of solutions to problem (1.1) without Choquard-type term in the non-degenerate case. We draw the attention of the reader to the degenerate case involving the magnetic operator in Liang et al. [24] and Mingqi et al. [30].

For the case $A\equiv 0$ in problem (1.1), there exist numerous articles dedicated to the study of the following Choquard equation,

$$-\Delta u + V(x)u = (|x|^{-\mu} * F(u))f(u), \quad x\in\mathbb{R}^N. \tag{1.4}$$

Eq. (1.4) can be used to describe many physical models. For example, it was proposed by Laskin [16] as a result of expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The study of existence and uniqueness of positive solutions to Choquard type equations attracted a lot of attention of researchers due to its applications in physical models Pekar [36]. In d'Avenia et al. [5], the authors obtained the existence of ground state solutions for following fractional Choquard equation of the form

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\mu * |u|^p)\,|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N), \quad N \geq 3, \tag{1.5}$$

where $s \in (0,1)$, $\omega > 0$ is a given parameter, $\mu \in (0,N)$, and $\mathcal{K}_\mu(x) = |x|^{N-\mu}$ is the Riesz potential. In Pucci et al. [37], the authors obtained the existence of nonnegative solutions to a class of Schrödinger–Choquard–Kirchhoff-type fractional equation using the Mountain pass theorem and the Ekeland variational principle. For more results for problems with Hardy–Littlewood–Sobolev critical nonlinearity without the magnetic operator case, see Cassani and Zhang [3], Ma and Zhang [29], and Song and Shi [38,39].

Once we turn our attention to the critical nonlocal system with critical nonlinearity, we immediately see that the literature is relatively scarce. In this case, we can cite recent works [9,10,45]. We call attention to Furtado et al. [10] who dealt with the following non–degenerate Kirchhoff system

$$\begin{cases} -m\left(\int_\Omega |\nabla u|^2\right)\Delta u = F_u(x,u,v) + \mu_1|u|^4u, \text{ in } \Omega, \\ -l\left(\int_\Omega |\nabla v|^2\right)\Delta v = F_v(x,u,v) + \mu_2|v|^4v, \quad \text{in } \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, the nonlinearity F is subcritical and locally superlinear at infinity, and they obtained multiple solutions with the aid of the symmetric mountain-pass theorem. For the degenerate case, Xiang et al. [45] investigated the existence and asymptotic behaviour of solutions to critical Schrödinger–Kirchhoff type systems by applying the mountain-pass theorem and Ekeland's variational principle.

It is well known that the Limit Index Theory due to Li [18] is one of the most effective methods to study the existence of infinitely many solutions for the noncooperative system. For example, Song and Shi [40] considered the noncooperative critical nonlocal system with variable exponents, Fang and Zhang [8] studied systems of $p\&q$-Laplacian elliptic equations with critical Sobolev exponent, Liang et al. [22] dealt with a class of noncooperative Kirchhoff-type system involving the fractional p-Laplacian and critical exponents. We also refer the interested reader to Huang and Li [12], Liang and Shi [20], and Liang and Zhang [21] for some applications of this method. However, to the best of our knowledge, none of the cited works address the system with upper critical exponent and magnetic operator in $\mathbb{R}^N$ in the degenerate case.

Inspired by the previously mentioned works, our main objective is to study the existence and multiple solutions to problem (1.1), by means of the Limit Index Theory. To the best of our knowledge, this is the first time in the literature to use the Limit Index Theory to investigate the degenerate fractional Kirchhoff-type system with magnetic fields and upper critical growth. Our main result is the following.

Theorem 1.1. *Suppose that assumptions* $(\mathcal{M})$ *and* $(\mathcal{G})$ *are fulfilled. Then problem* (1.1) *has infinitely many solutions.*

Remark 1.2. The main difficulty of this paper lies in the following three aspects: First, to recover the compactness of the Palais-Smale sequence, we shall use the second concentration-compactness principle for the convolution type nonlocal problem in the fractional Sobolev space. Second, the appearance of the magnetic field also brings additional difficulties to the study of problem (1.1), such as effects of the magnetic fields on the linear spectral sets and on the solution structure. Finally, since we consider problem (1.1) in the whole space, to apply the Limit Index Theory, we need to establish new techniques to overcome this difficulty. To the best of our knowledge, our theorem is also valid for $s = 1$, hence the corresponding result in this case is new as well.

The organization of this paper is as follows: In Sect. 2, we give some basic definitions of fractional Sobolev space and the well known Hardy–Littlewood–Sobolev inequality. In Sect. 3, we mainly introduce the Limit Index Theory. In Sect. 4, we prove some compactness lemmas for the functional associated to our problem. The proof of the main result Theorem 1.1 is given in Sect. 5.

2. Preliminaries

In this section, we collect some known results for the readers convenience and the later use. First, we shall give some useful facts for the fractional order Sobolev spaces. Let $H^s(\mathbb{R}^N)$ be a fractional order Sobolev spaces which is defined as follows

$$H^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N) : [u]_s < \infty\right\},$$

where $[u]_s$ denotes the Gagliardo semi-norm

$$[u]_s := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y\right)^{1/2},$$

equipped with the inner product

$$\langle u, v\rangle := \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} \xi\eta \mathrm{d}x \quad \text{for all} \quad u, v \in H^s(\mathbb{R}^N)$$

and the norm

$$\|u\|_s := \left([u]_s^2 + \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Here, $L^2(\mathbb{R}^N)$ denotes the Lebesgue space of real-valued functions with $\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x < \infty$. From [6, Theorem 6.7], we know that the embedding $H^s(\mathbb{R}^N) \hookrightarrow$

$L^t(\mathbb{R}^N)$ is continuous for any $t \in [2, 2_s^*]$. Moreover, there exists a positive constant C_t such that

$$|u|_{L^t(\mathbb{R}^N)} \le C_t \|u\|_s \quad \text{for all } u \in H^s(\mathbb{R}^N). \tag{2.1}$$

To obtain the existence of radial weak solutions to system (1.1), we shall use the following embedding theorem due to Lions [26].

Theorem 2.1. *Assume that $0 < s < 1$ and $2s < N$. Then the embedding*

$$H_r^s(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^t(\mathbb{R}^N),$$

is compact for any $2 < t < 2_s^*$, where $H_r^s(\mathbb{R}^N)$ is radial symmetric space, defined by

$$H_r^s(\mathbb{R}^N) := \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|), x \in \mathbb{R}^N\}.$$

Suppose that $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function. Then

$$[\,u]_{s,A} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{1/2}$$

is the Gagliardo semi-norm. Define

$$\mathcal{H}^s := \left\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) : [u]_{s,A} < \infty \right\}.$$

It can be endowed with the norm

$$\|u\|_{s,A} := \left([u]_{s,A}^2 + \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

The scalar product on $\mathcal{H}^s$ is defined by

$$(\xi, \eta)_{s,A} := \langle \xi, \eta \rangle_{L^2} + \langle \xi, \eta \rangle_{s,A},$$

where

$$\langle \xi, \eta \rangle_{s,A} = \mathscr{R} \iint_{\mathbb{R}^{2N}} \frac{(\xi(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} \xi(y)) \overline{(\eta(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} \eta(y))}}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y.$$

From [4, Proposition 2.1], one knows that $(\mathcal{H}^s, (\cdot,\cdot)_{s,A})$ is a real Hilbert space. Moreover, the space $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ is a subspace of $\mathcal{H}^s$, see [4, Proposition 2.2].

Let $H_A^s(\mathbb{R}^N)$ be the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ in $\mathcal{H}^s$. Then we have the following lemma, and its proof can be found in d'Avenia and Squassina [4].

