Nonlinear Analysis 75 (2012) 1524–1530

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Combined effects in nonlinear problems arising in the study of anisotropic continuous media

Vicenţiu Rădulescu [a,b,*], Dušan Repovš [c,d]

[a] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania
[b] Department of Mathematics, University of Craiova, 200585 Craiova, Romania
[c] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, P. O. Box 2964, 1001 Ljubljana, Slovenia
[d] Faculty of Education, University of Ljubljana, Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia

ARTICLE INFO

This paper is dedicated with esteem to Professor Marius Iosifescu on his 75th birthday

MSC:
35A05
35B40
35J60
58E05

Keywords:
Lane–Emden–Fowler equation
Bifurcation problem
Anisotropic continuous media
Perturbation
Positive solution

ABSTRACT

We are concerned with the Lane–Emden–Fowler equation $-\Delta u = \lambda k(x)u^q \pm h(x)u^p$ in Ω, subject to the Dirichlet boundary condition $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, k and h are variable potential functions, and $0 < q < 1 < p$. Our analysis combines monotonicity methods with variational arguments.

1. Introduction and the main results

In this paper we are concerned with the study of combined effects in a class of nonlinear bifurcation problems that arise in anisotropic continuous media. Bifurcation problems have a long history and their treatment goes back to the XVIIIth century. To the best of our knowledge, the first bifurcation problem is related to the buckling of a thin rod under thrust and was investigated by Daniel Bernoulli and Euler around 1744. In the case in which the rod is free to rotate at both end points, this yields the one-dimensional bifurcation problem

$$\begin{cases} u'' + \lambda \sin u = 0 & \text{in } (0, L) \\ 0 \le u \le \pi \\ u'(0) = u'(L) = 0. \end{cases}$$

Consider the bifurcation problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

* Corresponding author at: Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania. Tel.: +40 723049852.
E-mail addresses: vicentiu.radulescu@imar.ro (V. Rădulescu), dusan.repovs@guest.arnes.si (D. Repovš).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.01.037

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. Problems of this type have been studied starting with the pioneering paper [1] by Gelfand, who considered the case $f(u) = e^u$. We also refer to the important contributions due to Amann [2] and Keller and Cohen [3], who assumed that f is a function of class C^1 which is convex and positive, such that $f'(0) > 0$. They proved the following basic facts:

(i) there exists $\lambda^* \in (0, \infty)$ such that problem (1) has (resp., has no) classical solution if $\lambda \in (0, \lambda^*)$ (resp., $\lambda > \lambda^*$);
(ii) for $\lambda \in (0, \lambda^*)$, among the solutions of (1) there exists a minimal one, say $u(\lambda)$;
(iii) the mapping $\lambda \longmapsto u(\lambda)$ is convex, increasing and of class C^1;
(iv) $u(\lambda)$ is the only solution of problem (1) such that the operator $-\Delta - \lambda f'(u)$ is coercive.

Motivated by a problem raised by Brezis, Mironescu and Rădulescu studied in [4,5] the case where f fulfills the above hypotheses and has a linear growth at infinity, that is, there exists $\lim_{t\to\infty} f(t)/t =: a \in (0, \infty)$. The results in this framework are strongly related with the sign of $b := \lim_{t\to\infty}(f(t) - at)$. For instance, if $b \geq 0$ then the following properties hold: (i) $\lambda^* = \lambda_1/a$; (ii) $u(\lambda)$ is the unique solution of problem (1) for all $\lambda \in (0, \lambda^*)$; (iii) $\lim_{\lambda\to\lambda^*-} u(\lambda) = +\infty$ uniformly on compact subsets of Ω; and (iv) problem (1) has no solution if $\lambda = \lambda^*$. A different behavior holds if $b \in [-\infty, 0)$.

