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POSITIVE SOLUTIONS FOR THE ROBIN p-LAPLACIAN PLUS AN INDEFINITE POTENTIAL

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Abstract. We consider a nonlinear elliptic equation driven by the Robin p-Laplacian plus an indefinite potential. In the reaction we have the competing effects of a strictly $(p-1)$-sublinear parametric term and of a $(p-1)$-linear and nonuniformly nonresonant term. We study the set of positive solutions as the parameter $\lambda > 0$ varies. We prove a bifurcation-type result for large values of the positive parameter λ. Also, we show that for all admissible $\lambda > 0$, the problem has a smallest positive solution $\overline{u}_\lambda$ and we study the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem:

$$(P_\lambda) \qquad \left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = f(z,u(z),\lambda) + g(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u(z)^{p-1} = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega, \ \lambda > 0. \end{array} \right\}$$

In this problem, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega), \ 1 < p < \infty.$$

The potential function $\xi \in L^\infty(\Omega)$ is in general indefinite (that is, sign-changing). Therefore the differential operator (the left-hand side of (P_λ)) need not be coercive. In the reaction (the right-hand side of (P_λ)), we have the competing effects of two terms. The first is a parametric function which is strictly $(p-1)$-sublinear near $+\infty$. The second function (the perturbation of the parametric term), is $(p-1)$-linear near $+\infty$. Both functions are Carathéodory (that is, for all $x \in \mathbb{R}$ the mappings $z \mapsto f(z,x,\lambda)$ and $z \mapsto g(z,x)$ are measurable and for all $z \in \Omega$ the functions $x \mapsto f(z,x,\lambda)$ and $x \mapsto g(z,x)$ are continuous). In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u, defined by extension (according to the nonlinear Green's identity) of the map

$$C^1(\overline{\Omega}) \ni u \mapsto |Du|^{p-2}(Du,n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

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with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This map is uniformly continuous from $C^1(\overline{\Omega})$ into $L^p(\partial\Omega)$ (in fact, locally Lipschitz if $p \geq 2$ and Hölder continuous if $1 < p < 2$). Also, $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. So, this map admits a unique extension to the whole Sobolev space. We refer for details to Lemma 3 and Theorem 1 in Casas & Fernández [5] (see also Papageorgiou, Rădulescu & Repovš [20, p. 28] for the classical case).

Our aim in this paper is to study the nonexistence, existence and multiplicity of positive solutions for problem (P_λ) as the parameter λ moves on the positive semi-axis $(0, +\infty)$. We prove a bifurcation-type result for large values of the parameter. More precisely, we show that there is a critical parameter value $\lambda^* > 0$ such that
(i) for all $\lambda > \lambda^*$, problem (P_λ) has at least two positive solutions;
(ii) for all $\lambda = \lambda^*$, problem (P_λ) has at least one positive solution;
(iii) for all $0 < \lambda < \lambda^*$, problem (P_λ) has no positive solutions.

Moreover, we show that for every admissible parameter $\lambda \in [\lambda^*, +\infty)$, problem (P_λ) has a smallest positive solution $\overline{u}_\lambda$ and we examine the continuity and monotocicity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

The first such bifurcation-type result for parametric elliptic equations with competing nonlinearities was proved by Ambrosetti, Brezis & Cerami [2] (semilinear Dirichlet problems with concave-convex reaction). Their work was extended to Dirichlet p-Laplace equations by Garcia Azorero, Manfredi & Peral Alonso [7], Guo & Zhang [10], Hu & Papageorgiou [12]. For equations of logistic type there are the works of Rădulescu & Repovš [21] (semilinear Dirichlet problems) and Cardinali, Papageorgiou & Rubbioni [4] (nonlinear Neumann problems). For Robin problems, we mention the work of Papageorgiou & Rădulescu [16]. In all aforementioned works the differential operator is coercive and the reaction has a different pair of competing nonlinearities. In the present paper we distinguish a new class of competition phenomena, which lead to bifurcation-type results. In fact, the behaviour of the set of positive solutions as the parameter $\lambda > 0$ varies, is similar to that of superdiffusive logistic equations, since the "bifurcation" occurs for large values of $\lambda > 0$.

Our method of proof uses variational tools from critical point theory together with suitable truncation, perturbation and comparison arguments.

2. Mathematical background and hypotheses

Suppose that X is a Banach space. We denote by X^* the topological dual of X and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X).

Given $\varphi \in C^1(X, \mathbb{R})$ we say that φ satisfies the "Palais-Smale condition" (the "PS-condition" for short) if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that
$\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ. Using this condition, one can prove a deformation theorem from which follows the minimax theory for the critical values of φ. Prominent in this theory is the so-called "mountain pass theorem", which we recall here because we will use it in the sequel.

Theorem 2.1. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the PS-condition, u_0, $u_1 \in X$, $||u_1 - u_0|| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : ||u - u_0|| = \rho\} = m_\rho$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \varphi(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \ge m_\rho$ and c is a critical value of φ (that is, we can find $\hat{u} \in X$ such that $\varphi'(\hat{u}) = 0$ and $\varphi(\hat{u}) = c$).

Remark 2.2. We mention that if $\varphi' = A + K$, with $A : X \to X^*$ a continuous map of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in X) and $K : X \to X^*$ is completely continuous (that is, if $u_n \xrightarrow{w} u$ in X, then $K(u_n) \to K(u)$ in X^*), then φ satisfies the PS-condition (see Marano & Papageorgiou [14, Proposition 2.2]). This is the case in our setting.

The analysis of problem (P_λ) involves the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue space $L^p(\partial\Omega)$.

We denote by $||\cdot||$ the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$||u|| = \left(||u||_p^p + ||Du||_p^p\right)^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we introduce the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using $\sigma(\cdot)$ we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega), 1 \le q \le \infty$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map gives meaning to the notion of "boundary values" for any Sobolev function. The trace map is not surjective (in fact, $\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$) and $\ker \gamma_0 = W_0^{1,p}(\Omega)$. Moreover, γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^p(\partial\Omega)$ for all $1 \le q < \infty$ if $N \le p$. In the sequel, for the sake of notational simplicity, we will drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,p}(\Omega).$$

In the next proposition, we have collected the main properties of this map (see Gasinski & Papageorgiou [9, p. 279]).

Proposition 2.3. *The map $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (thus, maximal monotone, too) and of type $(S)_+$.*

Now we introduce our conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

$H(\xi)$: $\xi \in L^\infty(\Omega)$

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 2.4. When $\beta \equiv 0$, we have the Neumann problem.

Let $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma_p(u) = ||Du||_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

Also, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies

$$|f_0(z,x)| \leq a(z)(1 + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega), 1 < r \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$ (the critical Sobolev exponent).

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega F_0(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

In the framework of variational methods, the local minimizers of φ_0 play an important role. As we will see in the sequel, solutions of the problem are often generated by minimizing φ_0 on a constrained set defined by using the usual pointwise order on $W^{1,p}(\Omega)$ (this is done, via truncation of $f_0(z,\cdot)$). It is well-known that the order cone

$$W_+ = \{u \in W^{1,p}(\Omega) : u(z) \geq 0 \text{ for almost all } z \in \Omega\}$$

of $W^{1,p}(\Omega)$ has an empty interior. So, it is not clear if the constrained minimizer is in fact an unconstrained local minimizer of φ_0 over all of $W^{1,p}(\Omega)$.