Lemma 2.1. *Let $u \in H_A^s(\mathbb{R}^N)$. Then $|u| \in H^s(\mathbb{R}^N)$, that is,*

$$\||u|\|_s \le \|u\|_{s,A} \quad \textit{for all } u \in H_A^s(\mathbb{R}^N).$$

Following the same discussion as in d'Avenia and Squassina [4], together with Lemma 2.1, we arrive at the following embedding result.

Lemma 2.2. *The space $H_A^s(\mathbb{R}^N, \mathbb{C})$ is continuously embedded in $L^\vartheta(\mathbb{R}^N, \mathbb{C})$ for all $\vartheta \in [2, 2_s^*]$. Furthermore, the space $H_A^s(\mathbb{R}^N, \mathbb{C})$ is continuously compactly embedded in $L^\vartheta(K, \mathbb{C})$ for all $\vartheta \in [2, 2_s^*]$ and any compact set $K \subset \mathbb{R}^N$.*

From Lemma 2.1, Theorem 2.1, and the Brézis-Lieb Lemma, we obtain the following lemma.

Lemma 2.3. *Let*

$$H^s_{r,A}(\mathbb{R}^N,\mathbb{C}) := \{u \in H^s_A(\mathbb{R}^N,\mathbb{C}) : u(x) = u(|x|), x \in \mathbb{R}^N\}.$$

Then the space $H^s_{r,A}(\mathbb{R}^N,\mathbb{C})$ is continuously compactly embedded in $L^\tau(\mathbb{R}^N,\mathbb{C})$ for any $\tau \in (2, 2^*_s)$.

By [6, Proposition 3.6], we have

$$[u]_s = \|(-\Delta)^{\frac{s}{2}}\|_{L^2(\mathbb{R}^N)} \quad \text{for any } u \in H^s(\mathbb{R}^N),$$

i.e.

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x dy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 \mathrm{d}x.$$

Moreover,

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \mathrm{d}x \mathrm{d}y = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(x) \cdot (-\Delta)^{\frac{s}{2}} v(x) \mathrm{d}x.$$

Now, recall the well known Hardy–Littlewood–Sobolev inequality, see [25, Theorem 4.3].

Lemma 2.4. *Assume that $p, r > 1$ and $0 < \mu < N$ with $1/p + (N-\mu)/N + 1/r = 2$, $f \in L^p(\mathbb{R}^N)$, and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(p, r, \mu, N)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\mu}} \; dx dy \le C(p, r, \mu, N) \|f\|_{L^p} \|h\|_{L^r}. \tag{2.2}$$

Set $p = r = 2N/(N+\mu)$. Then

$$C(p, r, \mu, N) = C(N, \mu) = \pi^{\frac{N-\mu}{2}} \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{N+\mu}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{\frac{\mu}{N}}.$$

If $u = v = |w|^q$, then Lemma 2.4 implies that

$$\int_{\mathbb{R}^N} (\mathcal{I}_\mu * |w|^q)\, |w|^q \, \mathrm{d}x$$

is well defined, if $w \in L^{rq}(\mathbb{R}^N)$ for some $r > 1$ satisfying $2/r + (N-\mu)/N = 2$. Thus, if $w \in H^s(\mathbb{R}^N)$, then by the Sobolev embedding theorem we get that $q \in [p_*, p^*]$. In particular, in the upper critical case,

$$\int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u|^{p^*}\right) |u|^{p^*} \, \mathrm{d}x \le C(N, \mu) \|u\|_{2^*_s}^{2p^*} \tag{2.3}$$

and the equality holds if and only if

$$u = C \left(\frac{l}{l^2 + |x-m|^2} \right)^{\frac{N-2}{2}}, \tag{2.4}$$

for some $x_0 \in \mathbb{R}^N$, where $C > 0$ and $l > 0$, see [25]. Let

$$S = \inf_{u \in D^s(\mathbb{R}^N)\setminus\{0\}} \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x dy : \int_{\mathbb{R}^N} |u|^{2^*_s} \mathrm{d}x = 1 \right\} \tag{2.5}$$

and

$$S_H = \inf_{u\in D^s(\mathbb{R}^N)\setminus\{0\}} \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y : \right. \\ \left. \times \int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u|^{p^*}\right) |u|^{p^*} \,\mathrm{d}x = 1 \right\}. \tag{2.6}$$

By (2.5) and (2.3), S_H is achieved if and only if u satisfies (2.4) and $S_H = S/C(N,\mu)^{\frac{1}{p^*}}$, see Mukherjee and Sreenadh [34].

3. Limit Index Theory

In this section, we shall show that all hypotheses of the Limit Index Theory are satisfied and this will eventually yield the conclusion that there exist infinitely many solutions. To this end, we introduce some definitions from the Limit Index Theory that can be found in Li [18] and Willem [43], the reader may also refer to Fang and Zhang [8] and Liang and Shi [20].

Definition 3.1 [18,43]. The action of a topological group G on a normed space Z is a continuous map

$$G \times Z \to Z : [g, z] \mapsto gz$$

such that

$$1 \cdot z = z, \quad (gh)z = g(hz) \quad z \mapsto gz \text{ is linear}, \quad \text{for all } g, h \in G.$$

The action is isometric if

$$\|gz\| = \|z\| \quad \text{for all } g \in G, \quad z \in Z$$

and in this case Z is called a G-space.
The set of invariant points is defined by

$$\mathrm{Fix}(G) := \{z \in Z : gz = z, \quad \text{for all} \quad g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Assume that Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \text{ for all} \quad g \in G\}$$

be a family of all G-invariant closed subsets of Z, and let

$$\Gamma := \left\{h \in C^0(Z, Z) : h(gu) = g(hu), \quad \text{for all} \quad g \in G\right\}$$

be the class of all G-equivariant mappings of Z. Finally, we call the set

$$O(u) := \{gu : g \in G\}$$

the G-orbit of u.

Definition 3.2 (*see* [18]). An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, $h \in \Gamma$, the following conditions are satisfied:
1. $i(A) = 0 \Leftrightarrow A = \emptyset$;
2. (Monotonicity) $A \subset B \Rightarrow i(A) \leq i(B)$;
3. (Subadditivity) $i(A \cup B) \leq i(A) + i(B)$;
4. (Supervariance) $i(A) \leq i(\overline{h(A)})$, for all $h \in \Gamma$;
5. (Continuity) If A is compact and $A \cap \mathrm{Fix}(G) = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighbourhood N of A such that $i(\overline{N}) = i(A)$;
6. (Normalization) If $x \notin \mathrm{Fix}(G)$, then $i(O(x)) = 1$.

Definition 3.3 (*see* [2]). An index theory is said to satisfy the d-dimensional property if there is a positive integer d such that

$$i(V^{dk} \cap S_1) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}(G) = \{0\}$, where S_1 is the unit sphere in Z.

Suppose that U and V are G-invariant closed subspaces of Z such that

$$Z = U \oplus V,$$

where V is infinite-dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \ldots$, and $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$. Let

$$Z_j = U \bigoplus V_j,$$

and for all $A \in \Sigma$, let

$$A_j = A \bigoplus Z_j.$$

Definition 3.4 (*see* [18]). Let i be an index theory satisfying the d-dimensional property. A limit index with respect to (Z_j) induced by i is a mapping

$$i^\infty : \Sigma \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^\infty(A) = \limsup_{j \to \infty} (i(A_j) - n_j).$$

Proposition 3.1 (see [18]). *Let $A, B \in \Sigma$. Then i^∞ satisfies:*
1. $A = \emptyset \Rightarrow i^\infty = -\infty$;
2. *(Monotonicity)* $A \subset B \Rightarrow i^\infty(A) \leq i^\infty(B)$;
3. *(Subadditivity)* $i^\infty(A \cup B) \leq i^\infty(A) + i^\infty(B)$;
4. *If* $V \cap \mathrm{Fix}(G) = \{0\}$, *then* $i^\infty(S_\rho \cap V) = 0$, *where* $S_\rho = \{z \in Z : \|z\| = \rho\}$;
5. *If* Y_0 *and* $\widetilde{Y_0}$ *are* G*-invariant closed subspaces of* V *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim(Y_0) = dm$, *then* $i^\infty(S_\rho \cap Y_0) \geq -m$.