The perturbed nonlinear problem

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

was studied in [6], where f has either a sublinear or a linear growth at infinity, g is a singular nonlinearity, and a is a non-negative potential. The framework discussed in [6] includes the case $f(t) = t^p$ and $g(t) = t^{-\gamma}$, where $0 < p \leq 1$ and $0 < \gamma < 1$. For instance, if f is sublinear and $\inf_{x\in\Omega} a(x) > 0$ then problem (2) has a unique solution for all $\lambda \in \mathbb{R}$. From a physical point of view, problem (2) arises in the context of chemical heterogeneous catalysis, in the theory of heat conduction in electrically conducting materials, as well as in the study of non-Newtonian fluids, boundary-layer phenomena for viscous fluids. Such equations are also encountered in glacial advance (see [7]), in transport of coal slurries down conveyor belts (see [8]), and in several other geophysical and industrial contents (see [9] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence).

The above results are summarized and completed in the recent works [10–13].

In this paper we are concerned with the competition between convex and concave nonlinearities and variable potentials. Such problems arise in the study of anisotropic continuous media. We point out that nonlinear elliptic equations with convex–concave nonlinearities have been studied starting with the seminal paper by Ambrosetti et al. [14]. They considered the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where λ is a positive parameter, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and $0 < q < 1 < p < 2^\star - 1$ ($2^\star = 2N/(N-2)$ if $N \geq 3$, $2^\star = +\infty$ if $N = 1, 2$). Ambrosetti et al. proved that there exists $\lambda^* > 0$ such that problem (3) admits at least two solutions for all $\lambda \in (0, \lambda^*)$, has one solution for $\lambda = \lambda^*$, and no solution exists provided that $\lambda > \lambda^*$. In [15], Alama and Tarantello studied the related Dirichlet problem with indefinite weights

$$\begin{cases} -\Delta u - \lambda u = k(x)u^q - h(x)u^p, & \text{if } x \in \Omega \\ u > 0, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega, \end{cases} \tag{4}$$

where h, k are non-negative and $1 < p < q$. We refer to [16] for a related problem with lack of compactness.

In the present paper we are concerned with the nonlinear elliptic problem

$$\begin{cases} -\Delta u = \lambda k(x)u^q \pm h(x)u^p, & \text{if } x \in \Omega \\ u > 0, & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega, \end{cases} \tag{$(P_\lambda)_\pm$}$$

under the basic assumption $0 < q < 1 < p$.

If $\lambda = 0$, Eq. $(P_\lambda)_\pm$ is called the Lane–Emden–Fowler equation and arises in the boundary-layer theory of viscous fluids (see [17]). This equation goes back to the paper by Lane [18] in 1869 and is originally motivated by Lane's interest in computing both the temperature and the density of mass on the surface of the sun. Eq. $(P_\lambda)_\pm$ describes the behavior of the density of a gas sphere in hydrostatic equilibrium and the index p, which is called the polytropic index in astrophysics, is related to the ratio of the specific heats of the gas. Problem $(P_\lambda)_\pm$ may be also viewed as a prototype of pattern formation in biology and is related to the steady-state problem for a chemotactic aggregation model introduced by Keller and Segel [19]. Problem $(P_\lambda)_\pm$ also plays an important role in the study of activator–inhibitor systems modeling biological pattern formation, as proposed by Gierer and Meinhardt [20]. Problems of this type, as well as the associated evolution equations, describe super-diffusivities phenomena. Such models have been proposed by de Gennes [21] to describe long range van der Waals interactions in thin films spread on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [22]), as well as to describe the flow over an impermeable plate (see [9]). Problems of this type are obtained from evolution equations of the form

$$u_{tt} = \mathrm{div}(u^{m-1}\nabla u) + h(x, u) \quad \text{in } \Omega \times (0, T)$$

through the implicit discretization in time arising in nonlinear semigroup theory (see [23]).

