

## PERTURBATIONS OF NONLINEAR EIGENVALUE PROBLEMS

NIKOLAOS S. PAPAGEORGIOU

National Technical University, Department of Mathematics,  
Zografou Campus, Athens 15780, Greece  
Institute of Mathematics, Physics and Mechanics,  
Jadranska 19, 1000 Ljubljana, Slovenia

VICENȚIU D. RĂDULESCU \*

Faculty of Applied Mathematics, AGH University of Science and Technology,  
al. Mickiewicza 30, 30-059 Kraków, Poland  
Institute of Mathematics, Physics and Mechanics,  
Jadranska 19, 1000 Ljubljana, Slovenia  
Institute of Mathematics “Simion Stoilow” of the Romanian Academy,  
P.O. Box 1-764, 014700 Bucharest, Romania

DUŠAN D. REPOVŠ

Institute of Mathematics, Physics and Mechanics,  
Jadranska 19, 1000 Ljubljana, Slovenia  
Faculty of Education and Faculty of Mathematics and Physics,  
University of Ljubljana, 1000 Ljubljana, Slovenia

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**ABSTRACT.** We consider perturbations of nonlinear eigenvalue problems driven by a nonhomogeneous differential operator plus an indefinite potential. We consider both sublinear and superlinear perturbations and we determine how the set of positive solutions changes as the real parameter  $\lambda$  varies. We also show that there exists a minimal positive solution  $\bar{u}_\lambda$  and determine the monotonicity and continuity properties of the map  $\lambda \mapsto \bar{u}_\lambda$ . Special attention is given to the particular case of the  $p$ -Laplacian.

**1. Introduction.** The aim of this paper is to study the following nonlinear non-homogeneous parametric Robin problem

$$\left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda u(z)^{p-1} + f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \quad u > 0, \quad \lambda \in \mathbb{R}, \quad 1 < p < \infty. \end{array} \right\} \quad (P_\lambda)$$

In this problem,  $\Omega \subseteq \mathbb{R}^N$  is a bounded domain with a  $C^2$ -boundary  $\partial\Omega$ . The map  $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$  involved in the differential operator, is continuous, strictly monotone (hence maximal monotone, too) and satisfies some other regularity and growth conditions listed in hypotheses  $H(a)$  below (see Section 2). These extra-conditions on

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\* Corresponding author.

$a(\cdot)$  are not restrictive and so our framework incorporates many differential operators of interest such as the  $p$ -Laplacian and the  $(p, q)$ -Laplacian (that is, the sum of a  $p$ -Laplacian and a  $q$ -Laplacian). The potential function  $\xi \in L^\infty(\Omega)$  is indefinite (that is, sign changing). In the reaction (right-hand side of the equation), we have a parametric term  $u \mapsto \lambda u^{p-1}$  and a perturbation  $f(z, x)$  which is a Carathéodory function (that is, for all  $x \in \mathbb{R}$  the mapping  $z \mapsto f(z, x)$  is measurable and for almost all  $z \in \Omega$  the mapping  $x \mapsto f(z, x)$  is continuous).

We consider two distinct cases. In the first one,  $f(z, \cdot)$  is  $(p-1)$ -sublinear near  $+\infty$ , while in the second we assume that  $f(z, \cdot)$  is  $(p-1)$ -superlinear. In the boundary condition,  $\frac{\partial u}{\partial n_a}$  denotes the conormal derivative of  $u$ , defined by extension of the map

$$L^1(\overline{\Omega}) \ni u \mapsto (a(Du), n)_{\mathbb{R}^N},$$

with  $n(\cdot)$  being the outward unit normal on  $\partial\Omega$ . The boundary coefficient  $\beta(\cdot)$  is non-negative and the case  $\beta \equiv 0$  is also included and corresponds to the Neumann problem.

We look for positive solutions of  $(P_\lambda)$  and we want to determine how the set of positive solutions changes as the parameter  $\lambda$  moves on the real line  $\mathbb{R}$ . More precisely, we show that there is a critical parameter value  $\lambda^* \in \mathbb{R}$  such that for  $\lambda < \lambda^*$  problem  $(P_\lambda)$  has

- at least one positive smooth solutions, when  $f(z, \cdot)$  is  $(p-1)$ -sublinear;
- at least two positive smooth solutions, when  $f(z, \cdot)$  is  $(p-1)$ -superlinear.