Definition 3.5 (*see* [43]). A functional $I \in C^1(Z, R)$ is said to satisfy the condition $(PS)_c^*$ if any sequence $\{u_{n_k}\}$, $u_{n_k} \in Z_{n_k}$ such that

$$I(u_{n_k}) \to c, \quad dI_{n_k}(u_{nk}) \to 0, \quad \text{as } k \to \infty,$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimensional subspace of Z, $I_{n_k} = I|_{Z_{n_k}}$.

Theorem 3.1 (see [18]). *Assume that*
(D_1) $I \in C^1(Z, R)$ *is* G*-invariant;*
(D_2) *There are* G*-invariant closed subspaces* U *and* V *such that* V *is infinite-dimensional and* $Z = U \oplus V$*;*
(D_3) *There is a sequence of* G*-invariant finite-dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \quad \dim(V_j) = dn_j,$$

such that $V = \overline{\cup_{j=1}^{\infty} V_j}$*;*
(D_4) *There is an index theory* i *on* Z *satisfying the* d*-dimensional property;*
(D_5) *There are* G*-invariant subspaces* Y_0, $\widetilde{Y_0}$, Y_1 *of* V *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $Y_1, \widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim(\widetilde{Y_0}) = dm < dk = \dim(Y_1)$*;*
(D_6) *There are* α *and* β, $\alpha < \beta$, *such that* f *satisfies* $(PS)_c^*$, *for all* $c \in [\alpha, \beta]$*;*

$$(\mathrm{D}_7) \quad \begin{cases} (a) \textit{ either } \mathrm{Fix}(G) \subset U \oplus Y_1, \quad \textit{or} \quad \mathrm{Fix}(G) \cap V = \{0\}, \\ (b) \textit{ there is } \rho > 0 \textit{ such that for all } u \in Y_0 \cap S_\rho, f(z) \geq \alpha, \\ (c) \textit{ for all } z \in U \oplus Y_1,\ f(z) \leq \beta, \end{cases}$$

If i^∞ *is the limit index corresponding to* i*, then the numbers*

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} f(u), \quad -k+1 \leq j \leq -m,$$

are critical values of f*, and* $\alpha \leq c_{-k+1} \leq \cdots \leq c_{-m} \leq \beta$*. Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \geq 0$*, then* $i(\mathbb{K}_c) \geq r+1$*, where* $\mathbb{K}_c = \{z \in Z : df(z) = 0, f(z) = c\}$.

4. Proof of $(PS)_c$ Condition

In this section, to overcome the lack of compactness caused by the critical exponents, we intend to employ the second concentration-compactness principle introduced in Li et al. [19]. In consideration of the appearance of convolution, it is natural to consider a variant of the concentration-compactness principle for the convolution type problem in the fractional Sobolev space. Since the proof is similar to that of [11,27,47], we omit the details.

Lemma 4.1 [19]. *Assume that* $\{u_n\}$ *be a bounded sequence in* $H^s(\mathbb{R}^N)$ *satisfying* $u_n \rightharpoonup u$ *weakly in* $H^s(\mathbb{R}^N)$, $u_n \to u$ *strongly in* $L^2_{loc}(\mathbb{R}^N)$ *and* $u_n \to u$ *a.e. on* $\mathbb{R}^N$*. Let* $|(-\Delta)^{\frac{s}{2}} u_n|^2 \rightharpoonup \omega$, $|u_n|^{2_s^*} \rightharpoonup \xi$ *and* $\left(\mathcal{I}_\mu * |u_n|^{p^*}\right)|u_n|^{p^*} \rightharpoonup \nu$ *weakly in the sense of measures, where* ω, ξ *and* ν *are bounded nonnegative measures on* $\mathbb{R}^N$*. Define*

$$\omega_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx,$$

$$\xi_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} |u_n|^{2_s^*} \, dx,$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} \left(\mathcal{I}_\mu * |u_n|^{p^*}\right) |u_n|^{p^*} \, dx.$$

Then there exists a (at most countable) set of distinct points $\{x_i\}_{i\in I} \subset \mathbb{R}^N$ *and a family of positive numbers* $\{\nu_i\}_{i\in I}$ *such that*

$$\nu = \left(\mathcal{I}_\mu * |u|^{p^*}\right) |u|^{p^*} + \sum_{i\in I} \nu_i \delta_{x_i}, \quad \sum_{i\in I} \nu_i^{\frac{N}{N+\mu}} < \infty, \tag{4.1}$$

$$\xi \geq |u|^{2_s^*} + C_\mu(N)^{-\frac{N}{N+\mu}} \sum_{i\in I} \nu_i^{\frac{N}{N+\mu}} \delta_{x_i}, \quad \xi_i \geq C_\mu(N)^{-\frac{N}{N+\mu}} \nu_i^{\frac{N}{N+\mu}}, \tag{4.2}$$

and

$$\omega \geq |(-\Delta)^{\frac{s}{2}} u|^2 + S_H \sum_{i\in I} \nu_i^{\frac{1}{p^*}} \delta_{x_i}, \quad \omega_i \geq S_H \nu_i^{\frac{1}{p^*}}, \tag{4.3}$$

where δ_{x_i} *is the Dirac-mass of mass 1 concentrated at* $x \in \mathbb{R}^N$*. For the energy at infinity, we have*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u_n|^{p^*}\right) |u_n|^{p^*} \, dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty, \tag{4.4}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx = \int_{\mathbb{R}^N} d\omega + \omega_\infty, \tag{4.5}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx = \int_{\mathbb{R}^N} d\xi + \xi_\infty, \tag{4.6}$$

$$\xi_\infty \leq \left(S^{-1} \omega_\infty\right)^{\frac{2_s^*}{2}}, \tag{4.7}$$

$$\nu_\infty \leq C_\mu(N) \left(\int_{\mathbb{R}^N} d\xi + \xi_\infty\right)^{\frac{N+\mu}{2N}} \xi_\infty^{\frac{N+\mu}{2N}}, \tag{4.8}$$

and

$$\nu_\infty \leq S_H^{-p^*} \left(\int_{\mathbb{R}^N} d\omega + \omega_\infty\right)^{\frac{p^*}{2}} \omega_\infty^{\frac{p^*}{2}}. \tag{4.9}$$

Now, to show that all hypotheses of the Limit Index Theory are satisfied 3.1, we denote $G_1 = O(N)$, where $O(N)$ is the group of orthogonal linear transformations in $\mathbb{R}^N$, $E = H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ and

$$E_{G_1} = H^s_{r,A,O(N)} := \{u \in H^s_{r,A}(\mathbb{R}^N, \mathbb{C}) : gu(x) = u(g^{-1}x) = u(x), g \in O(N)\}.$$