The next result is helpful in this direction. It is a particular case of a more general result that can be found in Papageorgiou & Rădulescu [17]. The first to prove this relation between Hölder and Sobolev local minimizers were Brezis & Nirenberg [3].

Proposition 2.5. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in C^1(\overline{\Omega}) \quad \textit{with } ||h||_{C^1(\overline{\Omega})} \leq \rho_0.$$

Then $u_0 \in C^{1,\vartheta}(\overline{\Omega})$ with $\vartheta \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in W^{1,p}(\Omega) \textit{with } ||h|| \leq \rho_1.$$

As we already mentioned in the first section of this paper, our approach involves also comparison arguments. The next proposition will be helpful in this direction. It is a special case of a more general result of Papageorgiou, Rădulescu & Repovš [19].

Proposition 2.6. *Assume that* $h_1, h_2, \vartheta \in L^\infty(\Omega), \vartheta(z) \geq 0$ *for almost all* $z \in \Omega$

$$0 < \eta \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega$$

and $u_1, u_2 \in C^{1,\mu}(\overline{\Omega})$ *with* $0 < \mu \leq 1$ *are such that* $u_1 \leq u_2$ *and*

$$-\Delta_p u_1 + \vartheta(z)|u_1|^{p-2}u_1 = h_1,$$
$$-\Delta_p u_2 + \vartheta(z)|u_2|^{p-2}u_2 = h_2 \text{ for almost all } z \in \Omega.$$

Then $u_2 - u_1 \in \operatorname{int}\widehat{C_+} = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}$.

Next, we consider the following nonlinear eigenvalue problem

$$\left\{\begin{array}{l} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega. \end{array}\right\} \tag{2.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" if problem (2.1) admits a nontrivial solution $\hat{u}$, which is known as an "eigenfunction" corresponding to $\hat{\lambda}$. We denote by $\hat{\sigma}(p)$ the set of eigenvalues of problem (2.1). It is easy to see that $\hat{\sigma}(p) \subseteq \mathbb{R}$ is closed and has a smallest element $\hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta) \in \mathbb{R}$ (first eigenvalue), which has the following properties (for details, we refer to Papageorgiou & Rădulescu [16] and Fragnelli, Mugnai & Papageorgiou [6]).

Proposition 2.7. *If hypotheses* $H(\xi), H(\beta)$ *are satisfied, then problem (2.1) has a smallest eigenvalue* $\hat{\lambda}_1 \in \mathbb{R}$ *such that*

(a) $\hat{\lambda}_1$ *is isolated in* $\hat{\sigma}(p)$ *(that is, there exists* $\epsilon > 0$ *such that* $(\hat{\lambda}_1, \hat{\lambda}, +\epsilon) \cap \hat{\sigma}(p) = \emptyset$*);*

(b) $\hat{\lambda}_1$ *is simple (that is, if* $\hat{u}, \hat{v}$ *are eigenfunctions corresponding to* $\hat{\lambda}_1$*, then* $\hat{u} = \eta\hat{v}$ *for some* $\eta \in \mathbb{R}\backslash\{0\}$*);*

(2.2)

(*c*)
$$\hat{\lambda}_1 = \inf\left\{\frac{\gamma_0(u)}{||u||_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right\}.$$

Remark 2.8. The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace.

It follows from (2.2) that the elements of this eigenspace have fixed sign. We denote by $\hat{u}_1$ the positive, L^p-normalized (that is, $||\hat{u}_1||_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. We know that $\hat{u}_1 \in D_+$ (see [16], [6]). Also, every eigenvalue different from $\hat{\lambda}_1$ has eigenfunctions in $C^1(\overline{\Omega})$ which are nodal (that is, sign-changing). Finally, if $\xi \in L^\infty(\Omega), \xi(z) \geq 0$ for almost all $z \in \Omega$ and either $\xi \not\equiv 0$ or $\beta \not\equiv 0$, then $\hat{\lambda}_1 > 0$.

An easy consequence of the above properties is the following lemma (see Mugnai & Papageorgiou [15, Lemma 4.11]).

Lemma 2.9. *If hypotheses* $H(\xi), H(\beta)$ *hold,* $\eta \in L^\infty(\Omega), \eta(z) \leq \hat{\lambda}_1$ *for almost all* $z \in \Omega$ *and the inequality is strict on a set of positive measure, then there exists* $c_0 > 0$ *such that*

$$c_0||u||^p \leq \gamma_p(u) - \int_\Omega \eta(z)|u|^p dz \text{ for all } u \in W^{1,p}(\Omega).$$

The hypotheses on the two terms of the reaction of (P_λ) are the following.

$H(f)$ $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a Carathéodory function such that for all $\lambda > 0$, $f(z, x, \lambda) \geq 0$ for almost all $z \in \Omega$, all $x \geq 0$, $f(z, 0, \lambda) = 0$ for almost all $z \in \Omega$, and

(i) for every $\rho > 0$ and every $\lambda_0 > 0$, there exists $a_{\rho,\lambda_0} \in L^\infty(\Omega)$ such that

$$0 \leq f(z, x, \lambda) \leq a_{\rho,\lambda_0}(z) \text{ for almost all } z \in \Omega, \text{ all } 0 \leq x \leq \rho, \ 0 < \lambda \leq \lambda_0;$$

(ii) for every $\lambda > 0$, we have

$$\lim_{x\to+\infty} \frac{f(z, x, \lambda)}{x^{p-1}} = \lim_{x\to 0^+} \frac{f(z, x, \lambda)}{x^{p-1}} = 0 \text{ uniformly for almost all } z \in \Omega;$$

(iii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda)ds$, then there exist $v_0 \in L^p(\Omega)$ and $\tilde{\lambda} > 0$ such that $\int_\Omega F(z, v_0(z), \lambda)dz > 0$ for all $\lambda > \tilde{\lambda}$;

(iv) • we have $f(z, x, \lambda) \to 0^+$ as $\lambda \to 0^+$ uniformly for almost all $z \in \Omega$, all $x \in C \subseteq \mathbb{R}_+$ bounded, $f(z, x, \lambda) \to +\infty$ as $\lambda \to +\infty$ for almost all $z \in \Omega$, all $x > 0$;
• for every $s > 0$, we can find $\tilde{\eta}_s > 0$ such that

$$0 < \tilde{\eta}_s \leq f(z, x, \mu) - f(z, x, \lambda) \text{ for almost all } z \in \Omega, \text{all } x \geq s, \ \text{all } 0 < \lambda < \mu.$$

Remark 2.10. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality that

$$f(z, \cdot, \lambda)|_{(-\infty,0]} = 0 \text{ for almost all } z \in \Omega, \text{all } \lambda > 0. \tag{2.3}$$

Note that hypothesis $H(f)(ii)$ implies that $f(z, \cdot, \lambda)$ is strictly $(p - 1)$-sublinear near $+\infty$ and also near 0^+. Hypothesis $H(f)(iii)$ is satisfied if there exists $\tilde{\lambda} > 0$ such that $L(z) = \{x \in \mathbb{R} : f(z, x, \lambda) > 0\}$ is nonempty for almost all $z \in \Omega$, all $\lambda > \tilde{\lambda}$. Finally, note that hypothesis $H(f)(iv)$ implies that for almost all $z \in \Omega$, all $x > 0$, the mapping $\lambda \mapsto f(z, x, \lambda)$ is strictly increasing.