For  $\lambda \geq \lambda^*$ , problem  $(P_\lambda)$  has no positive solutions.

In the particular case of the  $p$ -Laplace differential operator (that is,  $a(y) = |y|^{p-2}y$  for all  $y \in \mathbb{R}^N$ ), problem  $(P_\lambda)$  can be viewed as a perturbation of the classical eigenvalue problem for the  $p$ -Laplacian. In this particular case, we can identify  $\lambda^*$  as the principal eigenvalue  $\hat{\lambda}_1$  of the differential operator  $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$  with the Robin boundary condition. This was already observed by these authors for the semilinear problem (that is,  $p = 2$  hence  $a(y) = y$  for all  $y \in \mathbb{R}^N$ ), see Papageorgiou, Rădulescu & Repovš [21]. Also, for both cases (sublinear and superlinear), we establish the existence of a smallest positive solution  $\bar{u}_\lambda$  and determine the monotonicity and continuity properties of the map  $\lambda \mapsto \bar{u}_\lambda$ . Finally in the sublinear case we address the question of uniqueness of the solution.

Nonlinear, nonhomogeneous parametric Robin problems were also studied by Aurtori & Pucci [2], Colasuonno, Pucci & Varga [5], Fragnelli, Mugnai & Papageorgiou [7], Papageorgiou, Rădulescu & Repovš [22, 23], and Perera, Pucci & Varga [24].

Our approach is variational, using results from the critical point theory and also truncation, perturbation and comparison techniques.

**2. Mathematical background and hypotheses.** Let  $X$  be a Banach space and  $X^*$  its topological dual. We denote by  $\langle \cdot, \cdot \rangle$  the duality brackets for the pair  $(X^*, X)$ . Given  $\varphi \in C^1(X, \mathbb{R})$ , we say that  $\varphi$  satisfies the ‘‘Cerami condition’’ (the ‘‘C-condition’’ for short), if the following property holds:

‘‘Every sequence  $\{u_n\}_{n \geq 1} \subseteq X$  such that

$$\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R} \text{ is bounded and } (1 + \|u_n\|)\varphi'(u_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty,$$

admits a strongly convergent subsequence.’’

In what follows, we denote by  $K_\varphi$  the critical set of  $\varphi$ , that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Also, if  $c \in \mathbb{R}$ , then

$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Using the notion of the C-condition, we have the following minimax theorem, known in the literature as the “mountain pass theorem”.

**Theorem 2.1.** *If  $\varphi \in C^1(X, \mathbb{R})$  satisfies the C-condition,  $u_0, u_1 \in X$ ,  $\|u_1 - u_0\| > \rho > 0$ ,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$$

*and  $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t))$  with  $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$ , then  $c \geq m_\rho$  and  $c$  is a critical value of  $\varphi$  (that is, we can find  $\hat{u} \in X$  such that  $\varphi'(\hat{u}) = 0$  and  $\varphi(\hat{u}) = c$ ).*

In the analysis of problem  $(P_\lambda)$ , we will use the Sobolev space  $W^{1,p}(\Omega)$ , the Banach space  $C^1(\bar{\Omega})$  and the “boundary” Lebesgue spaces  $L^r(\partial\Omega)$ ,  $1 \leq r \leq \infty$ . We denote by  $\|\cdot\|$  the norm of  $W^{1,p}(\Omega)$  defined by

$$\|u\| = (\|u\|_p^p + \|Du\|_p^p)^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space  $C^1(\bar{\Omega})$  is an ordered Banach space with positive (order) cone  $C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$ . This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n} \Big|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Evidently,  $\text{int } C_+$  contains the open set  $D_+$  defined by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

In fact,  $D_+$  is the interior of  $C_+$  when  $C^1(\bar{\Omega})$  is furnished with the  $C(\bar{\Omega})$ -norm topology.