For convenience, let $G_2 = \mathbb{Z}_2$, $Y = E \times E$, $X = Y_{G_1} = E_{G_1} \times E_{G_1}$. The space Y is endowed with the norm $\|(u,v)\|_{s,A} = \|u\|_{s,A} + \|v\|_{s,A}$. Using d'Avenia and Squassina [4], it is easy to prove that $(Y, \|\cdot\|_{s,A})$ is a reflexive Banach space. The corresponding energy functional of problem (1.1) is given by

$$\mathcal{J}(u,v) = -\frac{1}{2}\mathscr{M}(\|u\|_{s,A}^2) + \frac{1}{2}\mathscr{M}(\|v\|_{s,A}^2) - \frac{1}{2p^*}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |u|^{p^*}\right)|u|^{p^*}\,\mathrm{d}x$$
$$-\frac{1}{2p^*}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v|^{p^*}\right)|v|^{p^*}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} G(|x|,|u|^2,|v|^2)\mathrm{d}x \quad (4.10)$$

for each $(u,v) \in Y$. By condition $(\mathcal{F})$, we know that the functional $\mathcal{J}$ is well defined on Y and belongs to $C^1(Y,\mathbb{R})$. Moreover, its Fréchet derivative is given by

$$\langle \mathcal{J}'(u,v),(z_1,z_2)\rangle = -\mathfrak{M}(\|u\|_{s,A}^2)\left(\langle u,z_1\rangle_{s,A} + \mathscr{R}\int_{\mathbb{R}^N} u\overline{z}_1\mathrm{d}x\right)$$
$$+\mathfrak{M}(\|v\|_{s,A}^2)\left(\langle v,z_2\rangle_{s,A} + \mathscr{R}\int_{\mathbb{R}^N} v\overline{z}_2\mathrm{d}x\right)$$
$$-\mathscr{R}\int_{\mathbb{R}^N}(\mathcal{I}_\mu * |u|^{p^*})|u|^{p^*-2}u\overline{z}_1\mathrm{d}x - \mathscr{R}\int_{\mathbb{R}^N}(\mathcal{I}_\mu * |v|^{p^*})|v|^{p^*-2}v\overline{z}_2\mathrm{d}x$$
$$-\mathscr{R}\int_{\mathbb{R}^N} F_u(|x|,|u|^2,|v|^2)u\overline{z}_1\mathrm{d}x - \mathscr{R}\int_{\mathbb{R}^N} F_v(|x|,|u|^2,|v|^2)v\overline{z}_2\mathrm{d}x = 0$$

for any $(u,v),(z_1,z_2) \in Y$, where

$$\langle \zeta, z_i\rangle_{s,A}$$
$$= \mathscr{R}\iint_{\mathbb{R}^{2N}} \frac{(\zeta(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}\zeta(y))\overline{(z_i(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}z_i(y))}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$

for any $\zeta, z_i \in H_A^s(\mathbb{R}^N,\mathbb{C})$ $(i = 1,2)$. By condition $(\mathcal{F})$, we know that the functional $\mathcal{J}$ is $O(N)$-invariant. Therefore, from the principle of symmetric criticality of Krawcewicz and Marzantowicz [15], we know that (u,v) is a critical point of $\mathcal{J}$ if and only if (u,v) is a critical point of $J = \mathcal{J}|_{X=E_{G_1}\times E_{G_1}}$. So, we just need to prove the existence of a sequence of critical points $\mathcal{J}$ on Y.

Now, we begin to prove the $(PS)_c$ condition.

Lemma 4.2. *Let* $(\mathcal{M})$ *and* $(\mathcal{G})$ *hold,* $\{(u_{n_k},v_{n_k})\}$ *be a sequence such that* $\{(u_{n_k},v_{n_k})\} \in X_{n_k}$,

$$J_{n_k}(u_{n_k},v_{n_k}) \to c < c^* \quad \text{as } k \to \infty,$$

where $J_{n_k} = \mathcal{J}|_{X_{n_k}}$ *and*

$$c^* = \min\left\{\left(\frac{1}{\theta} - \frac{1}{2p^*}\right)(\mathfrak{m}_1 S_H^\sigma)^{\frac{p^*}{p^*-\sigma}}, \left(\frac{1}{\theta} - \frac{1}{2p^*}\right)\right.$$
$$\left.\times\left(\mathfrak{m}_1\hat{C}(\mu,N)^{-1}S^{\frac{p^*}{2}}\right)^{\frac{2}{p^*-2\sigma}}\right\}.$$

Then there exists a subsequence of $\{(u_{n_k},v_{n_k})\}$ *strongly convergent in* X.

Proof. If $\inf_{n\in\mathbb{N}}\|u_{n_k}\|_{s,A} = 0$ or $\inf_{n\in\mathbb{N}}\|v_{n_k}\|_{s,A} = 0$, then there exists a subsequence of $\{u_{n_k}\}$ (or $\{v_{n_k}\}$) such that $u_{n_k} \to 0$ or $v_{n_k} \to 0$ in E_{G_1} as $n \to$

∞. Thus, we can assume that $\inf_{n\in\mathbb{N}} \|u_{n_k}\|_s = d_1 > 0$ and $\inf_{n\in\mathbb{N}} \|v_{n_k}\|_s = d_2 > 0$ in the sequel.

On the one hand, from (M_1) and (F_3), we get

$$\begin{aligned} o(1)\|u_{n_k}\|_{s,A} &\geq \langle -dJ_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= \mathfrak{M}(\|u_{n_k}\|^2_{s,A})\|u_{n_k}\|^2_{s,A} + \int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u_{n_k}|^{p^*}\right) |u_{n_k}|^{p^*} \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} G_u(|x|, |u_{n_k}|^2, |v_{n_k}|^2)|u_{n_k}|^2 \mathrm{d}x \\ &\geq \mathfrak{m}^* \|u_{n_k}\|^2_{s,A}. \end{aligned} \tag{4.11}$$

Therefore,$\{u_{n_k}\}$ is bounded in E_{G_1}. On the other hand, from (F_2) and the fact that $2\sigma < \theta < 2p^*$, we have

$$\begin{aligned} c + o(1)\|v_{n_k}\|_{s,A} &= J_{n_k}(0, v_{n_k}) - \frac{1}{\theta}\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle \\ &= \frac{1}{2}\mathscr{M}(\|v_{n_k}\|^2_{s,A}) - \frac{1}{\theta}\mathfrak{M}(\|v_{n_k}\|^2_{s,A})\|v_{n_k}\|^2_{s,A} \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{2p^*}\right) \int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |v_{n_k}|^{p^*}\right) |v_{n_k}|^{p^*} \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \left[G(|x|, 0, |v_{n_k}|^2) - \frac{1}{\theta} G_v(|x|, 0, |v_{n_k}|^2)|v_{n_k}|^2\right] \mathrm{d}x \\ &\geq \left(\frac{1}{2\sigma} - \frac{1}{\theta}\right) \mathfrak{M}(\|v_{n_k}\|^2_{s,A})\|v_{n_k}\|^2_{s,A} \\ &\geq \left(\frac{1}{2\sigma} - \frac{1}{\theta}\right) \mathfrak{m}^* \|v_{n_k}\|^2_{s,A}. \end{aligned} \tag{4.12}$$

This fact implies that $\{v_{n_k}\}$ is bounded in E_{G_1}. Thus $\|u_{n_k}\|_{s,A} + \|v_{n_k}\|_{s,A}$ is bounded in X.

Next, we shall prove that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence strongly convergent in X. Since $\{u_{n_k}\}$ is bounded in E_{G_1} it follows that, up to a subsequence, $u_{n_k} \rightharpoonup u_0$ weakly in E_{G_1} and $u_{n_k} \to u_0$, a.e. in $\mathbb{R}^N$. Thus, it follows from (M_1) and (F_3) that

$$\begin{aligned} 0 &\leftarrow \langle -dJ_{n_k}(u_{n_k} - u_0, v_{n_k}), (u_{n_k} - u_0, 0)\rangle \\ &= \mathfrak{M}(\|u_{n_k} - u_0\|^2_{s,A})\|u_{n_k} - u_0\|^2_{s,A} \\ &\quad + \int_{\mathbb{R}^N} G_u(|x|, |u_{n_k} - u_0|^2, |v_{n_k}|^2)|u_{n_k} - u_0|^2 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u_{n_k} - u_0|^{p^*}\right) |u_{n_k} - u_0|^{p^*} \,\mathrm{d}x \\ &\geq \mathfrak{m}^* \|u_{n_k} - u_0\|^2_{s,A}, \end{aligned}$$

which implies that

$$u_{n_k} \to u_0 \quad \text{strongly in } E_{G_1}. \tag{4.13}$$

It suffices to prove that there exists $v_0 \in E_{G_1}$ such that

$$v_{n_k} \to v_0 \quad \text{strongly in } E_{G_1}. \tag{4.14}$$

Next, to prove (4.14), we divide the following proof into three claims.
Claim 1. Fix $i \in I$. Then either $\nu_i = 0$ or

$$\nu_i \geq (\mathfrak{m}_1 S_H^{\sigma})^{\frac{p^*}{p^*-\sigma}}. \tag{4.15}$$