$H(g)$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)$ and $p \leq r < p^*$ such that

$$(g(z, x)) \leq a(1)(1 + x^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \geq 0;$$

(ii) there exists a function $\eta_0 \in L^\infty(\Omega)$ such that $\eta_0(z) \leq \hat{\lambda}_1$ for almost all $z \in \Omega$, $\eta_0 \not\equiv \hat{\lambda}_1$, $\limsup_{x\to+\infty} \frac{g(z,x)}{x^{p-1}} \leq \eta_0(z)$ and $\limsup_{x\to 0^+} \frac{g(z,x)}{x^{p-1}} \leq \eta_0(z)$ uniformly for almost all $z \in \Omega$;

(iii) for almost all $z \in \Omega$ the mapping $x \mapsto \frac{g(z,x)}{x^{p-1}}$ is nondecreasing on $(0, +\infty)$.

Remark 2.11. As we did for $f(z, \cdot, \lambda)$, without any loss of generality, we may assume that

$$g(z, \cdot)|_{(-\infty,0]} = 0 \text{ for almost all } z \in \Omega. \tag{2.4}$$

Hypothesis $H(g)(ii)$ says that asymptotically at $+\infty$ and at 0^+ we have nonuniform nonresonance with respect to $\hat{\lambda}_1$ from the left.

H_0 : for every $\rho > 0$ and every $\tilde{\lambda} > 0$, we can find $\hat{\xi}_0^{\tilde{\lambda}} > 0$ such that for almost all $z \in \Omega$ and all $0 < \lambda \leq \lambda_0$, the function $x \mapsto f(z, x, \lambda) + g(z, x) + \hat{\xi}_\rho^{\hat{\lambda}} x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 2.12. This hypothesis is satisfied if, for example, for almost all $z \in \Omega$ and every $\lambda > 0$, the functions $f(z, \cdot, \lambda)$ and $g(z, \cdot)$ are differentiable and for every $\rho > 0$ and $\hat{\lambda} > 0$, there exists $\hat{\xi}_\rho^{\tilde{\lambda}} > 0$ such that

$$(f'(z, x, \lambda) + g'_x(z, x))x \geq -\hat{\xi}_\rho^{\tilde{\lambda}} x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

Examples. The following pairs of functions f and g satisfy hypotheses $H(f), H(g), H_0$. For the sake of simplicity we drop the z-dependence. Also recall (2.3) and (2.4).

$$f_1(x, \lambda) = \begin{cases} \lambda x^{p-1} \ln(1 + x) & \text{if } 0 \leq x \leq 1 \\ \lambda x^{q-1} & \text{if } 1 < x \end{cases} \qquad 1 < q < p$$

$$g_1(x) = \eta x^{p-1} \qquad \text{for } x \geq 0,\ \eta < \hat{\lambda}_1,$$

$$f_2(x, \lambda) = \begin{cases} \lambda x^{r-1} & \text{if } 0 \leq x \leq 1 \\ \lambda x^{q-1} & \text{if } 1 < x \end{cases} \qquad 1 < q < p < r,$$

$$g_2(x) = \begin{cases} c x^{\tau-1} - x^{q-1} & \text{if } 0 \leq x \leq 1 \\ \eta x^{p-1} + (c - 1 - \eta) & \text{if } 1 < x \end{cases} \qquad 1 < q < p \leq \tau, \eta < \hat{\lambda}_1, \quad c > \max\{\eta + 1, 0\},$$

$$f_3(x, \lambda) = \begin{cases} \lambda(x^{\tau-1} - x^{r-1}) & \text{if } 0 \leq x \leq 1 \\ \lambda x^{q-1} \ln x & \text{if } 1 < x \end{cases} \qquad 1 < q < p < \tau < r,$$

$$g_3(x) = \begin{cases} \eta(x^{p-1} + x^{r-1}) & \text{if } 0 \leq x \leq 1 \\ \eta(x^{p-1} + x^{q-1}) & \text{if } 1 < x \end{cases} \qquad 1 < q < p < r,\ \eta < \hat{\lambda}_1,$$

$$f_4(x, \lambda) = \begin{cases} x^{\tau-1} & \text{if } 0 \leq x \leq \rho(\lambda) \\ x^{q-1} + \mu(\lambda) & \text{if } \rho(\lambda) < x \end{cases}$$

$$g_4(x) = \eta x^{p-1}$$

with $\rho : (0, +\infty) \to (0, +\infty)$ strictly increasing, continuous, $\rho(\lambda) \to 0^+$ as $\lambda \to 0^+$, $\rho(\lambda) \to +\infty$ as $\lambda \to +\infty$, $\mu(\lambda) = [\rho(\lambda)^{\tau-1} - 1]\rho(\lambda)^{q-1}$, $1 < q < p < \tau$ and $\eta < \hat{\lambda}_1$.

Finally, we fix some basic notation which we will use throughout this work. Let $x \in \mathbb{R}$ and set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Also, if $u, \hat{u} \in W^{1,p}(\Omega)$ and $u \leq \hat{u}$, then

$$[u, \hat{u}] = \{v \in W^{1,p}(\Omega) : u(z) \leq v(z) \leq \hat{u}(z) \text{ for almost all } z \in \Omega\}.$$

We denote by $\text{int}_{C^1(\overline{\Omega})}[u, \hat{u}]$ the interior in $C^1(\overline{\Omega})$ of $[u, \hat{u}] \cap C^1(\overline{\Omega})$.

Under the hypotheses on the data of problem (P_λ), the main result of this paper is the following bifurcation-type theorem.

Theorem. *Assume that hypotheses $H(\xi), H(\beta), H(f), H(g), H_0$ hold. Then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda > \lambda^*$ problem (P_λ) has at least two positive solutions*
$$u_0,\ \hat{u} \in D_+;$$
(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution*
$$u_{\lambda^*} \in D_+;$$
(c) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has no positive solution.*

Finally, if $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of φ, that is,
$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

3. Positive solutions

Throughout the rest of the paper we assume that hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, $H(g)$, H_0 are fulfilled.

We introduce the two following two sets:
$$\begin{aligned}\mathcal{L} &= \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ admits a positive solution}\},\\ S(\lambda) &= \text{ the set of positive solutions for problem } (P_\lambda).\end{aligned}$$

We set $\lambda^* = \inf \mathcal{L}$ with the usual convention that $\inf \emptyset = +\infty$.