On  $\partial\Omega$  we consider the  $(N - 1)$ -dimensional Hausdorff (surface) measure  $\sigma(\cdot)$ . Using this measure on  $\partial\Omega$ , we can define in the usual way the “boundary” Lebesgue spaces  $L^r(\partial\Omega)$ . From the theory of Sobolev spaces, we know that there exists a unique continuous linear map  $\gamma_0 : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ , known as the “trace map”, such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}).$$

So, the trace map extends the notion of boundary value to all Sobolev functions. The map  $\gamma_0(\cdot)$  is compact into  $L^r(\partial\Omega)$  for all  $r \in [1, (N - 1)p/(N - p)]$  when  $p < N$ , and into  $L^r(\partial\Omega)$  for all  $1 \leq r < \infty$  when  $p \geq N$ . Also, we have

$$\text{im } \gamma_0 = W^{\frac{1}{p},p}(\partial\Omega) \left( \frac{1}{p} + \frac{1}{p'} = 1 \right) \text{ and } \ker \gamma_0 = W_0^{1,p}(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map  $\gamma_0(\cdot)$ . All restrictions of Sobolev functions on  $\partial\Omega$ , are understood in the sense of traces.

Let  $\vartheta \in C^1(0, \infty)$  and assume that it satisfies the following growth conditions:

$$0 < \hat{c} \leq \frac{\vartheta'(t)t}{\vartheta(t)} \leq c_0 \text{ and } c_1 t^{p-1} \leq \vartheta(t) \leq c_2 (t^{\tau-1} + t^{p-1}) \text{ for all } t > 0, \quad (1)$$

with  $0 < c_1, c_2$  and  $1 \leq \tau < p$ .

The hypotheses on the map  $a(\cdot)$  involved in the differential operator of  $(P_\lambda)$ , are as follows:

- $H(a) : a(y) = a_0(|y|)y$  for all  $y \in \mathbb{R}^N$  with  $a_0(t) > 0$  for all  $t > 0$  and
- (i)  $a_0 \in C^1(0, \infty)$ ,  $t \mapsto a_0(t)t$  is strictly increasing on  $(0, \infty)$ ,  $a_0(t)t \rightarrow 0^+$  as  $t \rightarrow 0^+$  and  $\lim_{t \rightarrow 0^+} \frac{a'_0(t)t}{a_0(t)} > -1$ ;
  - (ii)  $|\nabla a(y)| \leq c_3 \frac{\vartheta(|y|)}{|y|}$  for all  $y \in \mathbb{R}^N \setminus \{0\}$ , and for some  $c_3 > 0$ ;
  - (iii)  $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\vartheta(|y|)}{|y|} |\xi|^2$  for all  $y \in \mathbb{R}^N \setminus \{0\}$ ,  $\xi \in \mathbb{R}^N$ ;
  - (iv) if  $G_0(t) = \int_0^t a_0(s)s ds$  then there exists  $q \in (1, p]$  such that  $t \mapsto G_0(t^{1/q})$  is convex,  $\lim_{t \rightarrow 0^+} \frac{qG_0(t)}{t^q} = \tilde{c} > 0$  and  $0 \leq pG_0(t) - a_0(t)t^2$  for all  $t \geq 0$ .

**Remark 1.** Hypotheses  $H(a)(i)(ii)(iii)$  permit the use of the nonlinear regularity theory of Lieberman [13] and of the nonlinear maximum principle of Pucci & Serrin [25]. Hypothesis  $H(a)(iv)$  serves the needs of our problem. It is a mild condition which is satisfied in all cases of interest (see the examples below). These hypotheses imply that  $G_0(\cdot)$  is strictly increasing and strictly convex. We set  $G(y) = G_0(|y|)$  for all  $y \in \mathbb{R}^N$ . We have

$$\nabla G(y) = G'_0(|y|) \frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N, \nabla G(0) = 0.$$

So,  $G_0(\cdot)$  is the primitive of  $a(\cdot)$  and  $G(\cdot)$  is convex,  $G(0) = 0$ . Therefore

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2}$$

The next lemma summarizes the main properties of the map  $a(\cdot)$ . It is an easy consequence of hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$ .