Throughout this paper we assume that the variable weight functions $k, h \in L^\infty(\Omega)$ satisfy

$$\mathrm{ess\,inf}_{x\in\Omega} k(x) > 0 \quad \text{and} \quad \mathrm{ess\,inf}_{x\in\Omega} h(x) > 0.$$

We are concerned with the existence of weak solutions of problems $(P_\lambda)_+$ and $(P_\lambda)_-$, that is, functions $u \in H_0^1(\Omega)$ such that

(i) $u \geq 0$ a.e. on Ω and $u > 0$ on a subset of Ω with positive measure;
(ii) for all $\varphi \in H_0^1(\Omega)$ the following identity holds

$$\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x = \int_\Omega \left[\lambda k(x) u^q \pm h(x) u^p\right] \varphi \,\mathrm{d}x.$$

The main results of this paper give a complete description of both cases arising in problem $(P_\lambda)_\pm$. The first theorem is concerned with problem $(P_\lambda)_+$ and establishes the existence of a minimal solution, provided that $\lambda > 0$ is small enough. This result gives a complete qualitative analysis of the problem and the proof combines monotonicity arguments with variational techniques. We refer to Bartsch and Willem [24] for a related problem that is treated by means of a new critical point theorem, which guarantees the existence of infinitely many critical values of an even functional in a bounded range. In this framework, Bartsch and Willem proved that there exists a sequence (u_n) of solutions (not necessarily positive!) with corresponding energy $\mathcal{E}(u_n) < 0$ and such that $\mathcal{E}(u_n) \to 0$ as $n \to \infty$.

Theorem 1.1. *Assume* $0 < q < 1 < p < 2^\star - 1$. *Then there exists a positive number* λ^* *such that the following properties hold:*

(a) *for all* $\lambda \in (0, \lambda^*)$ *problem* $(P_\lambda)_+$ *has a minimal solution* $u(\lambda)$. *Moreover, the mapping* $\lambda \longmapsto u(\lambda)$ *is increasing.*
(b) *Problem* $(P_\lambda)_+$ *has a solution if* $\lambda = \lambda^*$*;*
(c) *problem* $(P_\lambda)_+$ *does not have any solution if* $\lambda > \lambda^*$.

The next result is concerned with problem $(P_\lambda)_-$ and asserts that there is some $\lambda^* > 0$ such that $(P_\lambda)_-$ has a nontrivial solution if $\lambda > \lambda^*$ and no solution exists provided that $\lambda < \lambda^*$.

Theorem 1.2. *Assume* $0 < q < 1 < p < 2^\star - 1$. *Then there exists a positive number* λ^* *the following properties hold:*

(a) *if* $\lambda > \lambda^*$, *then problem* $(P_\lambda)_-$ *has at least one solution;*
(b) *if* $\lambda < \lambda^*$, *then problem* $(P_\lambda)_-$ *does not have any solution.*

We point out that related results have been established by Il'yasov [25] and Lubyshev [26,27]. However, the proof techniques of [25–27] are completely different from the arguments in the present paper and they rely on the global fibering method introduced by Pohozaev [28], which is among the most powerful tools for studying nonlinear differential equations. The fibering scheme allows Il'yasov and Lubyshev to find constructively constrained minimization problems which have the property of ground among all constrained minimization problems corresponding to the given energy functional. We also refer to Bozhkov and Mitidieri [29,30] for relatively closed "convex–concave" settings, both for nonlinear differential equations and for quasilinear elliptic systems with Dirichlet boundary condition. Garcia Azorero and Peral Alonso [31] used the mountain pass theorem to obtain the existence of a sign-changing solution in a related quasilinear setting.

2. Proof of Theorem 1.1

We first prove that if $\lambda > 0$ is sufficiently small then problem $(P_\lambda)_+$ has a solution. For this purpose we use the method of sub- and super-solutions.

Consider the problem

$$\begin{cases} -\Delta w = \lambda k(x) w^q, & \text{in } \Omega \\ w > 0, & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Then, by [32], problem (5) has a unique solution w. We prove that the function $\underline{u} := \varepsilon w$ is a sub-solution of problem $(P_\lambda)_+$ provided that $\varepsilon > 0$ is small enough. For this purpose it suffices to show that

$$\varepsilon \lambda k(x) w^q \leq \lambda k(x) \varepsilon^q w^q + h(x) \varepsilon^p w^p \quad \text{in } \Omega,$$

which is true for all $\varepsilon \in (0, 1)$.