Proposition 3.1. *We have $\mathcal{L} \neq \emptyset$ and so $0 \leq \lambda^* < +\infty$.*

Proof. From hypotheses $H(f)(i), (ii)$, we see that given $\epsilon > 0$ and $\lambda > 0$, we can find $c_1 = c_1(\epsilon, \lambda) > 0$ such that
$$F(z, x, \lambda) \leq \frac{\epsilon}{p} x^p + c_1 \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.1}$$

Similarly, hypotheses $H(g)(i), (ii)$ imply that we can find $c_2 = c_2(\epsilon) > 0$ such that
$$G(z, x) \leq (\eta_0(z) + \epsilon) x^p + c_2 \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.2}$$

Let $\mu > ||\xi||_\infty$ (see hypothesis $H(\xi)$) and consider the Carathéodory function $k_\lambda(z, x)$ defined by
$$k_\lambda(z, x) = f(z, x, \lambda) + g(z, x) \text{ for all } (z, x) \in \Omega \times \mathbb{R},\ \lambda > 0 \text{ (see (2.3), (2.4))}.$$

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s) ds$ and consider the C^1-functional $\Psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$\Psi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}||u^-||_p^p - \int_\Omega K_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (3.1) and (3.2), we have for all $u \in W^{1,p}(\Omega)$.
$$\begin{aligned}\Psi_\lambda(u) &\geq c_3||u^-||^p + \frac{1}{p}\gamma_p(u^+) - \frac{1}{p}\int_\Omega (\eta_0(z) + 2\epsilon)(u^+)^p dz - c_4\\ &\quad \text{for some } c_3, c_4 > 0 \text{ (recall that } \mu > ||\xi||_\infty)\\ &\geq c_3||u^-||^p + (c_0 - 2\epsilon)\,||u^+||^p - c_4.\end{aligned}$$

Choosing $\epsilon \in (0, \frac{c_0}{2})$, we obtain
$$\Psi_\lambda(u) \geq c_5||u||^p - c_4 \text{ for some } c_5 > 0,\ \text{all } u \in W^{1,p}(\Omega),$$

$$\Rightarrow \quad \Psi_\lambda(\cdot) \text{ is coercive.}$$

Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that

$$\Psi_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

By the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\Psi_\lambda(u_\lambda) = \inf\left\{\Psi_\lambda(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.3}$$

Hypotheses $H(f)(i), (ii)$ imply that for every $\lambda > 0$, we can find $c_6 = c_6(\lambda) > 0$ such that

$$0 \leq F(z,x,\lambda) \leq c_6 x^p \text{ for almost all } z \in \Omega, \text{ all } x \geq 0.$$

Evidently, in hypothesis $H(f)(iii)$ we can have $v_0 \geq 0$ (see (2.3)). Consider the continuous integral functional $i_\lambda : L^p(\Omega) \to \mathbb{R}$ defined by

$$i_\lambda(v) = \int_\Omega F(z, v(z), \lambda)dz \text{ for all } v \in L^p(\Omega),$$

$$\Rightarrow \quad i_\lambda(v_0) > 0 \text{ for all } \lambda > \tilde{\lambda} > 0 \text{ (see hypothesis } H(f)(iii)).$$

Exploiting the density of $W^{1,p}(\Omega)$ in $L^p(\Omega)$, we can find $\tilde{v}_0 \in W^{1,p}(\Omega)$, $\tilde{v}_0 \geq 0$, $\tilde{v}_0 \neq 0$ such that

$$i_\lambda(\tilde{v}_0) > 0 \text{ for all } \lambda > \tilde{\lambda}.$$

Then using hypothesis $H(f)(iv)$ and Fatou's lemma, we infer that

$$\lim_{\lambda\to+\infty} \int_\Omega F(z, \tilde{v}_0, \lambda)dz = +\infty. \tag{3.4}$$

On the other hand, hypothesis $H(g)(i)$ implies that if $G(z,x) = \int_0^x g(z,s)ds$, then

$$\left|\int_\Omega G(z, \tilde{v}_0)dz\right| \leq c_7 \text{ for some } c_7 > 0. \tag{3.5}$$

Then from (3.4) and (3.5) we see that for large enough $\lambda > \tilde{\lambda}$, we have

$$\begin{aligned} & \Psi_\lambda(\tilde{v}_0) < 0, \\ \Rightarrow \quad & \Psi_\lambda(u_\lambda) < 0 = \Psi_\lambda(0) \text{ (see (3.3))} \\ \Rightarrow \quad & u_\lambda \neq 0. \end{aligned}$$

From (3.3) we have

$$\Psi'_\lambda(u_\lambda) = 0,$$

$$\Rightarrow \quad \langle A(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma - \int_\Omega \mu(u_\lambda^-)^{p-1}h d\sigma$$

$$\int_\Omega [f(z,u_\lambda,\lambda) + g(z,u_\lambda)]h dz \text{ for all } h \in W^{1,p}(\Omega). \tag{3.6}$$

In (3.6) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \gamma_p(u_\lambda^-) + \mu||u_\lambda^-||_p^p = 0 \text{ (see (2.3), (2.4))}, \\ \Rightarrow \quad & c_8||u_\lambda^-||^p \leq 0 \text{ for some } c_8 > 0 \text{ (recall that } \mu > ||\xi||_\infty), \\ \Rightarrow \quad & u_\lambda \geq 0, u_\lambda \neq 0. \end{aligned}$$

Then it follows from (3.6) that $u_\lambda \in S_\lambda \subseteq D_+$ and so for large enough $\lambda > \tilde{\lambda}$, we have $\lambda \in \mathcal{L}$, hence $\mathcal{L} \neq \emptyset$. ☐

Proposition 3.2. *For every $\lambda \in \mathcal{L}$ we have $S(\lambda) \subseteq D_+$ and $\lambda^* > 0$.*

Proof. Let $\lambda \in \mathcal{L}$ and let $u \in S(\lambda)$. Reasoning as in Papageorgiou & Rădulescu [16] using the nonlinear Green identity, we have

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = f(z,u(z),\lambda) + g(z,u(z)) \text{ for almost all } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{3.7}$$

By (3.7) and Papageorgiou & Rădulescu [17] (see Proposition 2.10) we have

$$u \in L^\infty(\Omega).$$

Invoking Theorem 2 of Lieberman [13], we infer that

$$u \in C_+ \backslash \{0\}.$$

Let $\rho = ||u||_\infty$ and let $\hat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis H_0. Then

$$\Delta_p u(z) \leq \left(||\xi||_\infty + \hat{\xi}_\rho^\lambda \right) u(z)^{p-1} \text{ for almost all } z \in \Omega. \tag{3.8}$$

From (3.8) and the nonlinear maximum principle (see, for example, Gasinski & Papageorgiou [8, p. 738]), we have

$$\begin{array}{rl} & u \in D_+, \\ \Rightarrow & S(\lambda) \subseteq D_+ \text{ for all } \lambda > 0. \end{array}$$

Next, we show that $\lambda^* = \inf \mathcal{L} > 0$. Hypotheses $H(f)(i), (ii), (iv)$ imply that given $\epsilon > 0$, we can find $\overline{\lambda} > 0$ such that

$$0 \leq f(z,x,\overline{\lambda}) \leq \epsilon x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.9}$$

Hypothesis $H(g)(ii)$ implies that we can find $M, \delta > 0$ such that

$$g(z,x) \leq (\eta_0(z) + \epsilon)x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geq M, \ 0 \leq x \leq \delta. \tag{3.10}$$

On the other hand, by hypothesis $H(g)(iii)$, we have for almost all $z \in \Omega$ and all $\delta \leq x \leq M$

$$\begin{array}{rrl} & \frac{g(z,x)}{x^{p-1}} & \leq \dfrac{g(z,M)}{M^{p-1}}, \\ \Rightarrow & g(z,x) & \leq \dfrac{g(z,M)}{M^{p-1}} x^{p-1} \\ & & \leq (\eta_0(z) + \epsilon)x^{p-1} \text{ (see (3.10))}. \end{array} \tag{3.11}$$