**Lemma 2.2.** *If hypotheses  $H(a)(i)(ii)(iii)$  hold, then*

- (a)  $y \mapsto a(y)$  is continuous and strictly monotone (thus, maximal monotone, too);
- (b)  $|a(y)| \leq c_4 (1 + |y|^{p-1})$  for all  $y \in \mathbb{R}^N$ , and for some  $c_4 > 0$ ;
- (c)  $(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1} |y|^p$  for all  $y \in \mathbb{R}^N$ .

Using this lemma and relation (2), we obtain the following growth properties for the primitive  $G(\cdot)$

**Corollary 1.** *If hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  hold, then  $\frac{c_1}{p(p-1)} |y|^p \leq G(y) \leq c_5 (1 + |y|^p)$  for all  $y \in \mathbb{R}^N$ , and for some  $c_5 > 0$ .*

*Examples.* The following maps  $a(\cdot)$  satisfy hypotheses  $H(a)$  (see also Papageorgiou & Rădulescu [18, 19]):

- (a)  $a(y) = |y|^{p-2}y$ ,  $1 < p < \infty$ .

This map corresponds to the  $p$ -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div} [|Du|^{p-2} Du] \text{ for all } u \in W^{1,p}(\Omega).$$

- (b)  $a(y) = |y|^{p-2}y + |y|^{q-2}y$ ,  $1 < q < p < \infty$ .

This map corresponds to the  $(p, q)$ -Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \text{ for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics (see Cherfilis & Ilyasov [4]). A survey of some recent results on such equations with several relevant references, can be found in Marano & Mosconi [14].

- (c)  $a(y) = [1 + |y|^2]^{\frac{p-2}{2}}y$ ,  $1 < p < \infty$ .

This map corresponds to the generalized  $p$ -mean curvature differential operator defined by

$$\operatorname{div} [(1 + |Du|^2)^{\frac{p-2}{2}} Du] \text{ for all } u \in W^{1,p}(\Omega).$$

(d)  $a(y) : |y|^{p-2}y \left(1 + \frac{1}{1+|y|^p}\right)$ ,  $1 < p < \infty$

Let  $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$  be the nonlinear map defined by

$$\langle A(u), h \rangle = \int_{\Omega} (a(Du), Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,p}(\Omega).$$

The next proposition establishes the properties of  $A(\cdot)$  and is a special case of a more general result of Gasinski & Papageorgiou [9] (see also Gasinski & Papageorgiou [10, Problem 2.192]).

**Proposition 1.** *If hypotheses  $H(a)$  hold, then  $A(\cdot)$  is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (thus maximal monotone, too) and of type  $(S)_+$ , that is, if  $u_n \xrightarrow{w} u$  in  $W^{1,p}(\Omega)$  and  $\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$ , then  $u_n \rightarrow u$  in  $W^{1,p}(\Omega)$ .*

We will also need the following strong comparison principle due to Papageorgiou, Rădulescu & Repovš [21].

**Proposition 2.** *If hypotheses  $H(a)$  hold,  $k \in L^\infty(\Omega)$  with  $k(z) \geq 0$  for almost all  $z \in \Omega$ ,  $h_1, h_2 \in L^\infty(\Omega)$*

$$0 < \tilde{\gamma} \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega$$

and  $u, v \in C^{1,\alpha}(\bar{\Omega})$  with  $\alpha \in (0, 1], u \leq v$  and

$$-\operatorname{div} a(Du) + k(z)|u|^{p-2}u = h_1(z) \text{ for almost all } z \in \Omega,$$

$$-\operatorname{div} a(Dv) + k(z)|v|^{p-2}v = h_2(z) \text{ for almost all } z \in \Omega,$$

then  $v - u \in \operatorname{int} C_+$ .

We introduce the hypotheses on the potential function  $\xi(\cdot)$  and the boundary coefficient  $\beta(\cdot)$

$$H(\xi) : \xi \in L^\infty(\Omega);$$

$$H(\beta) : \beta \in C^{0,\alpha}(\partial\Omega) \text{ with } \alpha \in (0, 1) \text{ and } \beta(z) \geq 0 \text{ for all } z \in \partial\Omega.$$

**Remark 2.** The case  $\beta \equiv 0$  corresponds to the Neumann problem.

Let  $\mu : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  be the  $C^1$ -functional defined by

$$\mu(u) = \int_{\Omega} pG(Du)dz + \int_{\Omega} \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

Consider a Carathéodory function  $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$|f_0(z, x)| \leq a_0(z)(1 + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ and for all } x \in \mathbb{R},$$

with  $a_0 \in L^\infty(\Omega)$  and  $1 \leq r \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$  (the critical Sobolev exponent).