Let v be the unique solution of the linear problem

$$\begin{cases} -\Delta v = 1, & \text{in } \Omega \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

We prove that if $\lambda > 0$ is small enough then there is $M > 0$ such that $\overline{u} := Mv$ is a super-solution of $(P_\lambda)_+$. Therefore it suffices to show that

$$M \geq \lambda k(x)(Mv)^q + h(x)(Mu)^p \quad \text{in } \Omega. \tag{6}$$

Set

$$A := \|k\|_{L^\infty}\| \cdot \|v\|^q_{L^\infty} \quad \text{and} \quad B := \|h\|_{L^\infty}\| \cdot \|v\|^p_{L^\infty}.$$

Thus, by (6), it is enough to show that there is $M > 0$ such that

$$M \geq \lambda AM^q + BM^p$$

or, equivalently,

$$1 \geq \lambda AM^{q-1} + BM^{p-1}. \tag{7}$$

Consider the mapping $(0, \infty) \ni t \longmapsto \lambda At^{q-1} + Bt^{p-1}$. A straightforward computation shows that this function attains its minimum for $t = C\lambda^{1/(p-q)}$, where $C = [AB^{-1}(1-q)(p-1)^{-1}]^{(q-1)/(p-q)}$. Moreover, the global minimum of this mapping is

$$(AC^{q-1} + BC^{p-1})\lambda^{(p-1)/(p-q)}.$$

This shows that condition (7) is fulfilled for all $\lambda \in (0, \lambda_0]$ and $M = C\lambda^{1/(p-q)}$, where $\lambda_0 > 0$ satisfies

$$(AC^{q-1} + BC^{p-1})\lambda_0^{(p-1)/(p-q)} = 1.$$

It remain to argue that $\varepsilon w \leq Mv$. This is a consequence of the maximum principle (see [33]), provided that $\varepsilon > 0$ is small enough. Thus, problem $(P_\lambda)_+$ has at least one solution $u(\lambda)$ for all $\lambda < \lambda^*$.

Set

$$\lambda^* := \sup\{\lambda > 0;\ \text{problem } (P_\lambda)_+ \text{ has a solution}\}.$$

By the definition of λ^*, problem $(P_\lambda)_+$ does not have any solution if $\lambda > \lambda^*$. In what follows we claim that λ^* is finite. Denote

$$m := \min\{\operatorname{ess\,inf}_{x\in\Omega} k(x), \operatorname{ess\,inf}_{x\in\Omega} h(x)\} > 0.$$

Let $\lambda' > 0$ be such that $m(\lambda' + t^{p-q}) > \lambda_1 t^{1-q}$ for all $t \geq 0$, where λ_1 stands for the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. Denote by $\varphi_1 > 0$ an eigenfunction of the Laplace operator corresponding to λ_1. Since $u(\lambda)$ solves $(P_\lambda)_+$ we have for all $\lambda > \lambda'$,

$$\begin{aligned} \lambda_1 \int_\Omega u(\lambda)\varphi_1 dx &= \int_\Omega \left(\lambda k(x)u(\lambda)^q + h(x)u(\lambda)^p\right)\varphi_1 dx \\ &\geq \int_\Omega m(\lambda u(\lambda)^q + u(\lambda)^p)\varphi_1 dx > \lambda_1 \int_\Omega u(\lambda)\varphi_1 dx. \end{aligned}$$

This implies that $\lambda^* \leq \lambda' < +\infty$, which proves our claim.

Let us now prove that $u(\lambda)$ is a minimal solution of $(P_\lambda)_+$. Consider the sequence $(u_n)_{n\geq 0}$ defined by $u_0 = w$ (w is the unique solution of (5)) and u_n is the unique solution of the problem