So, by (3.10) and (3.11), we infer that

$$g(z,x) \leq (\eta_0(z) + \epsilon)x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.12}$$

Let $\lambda \in (0, \overline{\lambda})$ (see (3.9)) and assume that $\lambda \in \mathcal{L}$. Then from the first part of the proof, we know that we can find $u_\lambda \in S(\lambda) \subseteq D_+$. For every $h \in W^{1,p}(\Omega)$, $h \geq 0$ we have

$$\langle A(u_\lambda), h \rangle + \int_\Omega \xi(z) u_\lambda^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h d\sigma$$

$$= \int_\Omega [f(z,u_\lambda,\lambda)+g(z,u_\lambda)]hdz$$

$$\leq \int_\Omega (\eta_0(z)+2\epsilon)u_\lambda^{p-1}hdz \text{ (see (3.9), (3.12) and hypothesis } H(f)(iv)). \tag{3.13}$$

In (3.13) we choose $h=u_\lambda \in W^{1,p}(\Omega), u_\lambda \geq 0$. Then

$$\begin{aligned} & \gamma_p(u_\lambda) - \textstyle\int_\Omega \eta_0(z)u_\lambda^{p-1}dz \leq 2\epsilon ||u_\lambda||^p, \\ \Rightarrow \quad & c_0 \leq 2\epsilon \text{ (see Lemma 2.9)}, \end{aligned}$$

Choosing $\epsilon \in (0, \frac{c_0}{2})$, we get a contradiction. Therefore $\lambda \notin \mathcal{L}$ and so

$$0 < \overline{\lambda} \leq \lambda^*.$$

The proof is now complete. □

Next, we show that $\mathcal{L}$ is half-line.

Proposition 3.3. *Assume that $\lambda \in \mathcal{L}$. Then $[\lambda, +\infty) \subseteq \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S(\lambda) \subseteq D_+$ (see Proposition 3.2). Let $\vartheta > \lambda$ and consider the following truncation-perturbation of the reaction in problem (P_ϑ):

$$\hat{k}_\vartheta(z,x) = \begin{cases} f(z,u_\lambda(z),\vartheta)+g(z,u_\lambda(z))+\mu u_\lambda(z)^{p-1} & \text{if } x \leq u_\lambda(z) \\ f(z,x,\vartheta)+g(z,x)+\mu x^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.14}$$

Recall that $\mu > ||\xi||_\infty$. We set $\hat{K}_\vartheta(z,x) = \int_0^x \hat{k}_\vartheta(z,s)ds$ and consider the C^1-functional $\hat{\psi}_\vartheta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\vartheta(u) = \frac{1}{p}\gamma_p(u) + \frac{\mu}{p}||u||_p^p - \int_\Omega \hat{K}_\vartheta(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Reasoning as in the proof of Proposition 3.1, we can show that

- $\hat{\psi}_\vartheta(\cdot)$ is coercive;
- $\hat{\psi}_\vartheta(\cdot)$ is sequentially weakly lower semicontinuous.

So, we can find $u_\vartheta \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\psi}_\vartheta(u_\vartheta) = \inf\left\{\hat{\psi}_\vartheta(u) : u \in W^{1,p}(\Omega)\right\}, \\ \Rightarrow \quad & \hat{\psi}_\vartheta'(u_\vartheta) = 0, \end{aligned}$$

$$\Rightarrow \quad \langle A(u_\vartheta), h\rangle + \int_\Omega (\xi(z)+\mu)|u_\vartheta|^{p-2}u_\vartheta hdz + \int_{\partial\Omega} \beta(z)|u_\vartheta|^{p-2}u_\vartheta hd\sigma =$$

$$\int_\Omega \hat{k}_\vartheta(z,u_\vartheta)hdz \text{ for all } W^{1,p}(\Omega). \tag{3.15}$$

In (3.15) we choose $h=(u_\lambda - u_\vartheta)^+ \in W^{1,p}(\Omega)$. Then we have

$$\left\langle A(u_\vartheta), (u_\lambda - u_\vartheta)^+\right\rangle + \int_\Omega (\xi(z)+\mu)|u_\vartheta|^{p-2}u_\vartheta(u_\lambda - u_\vartheta)^+dz +$$

$$\int_{\partial\Omega} \beta(z)|u_\vartheta|^{p-2}u_\vartheta(u_\lambda - u_\vartheta)^+d\sigma$$

$$= \int_\Omega [f(z,u_\lambda,\vartheta)+g(z,u_\lambda)+\mu u_\lambda^{p-1}](u_\lambda - u_\vartheta)^+dz \text{ (see (3.14))}$$

$$\begin{aligned}
&\geq \int_\Omega [f(z,u_\lambda,\lambda)+g(z,u_\lambda)+\mu u_\lambda^{p-1}](u_\lambda-u_\vartheta)^+dz \text{ (since } \lambda<\vartheta,\\
&\quad\text{see hypothesis } H(f)(iv))\\
&= \langle A(u_\lambda),(u_\lambda-u_\vartheta)^+\rangle+\int_\Omega(\xi(z)+\mu)u_\lambda^{p-1}(u_\lambda-u_\vartheta)^+dz\\
&\quad+\int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\lambda-u_\vartheta)^+d\sigma\\
&\quad(\text{since } u_\lambda\in S(\lambda)),\\
&\Rightarrow u_\lambda\leq u_\vartheta \text{ (see Proposition 2.3 and recall that } \mu>||\xi||_\infty).
\end{aligned}$$

Then equation (3.15) becomes

$$\begin{aligned}
&\langle A(u_\vartheta),h\rangle+\int_\Omega\xi(z)u_\vartheta^{p-1}hdz+\int_{\partial\Omega}\beta(z)u_\vartheta^{p-1}hd\sigma\\
&= \int_\Omega[f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)]hdz\\
&\quad\text{for all } h\in W^{1,p}(\Omega),\\
&\Rightarrow u_\vartheta\in S(\vartheta)\subseteq D_+ \text{ and so } \vartheta\in\mathcal{L}.
\end{aligned}$$

Therefore we conclude that

$$[\lambda,+\infty)\subseteq\mathcal{L}.$$

The proof is now complete. ☐

An interesting byproduct of the above proof is the following corollary.

Corollary 3.4. *If hypotheses $H(\xi), H(\beta), H(f), H(g), H_0$ hold, $\lambda\in\mathcal{L},\vartheta>\lambda$ and $u_\lambda\in S(\lambda)\subseteq D_+$, then $\vartheta\in\mathcal{L}$ and we can find $u_\vartheta\in S(\vartheta)\subseteq D_+$ such that $u_\lambda\leq u_\vartheta, u_\vartheta\neq u_\lambda$.*

In fact, we can improve the conclusion of this corollary as follows.