We set  $F_0(z, x) = \int_0^x f_0(z, s)ds$  and consider the  $C^1$ -functional  $\varphi_0 : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varphi_0(v) = \frac{1}{p}\mu(u) - \int_{\Omega} F_0(z, u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From Papageorgiou & Rădulescu [20] we have the following proposition. The result is essentially an outgrowth of the nonlinear regularity theory of Lieberman [13].

**Proposition 3.** *Assume that hypotheses  $H(a)$  hold and  $u_0 \in W^{1,p}(\Omega)$  is a local  $C^1(\bar{\Omega})$ -minimizer of  $\varphi_0(\cdot)$ , that is, there exists  $\rho_1 > 0$  such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1(\bar{\Omega}) \text{ with } \|h\|_{C^1(\bar{\Omega})} \leq \rho_1.$$

*Then  $u_0 \in C^{1,\alpha}(\bar{\Omega})$  for some  $\alpha \in (0, 1)$  and  $u_0$  is a local  $W^{1,p}(\Omega)$ -minimizer of  $\varphi_0$ , that is, there exists  $\rho_2 > 0$  such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_2.$$

We will also use some facts about the spectrum of the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -\Delta_r u(z) + \xi(z)|u(z)|^{r-2}u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_r} + \beta(z)|u|^{r-2}u = 0 \text{ on } \partial\Omega, 1 < r < \infty. \end{array} \right\} \tag{3}$$

In this case, the conormal derivative  $\frac{\partial u}{\partial n_r}$  is defined by

$$\frac{\partial u}{\partial n_r} = |Du|^{r-2} \frac{\partial u}{\partial n} \text{ for all } u \in W^{1,r}(\Omega).$$

As before,  $n(\cdot)$  denotes the outward unit normal on  $\partial\Omega$ . We say that  $\hat{\lambda} \in \mathbb{R}$  is an “eigenvalue”, if problem (3) admits a nontrivial solution  $\hat{u} \in W^{1,r}(\Omega)$ , known as an “eigenfunction” corresponding to the eigenvalue  $\hat{\lambda}$ . The nonlinear regularity theory of Lieberman [13] (see also Gasinski & Papageorgiou [8, pp. 737-738]), implies that  $\hat{u} \in C^1(\bar{\Omega})$ . From Fragnelli, Mugnai & Papageorgiou [7] (see also Mugnai & Papageorgiou [16] and Papageorgiou & Rădulescu [17], where special cases of (3) are discussed), we have the following property.

**Proposition 4.** *If hypotheses  $H(\xi), H(\beta)$  hold, then problem (3) admits a smallest eigenvalue  $\hat{\lambda}_1 = \hat{\lambda}_1(r, \xi, \beta) \in \mathbb{R}$  such that*

- (a)  $\hat{\lambda}_1$  is isolated (that is, if  $\hat{\sigma}(r)$  denotes the spectrum of (3), then we can find  $\epsilon > 0$  such that  $(\hat{\lambda}_1, \hat{\lambda}_1 + \epsilon) \cap \hat{\sigma}(r) = \emptyset$ );
- (b)  $\hat{\lambda}_1$  is simple (that is, if  $\hat{u}, \hat{v} \in C^1(\bar{\Omega})$  are eigenfunctions corresponding to  $\hat{\lambda}_1$ , then  $\hat{u} = \hat{\xi}\hat{v}$  for some  $\hat{\xi} \in \mathbb{R} \setminus \{0\}$ );
- (c) we have

$$\hat{\lambda}_1 = \inf \left\{ \frac{\mu_r(u)}{\|u\|_r^r} : u \in W^{1,r}(\Omega), u \neq 0 \right\}, \tag{4}$$

with

$$\mu_r(u) = \|Du\|_r^r + \int_{\Omega} \xi(z)|u|^r dz + \int_{\partial\Omega} \beta(z)|u|^r d\sigma.$$