To prove (4.15), we take $\phi \in C_0^{\infty}(\mathbb{R}^N)$ be a radial symmetric function satisfying $0 \leq \phi \leq 1$; $\phi \equiv 1$ in $B(x_i, \epsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_i, 2\epsilon)$. For any $\epsilon > 0$, define $\phi_\epsilon := \phi\left(\frac{x-x_i}{\epsilon}\right)$, where $i \in I$. Clearly $\{\phi_\epsilon v_{n_k}\}$ is bounded in $H^s_{r,A}(\mathbb{R}^N, \mathbb{C})$ and $\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\phi_\epsilon)\rangle \to 0$ as $n \to \infty$. Hence

$$\begin{aligned}
&\mathfrak{M}\left(\|v_{n_k}\|^2_{s,A}\right)\left(\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|^2\phi_\epsilon(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right.\\
&\quad\left.+\int_{\mathbb{R}^N}|v_{n_k}|^2\phi_\epsilon(x)\mathrm{d}x\right) = -\mathscr{R}\left\{\mathfrak{M}\left(\|v_{n_k}\|^2_{s,A}\right)\right.\\
&\left.\iint_{\mathbb{R}^{2N}} \frac{(v_{n_k}(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y))\overline{v_{n_k}(x)(\phi_\epsilon(x)-\phi_\epsilon(y))}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right\}\\
&\quad+\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\phi_\epsilon\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N} G_v(|x|, |u_{n_k}|^2, |v_{n_k}|^2)|v_{n_k}|^2\phi_\epsilon\mathrm{d}x + o_n(1).
\end{aligned}\tag{4.16}$$

We deduce from (M_2) and diamagnetic inequality that

$$\begin{aligned}
&\mathfrak{M}\left(\|v_{n_k}\|^2_{s,A}\right)\left(\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|^2\phi_\epsilon(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right.\\
&\quad\left.+\int_{\mathbb{R}^N}|v_{n_k}|^2\phi_\epsilon(x)\mathrm{d}x\right)\\
&\geq \mathfrak{m}_1\left(\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|^2\phi_\epsilon(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right.\\
&\quad\left.+\int_{\mathbb{R}^N}|v_{n_k}|^2\phi_\epsilon(x)\mathrm{d}x\right)^\sigma\\
&\geq \mathfrak{m}_1\left(\iint_{\mathbb{R}^{2N}} \frac{\left||v_{n_k}(x)| - |v_{n_k}(y)|\right|^2\phi_\epsilon(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^\sigma.
\end{aligned}\tag{4.17}$$

We note that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{\left||v_{n_k}(x)| - |v_{n_k}(y)|\right|^2\phi_\epsilon(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\phi_\epsilon\mathrm{d}\omega = \omega_i \tag{4.18}$$

and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\phi_\epsilon\,\mathrm{d}x = \lim_{\epsilon\to 0}\int_{\mathbb{R}^N}\phi_\epsilon\mathrm{d}\nu = \nu_i. \tag{4.19}$$

By the Hölder inequality, we have

$$\begin{aligned}
&\Bigg|\mathscr{R}\Bigg\{\mathfrak{M}\left(\|v_{n_k}\|_{s,A}^2\right)\\
&\int\int_{\mathbb{R}^{2N}} \frac{(v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y))\overline{v_{n_k}(x)(\phi_\epsilon(x)-\phi_\epsilon(y))}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\Bigg\}\Bigg|\\
&\le C\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|\cdot|\phi_\epsilon(x)-\phi_\epsilon(y)|\cdot|v_{n_k}(x)|}{|x-y|^{N+2s}}\mathrm{d}xdy\\
&\le C\left(\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x)|^2|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^{N+2s}}\mathrm{d}xdy\right)^{1/2}. \qquad (4.20)
\end{aligned}$$

As the proof of Zhang et al. [47, Lemma 3.4], we get

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|v_{n_k}(x)|^2|\phi_\epsilon(x)-\phi_\epsilon(y)|^2}{|x-y|^{N+2s}}\mathrm{d}xdy=0. \qquad (4.21)$$

Furthermore, the Lebesgue dominated convergence theorem and (F_1) imply that

$$\begin{aligned}
&\int_{\mathbb{R}^N} G_v(|x|,|u_{n_k}|^2,|v_{n_k}|^2)|v_{n_k}|^2\phi_\epsilon(x)\mathrm{d}x\\
&\quad\to\int_{\mathbb{R}^N} G_v(|x|,|u_0|^2,|v_0|^2)|v_0|^2\phi_\epsilon(x)\mathrm{d}x \quad \text{as } n\to\infty. \qquad (4.22)
\end{aligned}$$

The definition of $\phi_\epsilon(x)$ gives us

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} G_v(|x|,|u_0|^2,|v_0|^2)|v_0|^2\phi_\epsilon\mathrm{d}x\right|\\
&\quad\le\int_{B_\epsilon(0)} |G_v(|x|,|u_0|^2,|v_0|^2)|v_0|^2\mathrm{d}x\to 0 \quad \text{as } \epsilon\to 0. \qquad (4.23)
\end{aligned}$$

Combining (4.16)–(4.21), we get that

$$\nu_i\ge\mathfrak{m}_1\omega_i^\sigma.$$

It follows from (4.3) that $\nu_i=0$ or

$$\nu_i\ge\left(\mathfrak{m}_1 S_H^\sigma\right)^{\frac{p^*}{p^*-\sigma}}.$$

Claim 2. $\nu_i=0,$ for all $i\in I$ and $\nu_\infty=0$.

Indeed, if Claim 2 were false, then there would exist a $i\in I$ such that (4.15) would hold. Similar to (4.12), by (M_3) and (F_2), we deduce

$$\begin{aligned}
c&=\lim_{\epsilon\to 0}\lim_{n\to\infty}\left(J_{n_k}(0,v_{n_k})-\frac{1}{\theta}\langle dJ_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\rangle\right)\\
&\ge\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu*|v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\,\mathrm{d}x\\
&\ge\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu*|v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\phi_\epsilon\,\mathrm{d}x\\
&\ge\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\nu_i\ge\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\left(\mathfrak{m}_1 S_H^\sigma\right)^{\frac{p^*}{p^*-\sigma}}. \qquad (4.24)
\end{aligned}$$

On the other hand, we show that $\nu_\infty = 0$. For this, we take a cut off function $\phi_R \in C^\infty(\mathbb{R}^N)$ such that

$$\phi_R(x) = \begin{cases} 0 & |x| < R, \\ 1 & |x| > R+1 \end{cases}$$

and $|\nabla \phi_R| \leq 2/R$. Using the Hardy–Littlewood–Sobolev and Hölder's inequality, we get

$$\begin{aligned} \nu_\infty &= \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\phi_R(y)\mathrm{d}x \\ &\leq C_\mu(N)\lim_{R\to\infty}\lim_{n\to\infty}|v_{n_k}|_{2_s^*}^{p^*}\left(\int_{\mathbb{R}^N}|v_{n_k}(x)|^{2_s^*}\phi_R(y)\mathrm{d}x\right)^{\frac{p^*}{2_s^*}} \\ &\leq \hat{C}(\mu,N)\xi_\infty^{\frac{p^*}{2_s^*}}. \end{aligned} \tag{4.25}$$