Proposition 3.5. *Assume that $\lambda\in\mathcal{L}$, $\vartheta>\lambda$ and $u_\lambda\in S(\lambda)\subseteq D_+$. Then $\vartheta\in\mathcal{L}$ and we can find $u_\vartheta\in S(\vartheta)\subseteq D_+$ such that $u_\vartheta-u_\lambda\in\operatorname{int}\widehat{C_+}$.*

Proof. From Corollary 3.4 we already know that $\vartheta\in\mathcal{L}$ and that there exists $u_\vartheta\in S(\vartheta)\subseteq D_+$ such that

$$u_\vartheta-u_\lambda\in C_+\backslash\{0\}.$$

Let $\rho=||u_\vartheta||_\infty$ and $\hat{\xi}_\rho^\vartheta>0$ as in H_0. We can always assume that $\hat{\xi}_\rho^\vartheta>||\xi||_\infty$. We have

$$\begin{aligned}
&-\Delta_p u_\lambda+(\xi(z)+\hat{\xi}_\rho^\vartheta)u_\lambda^{p-1}\\
&= f(z,u_\lambda,\lambda)+g(z,u_\lambda)+\hat{\xi}_\rho^\vartheta u_\lambda^{p-1}\\
&\leq f(z,u_\vartheta,\lambda)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1} \text{ (see hypothesis } H_0 \text{ and recall that } \lambda<\vartheta)\\
&= f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1}-[f(z,u_\vartheta,\vartheta)-f(z,u_\vartheta,\lambda)]\\
&\leq f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1}-\tilde{\eta}_s\\
&\quad\text{with } 0<s=\min_{\overline{\Omega}}u_\vartheta \text{ (recall that } u_\vartheta\in D_+ \text{ and see hypothesis } H(f)(iv))
\end{aligned}$$

$$< \quad f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1}$$

$$= \quad -\Delta_p u_\vartheta+(\xi(z)+\hat{\xi}_\rho^\vartheta)u_\vartheta^{p-1} \text{ for almost all } z\in\Omega \text{ (since } u_\vartheta\in S(\vartheta)). \tag{3.16}$$

Since $\tilde{\eta}_s>0$, from (3.16) and Proposition 2.6, we infer that

$$u_\vartheta-u_\lambda\in \operatorname{int}\widehat{C_+}.$$

The proof is complete. □

Now let $\lambda>\lambda^*$. By Proposition 3.3 we know that $\lambda\in\mathcal{L}$. We show that problem (P_λ) has at least two positive solutions.

Proposition 3.6. *If $\lambda>\lambda^*$, then problem (P_λ) has at least two positive solutions*

$$u_0,\ \hat{u}\in D_+,\ u_0\neq\hat{u}.$$

Proof. As we have already mentioned, $\lambda\in\mathcal{L}$. Let $\lambda^*<\eta<\lambda<\vartheta$. We have $\eta,\vartheta\in\mathcal{L}$ (see Proposition 3.3). According to Proposition 3.5, there are $u_\vartheta\in S(\vartheta)\subseteq D_+$ and $u_\mu\in S(\mu)\subseteq D_+$ such that

$$u_\vartheta-u_\mu\in\operatorname{int}\widehat{C_+}.$$

We introduce the Carathéodory function $l_\lambda(z,x)$ defined by

$$l_\lambda(z,x)=\begin{cases} f(z,u_\eta(z),\lambda)+g(z,u_\eta(z))+\mu u_\eta(z)^{p-1} & \text{if } x<u_\eta(z)\\ f(z,x,\lambda)+g(z,x)+\mu x^{p-1} & \text{if } u_\eta(z)\le x\le u_\vartheta(z)\\ f(z,u_\vartheta(z),\lambda)+g(z,x)+\mu u_\vartheta(z)^{p-1} & \text{if } u_\vartheta(z)<x.\end{cases} \tag{3.17}$$

Recall that $\mu>||\xi||_\infty$. We set $L_\lambda(z,x)=\int_0^x l_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u)=\frac{1}{p}\gamma_p(u)+\frac{\mu}{p}||u||_p^p-\int_\Omega L_\lambda(z,u)dz \text{ for all } u\in W^{1,p}(\Omega).$$

Since $\mu>||\xi||_\infty$, it is clear from (3.17) that $\hat{\varphi}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0\in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\varphi}_\lambda(u_0)=\inf\left\{\hat{\varphi}_\lambda(u):u\in W^{1,p}(\Omega)\right\},\\ \Rightarrow\ & \hat{\varphi}'_\lambda(u_0)=0,\\ \Rightarrow\ & \langle A(u_0),h\rangle+\int_\Omega(\xi(z)+\mu)|u_0|^{p-2}u_0hdz+\int_{\partial\Omega}\beta(z)|u_0|^{p-2}u_0hd\sigma=\\ & \int_\Omega l_\lambda(z,u_0)hdz \text{ for all } h\in W^{1,p}(\Omega). \end{aligned} \tag{3.18}$$

In (3.18) we first choose $h=(u_0-u_\vartheta)^+\in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \left\langle A(u_0),(u_0-u_\vartheta)^+\right\rangle+\int_\Omega(\xi(z)+\mu)u_0^{p-1}(u_0-u_\vartheta)^+dz\\ & +\int_{\partial\Omega}\beta(z)u_0^{p-1}(u_0-u_\vartheta)^+d\sigma\\ = & \int_\Omega[f(z,u_\vartheta,\lambda)+g(z,u_\vartheta)+\mu u_\vartheta^{p-1}](u_0-u_\vartheta)^+dz \text{ (see (3.17))}\\ \le & \int_\Omega[f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\mu u_\vartheta^{p-1}](u_0-u_\vartheta)^+dz \end{aligned}$$

$$
\begin{aligned}
&\text{(see hypothesis } H(f)(iv) \text{ and recall that } \lambda < \vartheta) \\
= \;& \langle A(u_\vartheta), (u_0 - u_\vartheta)^+ \rangle + \int_\Omega (\xi(z)+\mu) u_\vartheta^{p-1}(u_0-u_\vartheta)^+ dz \\
& + \int_{\partial\Omega} \beta(z) u_\vartheta^{p-1} (u_0 - u_\vartheta)^+ d\sigma \\
& \text{(since } u_\vartheta \in S(\vartheta)), \\
\Rightarrow \;& u_0 \le u_\vartheta \text{ (see Proposition 2.3 and recall that } \mu > ||\xi||_\infty).
\end{aligned}
$$

Similarly, if in (3.18) we choose $h = (u_\eta - u_0)^+ \in W^{1,p}(\Omega)$, we can show that

$$u_\eta \le u_0.$$

So, we have proved that

$$u_0 \in [u_\eta, u_\vartheta]. \tag{3.19}$$

Then it follows from (3.17), (3.18) and (3.19) that $u_0 \in S(\lambda) \subseteq D_+$. Moreover, arguing as in the proof of Proposition 3.5, via Proposition 2.6, we show that

$$
\begin{aligned}
& u_\vartheta - u_0 \in \operatorname{int} \widehat{C_+} \text{ and } u_0 - u_\eta \in \operatorname{int} \widehat{C_+}, \\
\Rightarrow \;& u_0 \in \operatorname{int}_{C^1(\overline{\Omega})} [u_\eta, u_\vartheta]. \qquad (3.20)
\end{aligned}
$$

Let $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional introduced in the proof of Proposition 3.1. From (3.17) it is clear that

$$\psi_\lambda|_{[u_\eta, u_\vartheta]} = \hat{\varphi}_\lambda|_{[u_\eta,u_\vartheta]} + \hat{k}_\lambda \text{ with } \hat{k}_\lambda \in \mathbb{R}. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$
\begin{aligned}
& u_0 \text{ is local } C^1(\overline{\Omega}) - \text{minimizer of } \psi_\lambda, \\
\Rightarrow \;& u_0 \text{ is local } W^{1,p}(\Omega) - \text{minimizer of } \psi_\lambda \text{ (see Proposition 2.5)}. \qquad (3.22)
\end{aligned}
$$