In (4), the infimum is realized on the corresponding one-dimensional eigenspace. The above properties imply that the elements of this eigenspace have fixed sign. We denote by  $\hat{u}_1 = \hat{u}_1(r, \xi, \beta)$  the positive,  $L^r$ -normalized (that is,  $\|\hat{u}_1\|_r = 1$ ) eigenfunction corresponding to  $\hat{\lambda}_1 = \hat{\lambda}(r, \xi, \beta)$ . The nonlinear Hopf lemma (see Pucci & Serrin [25, pp. 111, 120] and Gasinski & Papageorgiou [8, p. 738]) implies that  $\hat{u}_1 \in D_+$ . Moreover, if  $\hat{u}$  is an eigenfunction corresponding to an eigenvalue  $\hat{\lambda} \neq \hat{\lambda}_1$ , then  $\hat{u}$  is nodal (that is, sign-changing).

For every  $x \in \mathbb{R}$ , let  $x^\pm = \max\{\pm x, 0\}$ . Then given  $u \in W^{1,p}(\Omega)$ , we set  $u^\pm(\cdot) = u(\cdot)^\pm$ . We know that

$$u^\pm \in W^{1,p}(\Omega), |u| = u^+ + u^-, u = u^+ - u^-.$$

Given a measurable function  $k : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  (for example, a Carathéodory function), we denote by  $N_k(\cdot)$  the Nemytskii map corresponding to  $k(\cdot, \cdot)$ , that is,

$$N_k(u)(\cdot) = k(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega).$$

If  $v, u \in W^{1,p}(\Omega)$  and  $v \leq u$ , then we set

$$\begin{aligned} [v, u] &= \{y \in W^{1,p}(\Omega) : v(z) \leq y(z) \leq u(z) \text{ for almost all } z \in \Omega\} \\ [u] &= \{y \in W^{1,p}(\Omega) : u(z) \leq y(z) \text{ for almost all } z \in \Omega\}. \end{aligned}$$

**3.  $(p-1)$ -sublinear perturbation.** In this section, we examine the case where the perturbation  $f(z, x)$  in problem  $(P_\lambda)$  is  $(p-1)$ -sublinear near  $+\infty$ . More precisely, the hypotheses on  $f(z, x)$  are the following:

$H(f)_1$ :  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that  $f(z, 0) = 0$  for almost all  $z \in \Omega$  and

(i) for every  $\rho > 0$ , there exists  $a_\rho \in L^\infty(\Omega)$  such that

$$|f(z, x)| \leq a_\rho(z) \text{ for almost all } z \in \Omega, \text{ and for all } 0 \leq x \leq \rho;$$

(ii)  $\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = 0$  uniformly for almost all  $z \in \Omega$ ;  
 (iii) with  $q \in (1, p]$  as in hypothesis  $H(a)(iv)$  we have

$$\lim_{x \rightarrow 0^+} \frac{f(z, x)}{x^{q-1}} = +\infty \text{ uniformly for almost all } z \in \Omega;$$

(iv) for every  $\rho > 0$ , there exists  $\hat{\xi}_\rho > 0$  such that for almost all  $z \in \Omega$ , then function

$$x \mapsto f(z, x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on  $[0, \rho]$ .

**Remark 3.** Since we are looking for positive solutions and all the above hypotheses concern the positive semi-axis  $\mathbb{R}_+ = [0, +\infty)$ , we may assume without any loss of generality that  $f(z, x) = 0$  for almost all  $z \in \Omega$ , and for all  $x \leq 0$ . Hypothesis  $H(f)_1(ii)$  implies that  $f(z, \cdot)$  is  $(p-1)$ -superlinear near  $+\infty$ . Hypothesis  $H(f)_1(iii)$  implies that  $f(z, \cdot)$  is  $(q-1)$ -superlinear near  $0^+$  (that is,  $f(z, \cdot)$  exhibits a  $q$ -concave term near  $0^+$ ). Hypothesis  $H(f)_1(iv)$  is satisfied if for example  $f(z, \cdot)$  is differentiable and for every  $\rho > 0$ , there exists  $\eta_\rho > 0$  such that  $f'_x(z, x)x \geq -\eta_\rho x^{p-1}$  for almost all  $z \in \Omega$ , and for all  $0 \leq x \leq \rho$ . We stress that no global sign condition is imposed on  $f(z, \cdot)$ .