$$\begin{cases} -\Delta u_n = \lambda k(x)u_{n-1}^q + h(x)u_{n-1}^p, & \text{in } \Omega \\ u_n > 0, & \text{in } \Omega \\ u_n = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, by the maximum principle, $u_n \leq u_{n+1} \leq u(\lambda)$. Moreover, by the same argument as in [14], the sequence $(u_n)_{n\geq 0}$ converges to $u(\lambda)$. In order to show that $u(\lambda)$ is a minimal solution, let U be an arbitrary solution of problem $(P_\lambda)_+$. Thus, by the maximum principle, $w = u_0 \leq U$ and, by recurrence, $u_n \leq U$ for all $n \geq 1$. It follows that $u(\lambda) \leq U$. At this stage it is easy to deduce that the mapping $\lambda \longmapsto u(\lambda)$ is increasing. Fix $0 < \lambda_1 < \lambda_2 < \lambda^*$. Then $u(\lambda_2)$ is a super-solution of problem $(P_{\lambda_1})_+$, hence, by minimality, $u(\lambda_1) \leq u(\lambda_2)$. The fact that $\lambda_1 < \lambda_2$, combined with the maximum principle implies that $u(\lambda_1) < u(\lambda_2)$.

It remains to show that problem $(P_\lambda)_+$ has a solution if $\lambda = \lambda^*$. For this purpose it is enough to prove that $(u(\lambda))$ is bounded in $H_0^1(\Omega)$ as $\lambda \to \lambda^*$. Thus, up to a subsequence, $u(\lambda) \rightharpoonup u^*$ in $H_0^1(\Omega)$ as $\lambda \to \lambda^*$, which implies that u^* is a weak solution of $(P_\lambda)_+$ provided that $\lambda = \lambda^*$. Moreover, since the mapping $\lambda \longmapsto u(\lambda)$ is increasing, it follows that $u^* \geq 0$ a.e. on

Ω and $u^* > 0$ on a subset of Ω with positive measure. A key ingredient of the proof is that all solutions $u(\lambda)$ have negative energy. More precisely, if $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega k(x)|u|^{q+1}\mathrm{d}x - \frac{1}{p+1}\int_\Omega h(x)|u|^{p+1}\mathrm{d}x$$

then

$$\mathcal{E}(u(\lambda)) < 0 \quad \text{for all } \lambda \in (0, \lambda^*). \tag{8}$$

To deduce (8) we split the proof into the following steps:

(i) the solution $u(\lambda)$ is semi-stable, that is, the linearized operator $-\Delta - \lambda q k(x) u(\lambda)^{q-1} - p h(x) u(\lambda)^{p-1}$ is coercive:

$$\int_\Omega \left[|\nabla \psi|^2 - (\lambda q k(x) u(\lambda)^{q-1} + p h(x) u(\lambda)^{p-1})\psi^2\right] \mathrm{d}x \geq 0 \quad \text{for all } \psi \in H_0^1(\Omega).$$

Therefore

$$\int_\Omega \left[|\nabla u(\lambda)|^2 - (\lambda q k(x) u(\lambda)^{q+1} + p h(x) u(\lambda)^{p+1})\right] \mathrm{d}x \geq 0. \tag{9}$$

This follows by the same proof as in [13, Theorem 1.9].

(ii) Since $u(\lambda)$ is a solution of $(P_\lambda)_+$ we have

$$\int_\Omega |\nabla u(\lambda)|^2 \mathrm{d}x = \lambda \int_\Omega k(x) u(\lambda)^{q+1}\mathrm{d}x + \int_\Omega h(x) u(\lambda)^{p+1}\mathrm{d}x. \tag{10}$$

Combining relations (9) and (10) we deduce that

$$\lambda(1-q)\int_\Omega k(x) u(\lambda)^{q+1}\mathrm{d}x \geq (p-1)\int_\Omega h(x) u(\lambda)^{p+1}\mathrm{d}x. \tag{11}$$

Next, we observe that relation (10) implies

$$\begin{aligned}
\mathcal{E}(u(\lambda)) &= \lambda\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_\Omega k(x) u(\lambda)^{q+1}\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega h(x) u(\lambda)^{p+1}\mathrm{d}x \\
&= -\lambda \frac{1-q}{2(q+1)}\int_\Omega k(x) u(\lambda)^{q+1}\mathrm{d}x + \frac{p-1}{2(p+1)}\int_\Omega h(x) u(\lambda)^{p+1}\mathrm{d}x \\
&\leq -\lambda \frac{1-q}{2(q+1)}\int_\Omega k(x) u(\lambda)^{q+1}\mathrm{d}x + \lambda \frac{1-q}{2(p+1)}\int_\Omega k(x) u(\lambda)^{p+1}\mathrm{d}x \leq 0,
\end{aligned}$$

by (11).