Hypotheses $H(f)(ii)$ and $H(g)(ii)$ imply that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$F(z,x,\lambda) \le \frac{\epsilon}{p} x^p, \; G(z,x) \le \frac{1}{p}(\eta_0(z) + \epsilon) x^p \text{ for almost all } z \in \Omega, \text{ all } 0 \le x \le \delta. \tag{3.23}$$

For all $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \le \delta$, we have

$$
\begin{aligned}
\psi_\lambda(u) \ge \;& \frac{1}{p}\gamma_p(u^-) + \frac{\mu}{p}||u^-||_p^p + \frac{1}{p}\gamma_p(u^+) - \frac{1}{p}\int_\Omega \eta_0(z)(u^+)^p dz - \frac{2\epsilon}{p}||u^+||_p^p \\
& \text{(see (3.23) and recall the definition of } \psi_\lambda \text{ in the proof of Proposition 3.1)} \\
\ge \;& c_9 ||u^-||^p + \frac{1}{p}(c_0 - 2\epsilon)\,||u^+||^p \text{ for some } c_9 > 0 \\
& \text{(recall that } \mu > ||\xi||_\infty \text{ and use Lemma 2.9)}.
\end{aligned}
$$

Choosing $\epsilon \in \left(0, \frac{c_0}{2}\right)$, we conclude that

$$
\begin{aligned}
& \psi_\lambda(u) \ge c_{10}||u||^p \text{ for some } c_{10} > 0, \text{ all } u \in C^1(\overline{\Omega}) \text{ with } ||u||_{C^1(\overline{\Omega})} \le \delta, \\
\Rightarrow \;& u = 0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \psi_\lambda, \\
\Rightarrow \;& u = 0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \psi_\lambda \text{ (see Proposition 2.5)}. \qquad (3.24)
\end{aligned}
$$

Without any loss of generality, we may assume that

$$0 = \psi_\lambda(0) \leq \psi_\lambda(u_0).$$

The analysis is similar if the opposite inequality holds using (3.24) instead of (3.22). In addition, we may assume that K_{ψ_λ} is finite. Otherwise since $K_{\psi_\lambda} \subseteq D_+ \cup \{0\}$, we see that we already have an infinity of positive solutions for problem (P_λ) and so we are done. Then on account of (3.22), we can find $\rho \in (0,1)$ small such that

$$0 = \psi_\lambda(0) \leq \psi_\lambda(u_0) < \inf\{\psi_\lambda(u) : ||u - u_0|| = \rho\} = m_\lambda, \ ||u_0|| > \rho \tag{3.25}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 29).

From the proof of Proposition 3.1 we know that

$$\begin{aligned} & \psi_\lambda(\cdot) \text{ is coercive,} \\ \Rightarrow \quad & \psi_\lambda(\cdot) \text{ satisfies the PS-condition (see Section 2).} \end{aligned} \tag{3.26}$$

From (3.25) and (3.26) it follows that we can use Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{u} \in K_{\psi_\lambda} \subseteq D_+ \cup \{0\} \text{ and } 0 < m_\lambda \leq \psi_\lambda(\hat{u}), \\ \Rightarrow \quad & \hat{u} \in S(\lambda) \subseteq D_+ \text{ and } \hat{u} \neq u_0 \text{ (see (3.25)).} \end{aligned}$$

The proof is now complete. □

Next, we show that the critical parameter value $\lambda^* > 0$ is also admissible (that is, $\lambda^* \in \mathcal{L}$).

Proposition 3.7. *We have that* $\lambda^* \in \mathcal{L}$.

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq (\lambda^*, +\infty)$ be such that $\lambda_n \to (\lambda^*)^+$ as $n \to \infty$. From the proof of Proposition 3.5, we know that we can find $u_n \in S(\lambda_n) \subseteq D_+$ $(n \in \mathbb{N})$ decreasing. We have

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}, \tag{3.27}$$

$$\begin{aligned} & \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \\ & \qquad = \int_\Omega [f(z, u_n, \lambda_n) + g(z, u_n)] h dz \\ & \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.28}$$

In (3.28) we choose $h = u_n \in W^{1,p}(\Omega)$. Using (3.27) and hypotheses $H(\xi), H(\beta)$, $H(f)(i)$, $H(g)(i)$, we see that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Therefore, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_{\lambda^*} \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_{\lambda^*} \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.29}$$

For every $n \in \mathbb{N}$, we have
$-\Delta_p u_n(z) + \xi(z) u_n(z)^{p-1} = f(z, u_n(z), \lambda_n) + g(z, u_n(z))$ for almost all $z \in \Omega$,

$$\frac{\partial u}{\partial n_p} + \beta(z)u_n^{p-1} = 0 \text{ on } \partial\Omega \text{ (see Papageorgiou \& Rădulescu [16]).} \tag{3.30}$$

From Papageorgiou & Rădulescu [17, Proposition 7] and (3.30), we know that we can find $c_{11} > 0$ such that

$$||u_n||_\infty \leq c_{11} \text{ for all } n \in \mathbb{N}.$$

Then invoking Theorem 2 of Lieberman [13], we can find $\gamma \in (0,1)$ and $c_{12} > 0$ such that

$$u_n \in C^{1,\gamma}(\overline{\Omega}) \text{ and } ||u_n||_{C^{1,\gamma}(\overline{\Omega})} \leq c_{12} \text{ for all } n \in \mathbb{N}. \tag{3.31}$$

Since $C^{1,\gamma}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, from (3.29) and (3.31), we have

$$u_n \to u_{\lambda^*} \text{ in } C^1(\overline{\Omega}). \tag{3.32}$$

Passing to the limit as $n \to \infty$ in (3.28) and using (3.32), we obtain

$$\begin{aligned} &\langle A(u_{\lambda^*}), h\rangle + \int_\Omega \xi(z)u_{\lambda^*}^{p-1}hdz + \int_{\partial\Omega} \beta(z)u_{\lambda^*}^{p-1}hd\sigma = \\ &\int_\Omega [f(z, u_{\lambda^*}, \lambda^*) + g(z, u_{\lambda^*})]hdz \text{ for all } h \in W^{1,p}(\Omega), \\ \Rightarrow\ & u_{\lambda^*} \text{ is a nonnegative solution of } (P_{\lambda^*}). \end{aligned} \tag{3.33}$$

We need to show that $u_{\lambda^*} \neq 0$. Then we will have $u_{\lambda^*} \in S(\lambda^*) \subseteq D_+$ and $\lambda^* \in \mathcal{L}$.