Note that $\{\phi_R v_{n_k}\}$ is bounded in $H^s_{r,A}(\mathbb{R}^N,\mathbb{C})$. Hence, $\langle dJ_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k}\phi_R)\rangle \to 0$ as $n \to \infty$, which yields that

$$\begin{aligned} &\mathfrak{M}\left(\|v_{n_k}\|_{s,A}^2\right)\left(\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|^2\phi_R(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \right.\\ &\qquad\left. + \int_{\mathbb{R}^N}|v_{n_k}|^2\phi_R(x)\mathrm{d}x\right) \\ &= -\mathscr{R}\left\{\mathfrak{M}\left(\|v_{n_k}\|_{s,A}^2\right)\right. \\ &\qquad\left.\iint_{\mathbb{R}^{2N}}\frac{(v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y))\overline{v_{n_k}(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right\} \\ &\qquad + \int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\phi_R\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^N}G_v(|x|,|u_{n_k}|^2,|v_{n_k}|^2)|v_{n_k}|^2\phi_R\mathrm{d}x + o_n(1). \end{aligned} \tag{4.26}$$

We can easily deduce that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{||v_{n_k}(x)|-|v_{n_k}(y)||^2\phi_R(y)}{|x-y|^{N+2s}}\mathrm{d}xdy = \omega_\infty$$

and

$$\begin{aligned} &\left|\mathscr{R}\left\{\mathfrak{M}\left(\|v_{n_k}\|_{s,A}^2\right)\right.\right. \\ &\qquad\left.\left.\iint_{\mathbb{R}^{2N}}\frac{(v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y))\overline{v_{n_k}(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right\}\right| \\ &\leq C\left(\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}xdy\right)^{1/2}. \end{aligned}$$

Furthermore,

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\mathrm{d}xdy$$
$$=\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{N+2s}}\mathrm{d}xdy.$$

On the other hand, similar to the proof of Zhang et al. [47, Lemma 3.4], we have

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{N+2s}}\mathrm{d}xdy=0.$$

It follows from (M_2) that

$$\begin{aligned}
&\mathfrak{M}\left(\|v_{n_k}\|_{s,A}^2\right)\left(\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|^2\phi_R(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right.\\
&\qquad\left.+\int_{\mathbb{R}^N}|v_{n_k}|^2\phi_R(x)\mathrm{d}x\right)\\
&\quad\geq\mathfrak{m}_1\left(\iint_{\mathbb{R}^{2N}}\frac{|v_{n_k}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v_{n_k}(y)|^2\phi_R(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right.\\
&\qquad\left.+\int_{\mathbb{R}^N}|v_{n_k}|^2\phi_R(x)\mathrm{d}x\right)^{\sigma}\\
&\quad\geq\mathfrak{m}_1\left(\iint_{\mathbb{R}^{2N}}\frac{\left||u_n(x)|-|u_n(y)|\right|^2\phi_R(y)}{|x-y|^{N+2s}}\mathrm{d}xdy\right)^{\sigma}=\mathfrak{m}_1\omega_\infty^\sigma.
\end{aligned}$$

By the definition of ϕ_R and conditions (F_1)–(F_2), we have

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}F_v(|x|,|u_{n_k}|^2,|v_{n_k}|^2)|v_{n_k}|^2\phi_R\mathrm{d}x\\
&\quad=\lim_{R\to\infty}\int_{\mathbb{R}^N}F_v(|x|,|u_0|^2,|v_0|^2)|v_0|^2\phi_R\mathrm{d}x=0.
\end{aligned}$$

Therefore, by (4.26) together with (4.25), we can obtain that

$$\hat{C}(\mu,N)\xi_\infty^{\frac{p^*}{2_s^*}}\geq\nu_\infty\geq\mathfrak{m}_1\omega_\infty^\sigma.$$

It follows from (4.3) that $\nu_\infty=0$ or

$$\nu_\infty\geq\left(\mathfrak{m}_1\hat{C}(\mu,N)^{-1}S^{\frac{p^*}{2}}\right)^{\frac{2}{p^*-2\sigma}}.$$

Then we have

$$\begin{aligned}
c&=\lim_{R\to\infty}\lim_{n\to\infty}\left(J_{n_k}(0,v_{n_k})-\frac{1}{\theta}\langle dJ_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k})\rangle\right)\\
&\geq\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu*|v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\phi_R\,\mathrm{d}x\\
&\geq\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\nu_\infty\geq\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\left(\mathfrak{m}_1\hat{C}(\mu,N)^{-1}S^{\frac{p^*}{2}}\right)^{\frac{2}{p^*-2\sigma}}.
\end{aligned}\tag{4.27}$$

Invoking the arguments above and together with (4.24), (4.27), set

$$c^* = \min\left\{\left(\frac{1}{\theta}-\frac{1}{2p^*}\right)(\mathfrak{m}_1 S_H^{\sigma})^{\frac{p^*}{p^*-\sigma}}, \left(\frac{1}{\theta}-\frac{1}{2p^*}\right)\left(\mathfrak{m}_1 \hat{C}(\mu,N)^{-1} S^{\frac{p^*}{2}}\right)^{\frac{2}{p^*-2\sigma}}\right\}.$$

Then for any $c < c^*$ we have

$$\nu_i = 0 \quad \text{for all } i \in I \qquad \text{and} \qquad \nu_\infty = 0.$$

Claim 3. $v_{n_k} \to v_0$ strongly in E_{G_1}. Indeed, by Claim 2 and Lemma 3.1, we know

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v_{n_k}|^{p^*}\right)|v_{n_k}|^{p^*}\,\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v_0|^{p^*}\right)|v_0|^{p^*}\,\mathrm{d}x. \tag{4.28}$$

Now, we define the linear functional $\mathcal{L}(v)$ on E_{G_1} as follows

$$\begin{aligned}
[\mathcal{L}(v), w] &= \mathscr{R}\iint_{\mathbb{R}^{2N}}\frac{(v(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y))\overline{(w(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}w(y))}}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad + \mathscr{R}\int_{\mathbb{R}^N} v\overline{w}\mathrm{d}x\\
&= \langle v, w\rangle_{s,A} + \mathscr{R}\int_{\mathbb{R}^N} v\overline{w}\mathrm{d}x
\end{aligned}$$

for any $v \in E_{G_1}$. Obviously, $\mathcal{L}$ is a bounded bi-linear operator. By the Hölder inequality, we have

$$|[\mathcal{L}(v), w]| \le \|v\|_{s,A}\|w\|_{s,A}.$$

Since $v_{n_k} \rightharpoonup v_0$ weakly in E_{G_1} we have

$$\lim_{n\to\infty}[\mathcal{L}(v_0), v_{n_k} - v_0] = 0. \tag{4.29}$$

Clearly, $[\mathcal{L}(v_{n_k}), v_{n_k} - v_0] \to 0$ as $n \to \infty$. Hence by (4.29), one has

$$\lim_{n\to\infty}[\mathcal{L}(v_{n_k}) - \mathcal{L}(v_0), v_{n_k} - v_0] = 0. \tag{4.30}$$