To complete the proof, it remains to argue that $\sup_{\lambda<\lambda^*} \|u(\lambda)\|_{H_0^1} < +\infty$. This follows after combining relations (8) and (9), Sobolev embeddings, and using the fact that $k, h \in L^\infty(\Omega)$. This completes the proof. □

3. Proof of Theorem 1.2

The energy functional associated to problem $(P_\lambda)_-$ is $\mathcal{F}_\lambda : H_0^1(\Omega) \to \mathbb{R}$ and it is defined by

$$\mathcal{F}_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega k(x)|u|^{q+1}\mathrm{d}x + \frac{1}{p+1}\int_\Omega h(x)|u|^{p+1}\mathrm{d}x.$$

Set

$$\|u\| := \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}; \qquad \|u\|_{q+1} := \left(\int_\Omega |u|^{q+1}\mathrm{d}x\right)^{1/(q+1)}; \qquad \|u\|_{p+1} := \left(\int_\Omega |u|^{p+1}\mathrm{d}x\right)^{1/(p+1)}.$$

We first prove that $\mathcal{F}_\lambda$ is coercive. Indeed,

$$\mathcal{F}_\lambda(u) \geq \frac{1}{2}\|u\|^2 - C_1\|u\|_{q+1}^{q+1} + C_2\|u\|_{p+1}^{p+1},$$

where $C_1 = \lambda(q+1)^{-1}\|k\|_{L^\infty}$ and $C_2 = (p+1)^{-1}\mathrm{ess\,inf}_{x\in\Omega} h(x)$ are positive constants. Since $q < p$, a straightforward computation shows that the mapping $(0, +\infty) \ni t \longmapsto At^{p+1} - Bt^{q+1}$ attains its global minimum $m < 0$ at

$$t = \left[\frac{B(q+1)}{A(p+1)}\right]^{1/(p-q)}.$$

Therefore

$$\mathcal{F}_\lambda(u) \geq \frac{1}{2}\|u\|^2 + m,$$

hence $\mathcal{F}_\lambda(u) \to +\infty$ as $\|u\| \to \infty$.

Let (u_n) be a minimizing sequence of $\mathcal{F}_\lambda$ in $H_0^1(\Omega)$. Since $\mathcal{F}_\lambda$ is coercive, it follows that (u_n) is bounded. Without loss of generality, we may assume that u_n is non-negative and that (u_n) converges weakly to some u in $H_0^1(\Omega)$. Standard arguments based on the lower semi-continuity of the energy functional show that u is a global minimizer of $\mathcal{F}_\lambda$, hence a non-negative solution of problem $(P_\lambda)_-$.

In what follows we prove that the weak limit u is a non-negative weak solution of problem $(P_\lambda)_-$ if $\lambda > 0$ is large enough. We first observe that $\mathcal{F}_\lambda(0) = 0$. So, in order to prove that the non-negative solution is nontrivial, it suffices to prove that there exists $\Lambda > 0$ such that

$$\inf_{u \in H_0^1(\Omega)} \mathcal{F}_\lambda(u) < 0 \quad \text{for all } \lambda > \Lambda.$$

For this purpose we consider the constrained minimization problem

$$\Lambda := \inf\left\{\frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{p+1}\int_\Omega h(x)|v|^{p+1}dx;\ v \in H_0^1(\Omega) \text{ and } \frac{1}{q+1}\int_\Omega k(x)|v|^{q+1}dx = 1\right\}. \tag{12}$$

Let (v_n) be an arbitrary minimizing sequence for this problem. Then (v_n) is bounded, hence we can assume that it weakly converges to some $v \in H_0^1(\Omega)$ with

$$\frac{1}{q+1}\int_\Omega k(x)|v|^{q+1}dx = 1 \quad \text{and} \quad \Lambda := \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{p+1}\int_\Omega h(x)|v|^{p+1}dx.$$

Thus, $\mathcal{F}_\lambda(v) = \Lambda - \lambda < 0$ for all $\lambda > \Lambda$.