*Example.* The following function satisfies hypotheses  $H(f)_1$ . For the sake of simplicity we drop the  $z$ -dependence:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{\tau-1} - 2x^{q-1} & \text{if } 0 \leq x \leq 1 \\ x^{r-1} - 2x^{s-1} & \text{if } 1 < x \end{cases}$$

with  $\tau < q \leq p$  and  $1 < s < r < p$ . Note that  $f(\cdot)$  changes sign.

Let

$$\mathcal{L} = \{\lambda \in \mathbb{R} : \text{problem } (P_\lambda) \text{ has a positive solution}\},$$

$$S_\lambda = \text{the set of all positive solutions of problem } (P_\lambda).$$

**Proposition 5.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold, then  $\mathcal{L} \neq \emptyset$  and  $S_\lambda \subseteq D_+$ .*

*Proof.* Let  $\eta > \|\xi\|_\infty$  and consider the following Carathéodory function

$$e_\lambda(z, x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (\lambda + \eta)x^{p-1} + f(z, x) & \text{if } 0 < x, \end{cases} \tag{5}$$

for all  $\lambda \in \mathbb{R}$ .

We set  $E_\lambda(z, x) = \int_0^x e_\lambda(z, s)ds$  and consider the  $C^1$ -functional  $\varphi_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varphi_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}\|u\|_p^p - \int_\Omega E_\lambda(z, u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Let  $F(z, x) = \int_0^x f(z, s)ds$ . Hypotheses  $H(f)_1(i), (ii)$  imply that given  $\epsilon > 0$ , we can find  $c_6 = c_6(\epsilon) > 0$  such that

$$F(z, x) \leq \frac{\epsilon}{p}x^p + c_6 \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{6}$$

Using (5), (6), Corollary 1 and hypothesis  $H(\beta)$ , we have

$$\varphi_\lambda(u) \geq \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p} \int_\Omega [\xi(z) + \eta - (\lambda + \xi)] |u|^p dz - c_7 \text{ for some } c_7 > 0.$$

Choosing  $\lambda \in \mathbb{R}$  such that  $\lambda + \epsilon < \eta - \|\xi\|_\infty$ , we can write

$$\begin{aligned} \varphi_\lambda(u) &\geq \frac{c_1}{p(p-1)}\|Du\|_p^p + c_8\|u\|_p^p - c_7 \text{ for some } c_8 > 0, \\ \Rightarrow \varphi_\lambda(\cdot) &\text{ is coercive.} \end{aligned}$$

By the Sobolev embedding theorem and the compactness of the trace map we deduce that  $\varphi(\cdot)$  is sequentially weak lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find  $u_\lambda \in W^{1,p}(\Omega)$  such that

$$\varphi_\lambda(u_\lambda) = \inf \{ \varphi_\lambda(u) : u \in W^{1,p}(\Omega) \}. \tag{7}$$

Hypothesis  $H(a)(iv)$  implies that given  $\tilde{c}_0 > \tilde{c}$ , we can find  $\delta \in (0, 1)$  such that

$$G(y) \leq \frac{\tilde{c}_0}{q}|y|^q \text{ for all } |y| \leq \delta. \tag{8}$$

Hypothesis  $H(f)(iii)$  implies that given any  $\vartheta > 0$ , by choosing  $\delta > 0$  even smaller if necessary, we can also have

$$F(z, x) \geq \frac{\vartheta}{q}x^q \text{ for almost all } z \in \Omega, \text{ and for all } 0 \leq x \leq \delta. \tag{9}$$

Let  $\hat{\lambda}_1 = \hat{\lambda}_1(q, \xi_0, \beta_0)$  and  $\hat{u}_1 = \hat{u}_1(q, \xi_0, \beta_0) \in D_+$  with  $\xi_0 = \frac{1}{c_0}\xi, \beta_0 = \frac{1}{c_0}\beta$ . We choose small  $t \in (0, 1)$  such that

$$0 \leq t\hat{u}_1(z) \leq \delta \text{ and } |D(t\hat{u}_1)(z)| \leq \delta \text{ for all } z \in \bar{\Omega}. \tag{10}$$