Hence, by (4.29) and (4.30), one has

$$\begin{aligned}
o(1) &= \langle dJ_{n_k}(u_{n_k}, v_{n_k}) - dJ_{n_k}(u_0, v_0), (0, v_{n_k}) - (0, v_0)\rangle\\
&= \mathfrak{M}(\|v_{n_k}\|_{s,A}^2)\|v_{n_k}\|_{s,A}^2 - \mathfrak{M}(\|v_{n_k}\|_{s,A}^2)[L(v_{n_k}), v_0]\\
&\quad -\mathfrak{M}(\|v_0\|_{s,A}^2)[\mathcal{L}(v_0), v_{n_k} - v_0]\\
&\quad -\lambda\int_{\mathbb{R}^N}[G_v(|x|, u_{n_k}, v_{n_k}) - G_v(|x|, u_0, v_0)](v_{n_k} - v_0)\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^N}\left[(\mathcal{I}_\mu * |v_{n_k}|^{p^*})|v_{n_k}|^{p^*-2}v_{n_k} - (\mathcal{I}_\mu * |v_0|^{p^*})|v_0|^{p^*-2}v_0\right](v_{n_k} - v_0)\mathrm{d}x\\
&= \mathfrak{M}(\|v_{n_k}\|_{s,A}^2)[\mathcal{L}(v_{n_k}) - \mathcal{L}(v_0), v_{n_k} - v_0]\\
&\quad -\int_{\mathbb{R}^N}(\mathcal{I}_\mu * |v_{n_k} - v_0|^{p^*})|v_{n_k} - v_0|^{p^*}\,\mathrm{d}x + o(1).
\end{aligned} \tag{4.31}$$

This fact together with $\|v_{n_k}\|_{s,A} \to \beta$ implies that

$$\mathfrak{M}(\beta^2)\lim_{n\to\infty}\|v_{n_k} - v_0\|_{s,A}^2 = 0. \tag{4.32}$$

It follows from (M_1) that $v_{n_k} \to v_0$ strongly in E_{G_1}.

To sum up, we know that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in X and the proof of Lemma 4.2 is complete. □

5. Proof of Theorem 1.1

In this section, we prove that problem (1.1) has infinitely many solutions.

Proof of Theorem 1.1. To apply Theorem 3.1, we define

$$Y = U \oplus V, \quad U = E_{G_1} \times \{0\}, \quad V = \{0\} \times E_{G_1},$$
$$Y_0 = \{0\} \times E_{G_1}^{m^\perp}, \quad Y_1 = \{0\} \times E_{G_1}^{(k)},$$

where m and k are yet to be determined. Obviously, Y_0, Y_1 are G-invariant and

$$\operatorname{codim}_V Y_0 = m, \quad \dim Y_1 = k.$$

It's easy to verify that (D_1), (D_2), (D_4) in Theorem 3.1 are satisfied. Let

$$V_j = E_{G_1}^{(j)} = \operatorname{span}\{e_1, e_2, \ldots, e_j\}.$$

Hence (D_3) in Theorem 3.1 holds. To verify the conditions in (D_7) in Theorem 3.1, note that $\operatorname{Fix}(G) \cap V = 0$, thus (a) of (D_7) in Theorem 3.1 is satisfied. Now, we verify that (b) and (c) of (D_7) in Theorem 3.1 holds.

(i) Let $(0, v) \in Y_0 \cap S_{\rho_m}$, then from (F_1) and (F_3), we get

$$\begin{aligned} J(0,v) &= \frac{1}{2}\mathscr{M}(\|v\|_{s,A}^2) - \frac{1}{2p^*}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v|^{p^*}\right)|v|^{p^*}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} G(|x|, 0, |v|^2)\mathrm{d}x \\ &\geq \frac{\mathfrak{m}^*}{\sigma}\|v\|_{s,A}^2 - \frac{S_H^{-p^*}}{p_s^*}\|v\|_{s,A}^{2p^*} - c\|v\|_{s,A}^p. \end{aligned} \tag{5.1}$$

Therefore we can choose $\rho_m > 0$ such that $J(0, v) \geq \alpha$ for $\|v\|_{s,A} = \rho_m$ since $2 < p < 2p^*$. This fact implies that (b) of (D_7) in Theorem 3.1 holds.

(ii) From (F_1) we have

$$\begin{aligned} J(u,0) &= -\frac{1}{2}\mathscr{M}(\|u\|_{s,A}^2) - \frac{1}{2p^*}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |u|^{p^*}\right)|u|^{p^*}\,\mathrm{d}x - \int_{\mathbb{R}^N} G(|x|, |u|^2, 0)\mathrm{d}x \\ &\leq 0. \end{aligned}$$

On the other hand, we can take α such that

$$\alpha > \sup_{u \in E_{G_1}} J(u, 0).$$

Let $(u, v) \in U \oplus Y_1$, then we have

$$\begin{aligned} J(u,v) &= -\frac{1}{2}\mathscr{M}(\|u\|_{s,A}^2) + \frac{1}{2}\mathscr{M}(\|v\|_{s,A}^2) - \frac{1}{2p^*}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |u|^{p^*}\right)|u|^{p^*}\,\mathrm{d}x \\ &\quad - \frac{1}{2p^*}\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * |v|^{p^*}\right)|v|^{p^*}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} G(|x|, |u|^2, |v|^2)\mathrm{d}x \\ &\leq \frac{c}{2}\|v\|_{s,A}^2 - \frac{c}{2p^*}\|u\|_{s,A}^{2p^*} + \alpha. \end{aligned}$$

Note that all norms are equivalent on the finite-dimensional space Y_1, so we can choose $k > m$ and $\beta_k > \alpha_m$ such that

$$J_{U \oplus Y_1} \leq \beta_k.$$

Hence (c) of (D_7) in Theorem 3.1 holds. By Lemma 4.2, $J(u,v)$ satisfies the condition of $(PS)_c^*$ for any $c \in [\alpha_m, \beta_k]$. Therefore (D_6) in Theorem 3.1 holds. Hence, by Theorem 3.1, we know that

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} J(u,v), \quad -k+1 \leq j \leq -m, \quad \alpha_m \leq c_j \leq \beta_k,$$

are critical values of J. Letting $m \to \infty$, we can obtain an unbounded sequence of critical values c_j. Since the functional J is even, we can get two critical points $(\pm u_j, \pm v_j)$ of J corresponding to c_j. □

Acknowledgements

S. Shi was supported by NSFC grant (no. 11771177), China Automobile Industry Innovation and Development Joint Fund (no. U1664257), Program for Changbaishan Scholars of Jilin Province and Program for JLU Science, Technology Innovative Research Team (no. 2017TD-20). D.D. Repovš was supported by the Slovenian Research Agency (nos. P1-0292, N1-0114, N1-0083).

References

[1] Ambrosio, V.: Zero mass case for a fractional Berestycki–Lions-type problem. Adv. Nonlinear Anal. **7**, 365–374 (2018)

[2] Benci, V.: On critical point theory for indefinite functionals in presence of symmetries. Trans. Am. Math. Soc. **274**, 533–572 (1982)

[3] Cassani, D., Zhang, J.: Choquard-type equations with Hardy–Littlewood–Sobolev upper critical growth. Adv. Nonlinear Anal. **8**, 1184–1212 (2019)

[4] d'Avenia, P., Squassina, M.: Ground states for fractional magnetic operators. ESAIM COCV **24**, 1–24 (2018)

[5] d'Avenia, P., Siciliano, G., Squassina, M.: On fractional Choquard equations. Math. Models Methods Appl. Sci. **25**(8), 1447–1476 (2014)

[6] Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhike's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**, 521–573 (2012)

[7] Düzgün, F., Gamze, F., Iannizzotto, A.: Three nontrivial solutions for nonlinear fractional Laplacian equations. Adv. Nonlinear Anal. **7**, 211–226 (2018)

[8] Fang, Y., Zhang, J.: Multiplicity of solutions for a class of elliptic systems with critical Sobolev exponent. Nonlinear Anal. **73**, 2767–2778 (2010)

[9] Fiscella, A., Pucci, P., Zhang, B.: p-fractional Hardy–Schrödinger–Kirchhoff systems with critical nonlinearities. Adv. Nonlinear Anal. **8**, 1111–1131 (2019)

[10] Furtado, M.F., de Oliveira, L.D., da Silva, J.P.: Multiple solutions for a critical Kirchhoff system. Appl. Math. Lett. **91**, 97–105 (2019)