Set

$$\lambda^* := \inf\{\lambda > 0;\ \text{problem } (P_\lambda)_- \text{ admits a nontrivial weak solution}\} \geq 0.$$

The above remarks show that $\Lambda \geq \lambda^*$ and that problem $(P_\lambda)_-$ has a solution for all $\lambda > \Lambda$. We now argue that problem $(P_\lambda)_-$ has a solution for *all* $\lambda > \lambda^*$. Fix $\lambda > \lambda^*$. By the definition of λ^*, there exists $\mu \in (\lambda^*, \lambda)$ such that $\mathcal{F}_\mu$ has a nontrivial critical point $u_\mu \in H_0^1(\Omega)$. Since $\mu < \lambda$, it follows that u_μ is a sub-solution of problem $(P_\lambda)_-$. We now want to construct a super-solution that dominates u_μ. For this purpose we consider the constrained minimization problem

$$\inf\left\{\mathcal{F}_\lambda(v);\ v \in H_0^1(\Omega) \text{ and } v \geq u_\mu\right\}. \tag{13}$$

The same arguments as those used to treat (12) show that problem (13) has a solution $u_\lambda \geq u_\mu$. Moreover, u_λ is a solution of problem $(P_\lambda)_-$, for all $\lambda > \lambda^*$. With the arguments developed in [34, p. 712] we deduce that problem $(P_\lambda)_-$ has a solution if $\lambda = \lambda^*$. The same monotonicity arguments as above show that $(P_\lambda)_-$ does not have any solution if $\lambda < \lambda^*$.

Fix $\lambda > \lambda^*$. It remains to argue that the non-negative weak solution u is, in fact, positive. Indeed, using similar arguments as in [35], which are based on the Moser iteration, we obtain that $u \in L^\infty(\Omega)$. Next, by bootstrap regularity, u is a classical solution of problem $(P_\lambda)_-$. Since u is a non-negative smooth weak solution of the differential inequality $\Delta u - h(x)u^p \leq 0$ in Ω, with $p > 1$, we deduce that $u > 0$ in Ω. This follows by applying the methods developed in Section 4.8 of Pucci and Serrin [36] and the comments therein. This completes the proof. □

Acknowledgements

The authors are grateful to the anonymous referees for the careful reading of the paper and their suggestions. V. Rădulescu acknowledges the support through Grant CNCSIS PCCE-8/2010 "Sisteme diferenţiale în analiza neliniară şi aplicaţii". D. Repovš acknowledges the support by ARRS grant P1-0292-0101 "Topology and geometry".

References

[1] I.M. Gelfand, Some problems in the theory of quasi-linear equations, Uspekhi Mat. Nauk 14 (1959) 87–158.
[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.
[3] H.B. Keller, D.S. Cohen, Some positone problems suggested by nonlinear heat generation, J. Math. Mech. 16 (1967) 1361–1376.
[4] P. Mironescu, V. Rădulescu, A bifurcation problem associated to a convex, asymptotically linear function, C. R. Acad. Sci. Paris Sér. I 316 (1993) 667–672.
[5] P. Mironescu, V. Rădulescu, The study of a bifurcation problem associated to an asymptotically linear function, Nonlinear Anal. 26 (1996) 857–875.
[6] F. Cîrstea, M. Ghergu, V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type, J. Math. Pures Appl. 84 (2005) 493–508.
[7] G.B. Whitham, Linear and Nonlinear Waves, Wiley-Interscience, New York, 1974.
[8] T. Carleman, Problèmes Mathématiques Dans La Théorie Cinétique Des Gaz, Almquist-Wiksells, Uppsala, 1957.
[9] A. Callegari, A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, SIAM J. Appl. Math. 38 (1980) 275–281.
[10] C. Bandle, W. Reichel, Solutions of quasilinear second-order elliptic boundary value problems via degree theory, in: Stationary Partial Differential Equations, in: Handb. Differ. Equ., vol. I, North-Holland, Amsterdam, 2004, pp. 1–70.