Arguing by contradiction, suppose that $u_{\lambda^*} = 0$. Then from (3.32) we have

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}). \tag{3.34}$$

Hypotheses $H(f)(ii)$ and $H(g)(ii)$ imply that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

(3.35)
$$f(z, x, \lambda_1)x \leq \epsilon x^p, \ g(z,x)x \leq (\eta_0(z) + \epsilon)x^p \text{ for almost all } z \in \Omega, \text{ all } 0 \leq x \leq \delta.$$

In (3.33) we choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} \gamma_p(u_n) &= \int_\Omega [f(z, u_n, \lambda_n) + g(z, u_n)]u_n dz \\ &\leq \int_\Omega [f(z, u_n, \lambda_1) + g(z, u_n)]u_n dz \text{ for all } n \in \mathbb{N} \text{ (see hypothesis } H(f)(iv)). \end{aligned} \tag{3.36}$$

From (3.34), we see that we can find $n_0 \in \mathbb{N}$ such that

$$u_n(z) \in (0, \delta] \text{ for all } z \in \overline{\Omega}, \text{ all } n \geq n_0. \tag{3.37}$$

Then from (3.35), (3.36), (3.37), we see that

$$\begin{aligned} &\gamma_p(u_n) - \int_\Omega \eta_0(z)u_n^p dz \leq 2\epsilon||u_n||_p^p \text{ for all } n \geq n_0, \\ \Rightarrow\ & c_0||u_n||^p \leq 2\epsilon||u_n||_p^p \text{ for all } n \geq n_0 \text{ (see Lemma 2.9)}, \\ \Rightarrow\ & c_0 \leq 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, choosing $\epsilon \in \left(0, \frac{c_0}{2}\right)$, we have a contradiction. Therefore $u_{\lambda^*} \neq 0$ and so $u_{\lambda^*} \in S(\lambda^*) \subseteq D_+$, hence $\lambda^* \in \mathcal{L}$. ☐

So, we conclude that

$$\mathcal{L} = [\lambda^*, +\infty) .$$

4. Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L}$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in D_+$ and we study the monotonicity and continuity properties of the map $\lambda \mapsto \bar{u}_\lambda$.

From Papageorgiou, Rădulescu & Repovš [18] (see the proof of Proposition 7), we know that $S(\lambda)$ is downward directed, that is, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \le u_1, u \le u_2$.

Proposition 4.1. *Assume that $\lambda \in \mathcal{L} = [\lambda^*, +\infty)$. Then problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in S(\lambda) \subseteq D_+$ (that is, $\bar{u}_\lambda \le u$ for all $u \in S(\lambda)$).*

Proof. According to Lemma 3.10 of Hu & Papageorgiou [11, p. 178] and since $S(\lambda)$ is downward directed, we can find $\{u_n\}_{n\ge1} \subseteq S(\lambda)$ decreasing such that

$$\inf S(\lambda) = \inf_{n\ge1} u_n .$$

We have

$$0 \le u_n \le u_1 \text{ for all } n \in \mathbb{N}, \tag{4.1}$$

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega [f(z, u_n.\lambda) + g(z, u_n)] h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{4.2}$$

Then reasoning as in the proof of Proposition 3.7 (see the part of the proof after (3.28)) and using (4.1) and (4.2), we obtain

$$\begin{aligned} & u_n \to \bar{u}_\lambda \text{ in } C^1(\overline{\Omega}) \text{ with } \bar{u}_\lambda \in S(\lambda), \\ \Rightarrow \quad & \bar{u}_\lambda = \inf S(\lambda). \end{aligned}$$

The proof is complete. ☐

Proposition 4.2. *The map $\lambda \mapsto \bar{u}_\lambda$ from $\overset{o}{\mathcal{L}} = (\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$ has the following properties:*

- *is strictly monotone in the sense that*

$$\overset{o}{\mathcal{L}} \ni \lambda < \vartheta \Rightarrow \bar{u}_\vartheta - \bar{u}_\lambda \in \operatorname{int} \widehat{C_+};$$

- *it is left continuous.*

Proof. First, we show the strict monotonicity of the map. So, let $\lambda \in \overset{o}{\mathcal{L}}$ and $\vartheta > \lambda$. Then $\vartheta \in \mathcal{L}$ and let $\bar{u}_\vartheta \in S(\vartheta) \subseteq D_+$ be the minimal solution of problem (P_ϑ). From the proof of Proposition 3.6, we know that we can find $u_\lambda \in S(\lambda) \subseteq D_+$ such that

$$\begin{aligned} & \bar{u}_\vartheta - u_\lambda \in \operatorname{int} \widehat{C_+} \text{ (see (3.20))}, \\ \Rightarrow \quad & \bar{u}_\vartheta - \bar{u}_\lambda \in \operatorname{int} \widehat{C_+} \text{ (since } \bar{u}_\lambda \le u_\lambda). \end{aligned}$$

This proves the strict monotonicity of the map $\lambda \mapsto \bar{u}_\lambda$ from $\overset{o}{\mathcal{L}} = (\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$.

Next, we show the left continuity of this map. So, let $\{\lambda_n\}_{n\geq 1} \subseteq \overset{o}{\mathcal{L}}$ and assume that $\lambda_n \to \lambda^-$. From the first part of the proof, we have

$$0 \leq \bar{u}_{\lambda_n} \leq \bar{u}_\lambda \text{ for all } n \geq 1$$

Then as before (see the proof of Proposition 3.7), we can say that

$$\bar{u}_{\lambda_n} \to \tilde{u}_\lambda \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty \tag{4.3}$$

and

$$\tilde{u}_\lambda \in S(\lambda) \subseteq D_+.$$

Suppose that $\tilde{u}_\lambda \neq \bar{u}_\lambda$. Then we can find $z_0 \in \overline{\Omega}$ such that

$$\begin{aligned} & \bar{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0), \\ \Rightarrow \quad & \bar{u}_\lambda(z_0) < \bar{u}_{\lambda_n}(z_0) \text{ for all } n \geq n_0, \end{aligned}$$

which contradicts the first part of the proposition. Therefore

$$\begin{aligned} & \tilde{u}_\lambda = \bar{u}_\lambda, \\ \Rightarrow \quad & \lambda \mapsto \bar{u}_\lambda \text{ is continuous from } \overset{o}{\mathcal{L}} \text{ into } C^1(\overline{\Omega}). \end{aligned}$$

The proof is now complete. □

Remark 4.3. In our setting the equation was nonuniformly nonresonant as $x \to +\infty$ (see hypotheses $H(f)(ii), H(g)(ii)$). Is it possible to treat also the resonant case, that is,

$$\limsup_{x\to+\infty} \frac{g(z,x)}{x^{p-1}} \leq \hat{\lambda}_1 \text{ uniformly for almost all } z \in \Omega.$$

Moreover, what is the situation of asymptotical behavior as $x \to +\infty$ we are nonresonant with respect to $\hat{\lambda}_1$, but from above the principal eigenvalue, that is,

$$\liminf_{x\to+\infty} \frac{g(z,x)}{x^{p-1}} \geq \hat{\eta} > \hat{\lambda}_1 \text{ uniformly for almost all } z \in \Omega.$$

A careful inspection of the arguments of this paper, reveals that for the nonresonant case but from above $\hat{\lambda}_1$, if a bifurcation-type result holds, then it will be for small values of $\lambda > 0$. This also suggests that if we want to extend the results of this paper to the resonant case, we must have resonance from the left of $\hat{\lambda}_1$, in the sense that

$$\hat{\lambda}_1 x^{p-1} - [f(z,x,\lambda) + g(z,x)] \to +\infty \text{ uniformly for almost all } z \in \Omega, \text{ as } x \to +\infty.$$

In this way we can preserve the coercivity of the energy functional and we hope to be able to extend the results of paper to the resonant case.

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