[11] Gao, F., da Silva, E.D., Yang, M., Zhou, J.: Existence of solutions for critical Choquard equations via the concentration compactness method. Proc. R. Soc. Edinb. Sect. A. https://doi.org/10.1017/prm.2018.131

[12] Huang, D., Li, Y.: Multiplicity of solutions for a noncooperative p-Laplacian elliptic system in $\mathbb{R}^N$. J. Differ. Equ. **215**, 206–223 (2005)

[13] Ji, C., Rădulescu, V.: Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation. Calc. Var. Partial Differ. Equ. **59**, 28 (2020)

[14] Ji, C., Rădulescu, V.: Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$. Manuscr. Math. **164**, 509–542 (2021)

[15] Krawcewicz, W., Marzantowicz, W.: Some remarks on the Lusternik–Schnirelman method for non-differentiable functionals invariant with respect to a finite group action. Rocky Mt. J. Math. **20**, 1041–1049 (1990)

[16] Laskin, N.: Fractional quantum mechanics and Lévy path integrals. Phys. Lett. A **268**, 298–305 (2000)

[17] Ledesma, L., César, E.: Multiplicity result for non-homogeneous fractional Schrödinger-Kirchhoff-type equations in $\mathbb{R}^N$. Adv. Nonlinear Anal. **7**, 247–257 (2018)

[18] Li, Y.: A limit index theory and its application. Nonlinear Anal. **25**, 1371–1389 (1995)

[19] Li, X., Ma, S., Zhang, G.: Solutions to upper critical fractional Choquard equations with potential. Adv. Differ. Equ. **25**, 135–160 (2020)

[20] Liang, S., Shi, S.: Multiplicity of solutions for the noncooperative $p(x)$-Laplacian operator elliptic system involving the critical growth. J. Dyn. Control Syst. **18**, 379–396 (2012)

[21] Liang, S., Zhang, J.: Multiple solutions for noncooperative $p(x)$-Laplacian equations in $\mathbb{R}^N$ involving the critical exponent. J. Math. Anal. Appl. **403**, 344–356 (2013)

[22] Liang, S., Molica Bisci, G., Zhang, B.: Multiple solutions for a noncooperative Kirchhoff-type system involving the fractional p-Laplacian and critical exponents. Math. Nachr. **291**, 1553–1546 (2018)

[23] Liang, S., Repovš, D.D., Zhang, B.: On the fractional Schrödinger-Kirchhoff equations with electromagnetic fields and critical nonlinearity. Comput. Math. Appl. **75**, 1778–1794 (2018)

[24] Liang, S., Repovš, D.D., Zhang, B.: Fractional magnetic Schrödinger–Kirchhoff problems with convolution and critical nonlinearities. Math. Models Methods Appl. Sci. **43**, 2473–2490 (2020)

[25] Lieb, E.H., Loss, M.: Analysis, Volume 14 of Graduate Studies in Mathematics. American Mathematical Society, Providence (2001)

[26] Lions, P.L.: Symétrie et compacité dans les éspaces de Sobolev. J. Funct. Anal. **49**, 315–334 (1982)

[27] Lions, P.L.: The concentration compactness principle in the calculus of variations. The locally compact case. Part I and II, Ann. Inst. H. Poincaré Anal. Non. Lineaire. 1. pp. 109–145 and 223–283 (1984)

[28] Liu, J., Ji, C.: Concentration results for a magnetic Schrödinger–Poisson system with critical growth. Adv. Nonlinear Anal. **10**, 775–798 (2021)

[29] Ma, P., Zhang, J.: Existence and multiplicity of solutions for fractional Choquard equations. Nonlinear Anal. **164**, 100–117 (2017)

[30] Mingqi, X., Pucci, P., Squassina, M., Zhang, B.: Nonlocal Schrödinger–Kirchhoff equations with external magnetic field. Discrete Contin. Dyn. Syst. **37**, 503–521 (2017)

[31] Mingqi, X., Rădulescu, V., Zhang, B.: A critical fractional Choquard–Kirchhoff problem with magnetic field. Commun. Contemp. Math. **21**, 36 (2019)

[32] Mingqi, X., Rădulescu, V., Zhang, B.: Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity. Calc. Var. Partial Differ. Equ. **58**, 27 (2019)

[33] Molica Bisci, G., Rădulescu, V., Servadei, R.: Variational Methods for Nonlocal Fractional Equations, Encyclopedia of Mathematics and its Applications, vol. 162. Cambridge University Press, Cambridge (2016)

[34] Mukherjee, T., Sreenadh, K.: Fractional Choquard equation with critical nonlinearities. Nonlinear Differ. Equ. Appl. NoDEA **24**(6), 63 (2017)

[35] Papageorgiou, N., Rădulescu, V., Repovš, D.: Nonlinear Analysis—Theory and Methods. Springer, Berlin (2019)

[36] Pekar, S.: Untersuchung über die Elektronentheorie der Kristalle. Akademie, Berlin (1954)

[37] Pucci, P., Xiang, M., Zhang, B.: Existence results for Schrödinger–Choquard–Kirchhoff equations involving the fractional p-Laplacian. Adv. Calc. Vari. **12**, 253–275 (2019)

[38] Song, Y., Shi, S.: Infinitely many solutions for Kirchhoff equations with Hardy–Littlewood–Sobolev critical nonlinearity, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**, 3223–3232 (2019)

[39] Song, Y., Shi, S.: Existence and multiplicity of solutions for Kirchhoff equations with Hardy–Littlewood–Sobolev critical nonlinearity. Appl. Math. Lett. **92**, 170–175 (2019)

[40] Song, Y., Shi, S.: Multiple solutions for a class of noncooperative critical nonlocal system with variable exponents. Math. Models Methods Appl. Sci. **44**, 6630–6646 (2021)

[41] Squassina, M., Volzone, B.: Bourgain–Brézis–Mironescu formula for magnetic operators. C. R. Math. **354**, 825–831 (2016)

[42] Wang, F., Xiang, M.: Multiplicity of solutions to a nonlocal Choquard equation involving fractional magnetic operators and critical exponent. Electron. J. Diff. Equ. **2016**, 1–11 (2016)

[43] Willem, M.: Minimax Theorems, Progress in Nonlinear Differential Equations and their Applications, vol. 24. Birkhäser, Boston/Basel/Berlin (1996)

[44] Xia, A.: Multiplicity and concentration results for magnetic relativistic Schrödinger equations. Adv. Nonlinear Anal. **9**, 1161–1186 (2020)

[45] Xiang, M., Rădulescu, V., Zhang, B.: Combined effects for fractional Schrödinger–Kirchhoff systems with critical nonlinearities. ESAIM COCV **24**, 1249–1273 (2018)

[46] Xiang, M., Zhang, B., Rădulescu, V.: Superlinear Schrödinger–Kirchhoff type problems involving the fractional p-Laplacian and critical exponent. Adv. Nonlinear Anal. **9**, 690–709 (2020)

[47] Zhang, X., Zhang, B., Repovš, D.D.: Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials. Nonlinear Anal. **142**, 48–68 (2016)

[48] Zhang, Y., Tang, X., Rădulescu, V.: Small perturbations for nonlinear Schrödinger equations with magnetic potential. Milan J. Math. **88**, 479–506 (2020)

Mingzhe Sun and Shaoyun Shi
School of Mathematics
Jilin University
Changchun 130012
People's Republic of China
e-mail: mathmzhsun@163.com

Shaoyun Shi
e-mail: shisy@jlu.edu.cn

Mingzhe Sun
Department of Mathematics
Yanbian University
Yanji 133002
People's Republic of China

Dušan D. Repovš
Faculty of Education and Faculty of Mathematics and Physics
University of Ljubljana
1000 Ljubljana
Slovenia
e-mail: dusan.repovs@guest.arnes.si

Received: October 8, 2021.
Revised: October 27, 2021.
Accepted: May 21, 2022.