[11] M. Ghergu, V. Rădulescu, Singular Elliptic Problems: Bifurcation and Asymptotic Analysis, in: Oxford Lecture Series in Mathematics and its Applications, vol. 37, Oxford University Press, New York, 2008.
[12] A. Kristály, V. Rădulescu, C. Varga, Variational Principles in Mathematical Physics, Geometry and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, in: Encyclopedia of Mathematics, vol. 136, Cambridge University Press, Cambridge, 2010.
[13] V. Rădulescu, Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations, in: Contemporary Mathematics and its Applications, vol. 6, Hindawi Publ. Corp., 2008.
[14] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.
[15] S. Alama, G. Tarantello, Elliptic problems with nonlinearities indefinite in sign, J. Funct. Anal. 141 (1996) 159–215.
[16] P. Pucci, V. Rădulescu, Combined effects in quasilinear elliptic problems with lack of compactness, Rend. Lincei-Mat. Appl., in press (doi:10.4171/RLM).
[17] J.S.W. Wong, On the generalized Emden–Fowler equation, SIAM Rev. 17 (1975) 339–360.
[18] H. Lane, On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment, Am. J. Sci. 50 (1869) 57–74.
[19] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (1970) 399–415.
[20] A. Gierer, H. Meinhardt, A theory of biological pattern formation, Kybernetik 12 (1972) 30–39.
[21] P.G. de Gennes, Wetting: statics and dynamics, Rev. Modern Phys. 57 (1985) 827–863.
[22] J.T. Chayes, S.J. Osher, J.V. Ralston, On singular diffusion equations with applications to self-organized criticality, Comm. Pure Appl. Math. 46 (1993) 1363–1377.
[23] J.L. Vázquez, Nonexistence of solutions for nonlinear heat equations of fast-diffusion type, J. Math. Pures Appl. 71 (1992) 503–526.
[24] T. Bartsch, M. Willem, On an elliptic equation with concave and convex nonlinearities, Proc. Amer. Math. Soc. 123 (1995) 3555–3561.
[25] Y. Il'yasov, On nonlocal existence results for elliptic equations with convex–concave nonlinearities, Nonlinear Anal. 61 (2005) 211–236.
[26] V.F. Lubyshev, Multiple positive solutions of an elliptic equation with a convex–concave nonlinearity containing a sign-changing term, Tr. Mat. Inst. Steklova 269 (2010) 167–180.
[27] V.F. Lubyshev, Multiple solutions of an even order nonlinear problem with convex–concave nonlinearity, Nonlinear Anal. 74 (2011) 1345–1354.
[28] S.I. Pohozaev, On an approach to nonlinear equations, Dokl. Acad. Sci. USSR 247 (1979) 1327–1331.
[29] Y. Bozhkov, E. Mitidieri, Existence of multiple solutions for quasilinear systems via fibering method, J. Differential Equations 190 (2003) 239–267.
[30] Y. Bozhkov, E. Mitidieri, Existence of multiple solutions for quasilinear equations via fibering method, in: Contributions to Nonlinear Analysis, in: Progr. Nonlinear Differential Equations Appl., vol. 66, Birkhäuser, Basel, 2006, pp. 115–134.
[31] J. Garcia Azorero, I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, Trans. Amer. Math. Soc. 323 (1991) 877–895.
[32] H. Brezis, S. Kamin, Sublinear elliptic equations in $\mathbb{R}^n$, Manuscripta Math. 74 (1992) 87–106.
[33] P. Pucci, J. Serrin, The strong maximum principle revisited, J. Differential Equations 196 (2004) 1–66. Erratum, J. Differential Equations, 207, 2004, pp. 226–227.
[34] R. Filippucci, P. Pucci, V. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, Comm. Partial Differential Equations 33 (2008) 706–717.
[35] P. Pucci, R. Servadei, On weak solutions for p-Laplacian equations with weights, Rend. Lincei-Mat. Appl. 18 (2007) 257–267.
[36] P. Pucci, J. Serrin, On the strong maximum and compact support principles and some applications, in: M. Chipot (Ed.), in: Handbook of Differential Equations—Stationary Partial Differential Equations, vol. 4, Elsevier, 2007, pp. 355–483.