Using (5), (8), (9), (10), we have

$$\begin{aligned} \varphi_\lambda(t\hat{u}_1) &\leq \frac{\tilde{c}_0 t^q}{q} \|D\hat{u}_1\|_q^q + \frac{1}{p} \int_\Omega \xi(z) |t\hat{u}_1|^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z) |t\hat{u}_1|^p d\sigma \\ &\quad - \frac{\lambda t^p}{p} \|\hat{u}_1\|_p^p - \frac{\vartheta t^q}{q} \text{ (recall that } \|\hat{u}_1\|_q = 1) \\ &\leq \frac{\tilde{c}_0 t^q}{q} \left( \|D\hat{u}_1\|_q^q + \int_\Omega \xi_0 |\hat{u}_1|^q dz + \int_{\partial\Omega} \beta_0(z) |\hat{u}_1|^q d\sigma \right) - \frac{\vartheta}{q} t^q \\ &\quad \text{(since } 0 < \delta < 1 \text{ and } q \leq p) \\ &\leq \frac{t^q}{q} (\tilde{c}_0 \hat{\lambda}_1 - \vartheta). \end{aligned}$$



But  $\vartheta > 0$  is arbitrary. So, choosing  $\vartheta > \tilde{c}_0 \hat{\lambda}_1$ , we see that

$$\begin{aligned} & \varphi_\lambda(t\hat{u}_1) < 0, \\ \Rightarrow & \varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \text{ (see (7))}, \\ \Rightarrow & u_\lambda \neq 0. \end{aligned}$$

From (7) we have for all  $h \in W^{1,p}(\Omega)$

$$\begin{aligned} & \varphi'_\lambda(u_\lambda) = 0 \\ \Rightarrow & \langle A(u_\lambda), h \rangle + \int_\Omega (\xi + \eta)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta_\lambda|u_\lambda|^{p-2}u_\lambda h d\sigma \\ & = \int_\Omega e_\lambda(z, u_\lambda) h dz. \end{aligned} \tag{11}$$

In (11) we choose  $h = -u_\lambda^- \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} & \frac{c_1}{p-1} \|Du_\lambda^-\|_p^p + \frac{1}{p} \int_\Omega [\xi(z) + \eta](u_\lambda^-)^p dz \leq 0 \\ & \text{(see Lemma 2.2, hypothesis } H(\beta) \text{ and (5))} \\ \Rightarrow & u_\lambda \geq 0, u_\lambda \neq 0 \text{ (recall that } \eta > \|\xi\|_\infty \text{)}. \end{aligned}$$

It follows from (5) and (11) that

$$\begin{aligned} & -\operatorname{div} a(Du_\lambda(z)) + \xi(z)u_\lambda(z)^{p-1} = \lambda u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) \text{ for almost all } z \in \Omega \\ & \frac{\partial u}{\partial n_a} + \beta(z)u_\lambda^{p-1} = 0 \text{ on } \partial\Omega \text{ (see Papageorgiou \& R\u0105dulescu [17])}. \end{aligned} \tag{12}$$

From (12) and Papageorgiou & R\u0105dulescu [20], we have  $u_\lambda \in L^\infty(\Omega)$ . Then the nonlinear regularity theory of Lieberman [13] implies that  $u_\lambda \in C_+ \setminus \{0\}$ .

Let  $\rho = \|u_\lambda\|_\infty$  and let  $\hat{\xi}_\rho > 0$  be as postulated by hypothesis  $H(f)(iv)$ . Then from (35) we have

$$\begin{aligned} & \operatorname{div} a(Du_\lambda(z)) \leq (\|\xi\|_\infty + \hat{\xi}_\rho) u_\lambda(z)^{p-1} \text{ for almost all } z \in \Omega, \\ \Rightarrow & u_\lambda \in D_+ \text{ (see Pucci \& Serrin [25, pp. 111, 120])}. \end{aligned}$$

Therefore we conclude that  $\lambda \in \mathcal{L}$  and so  $\mathcal{L} \neq \emptyset$  and also  $S_\lambda \subseteq D_+$ . □

Next, we show that  $\mathcal{L}$  is a half-line.

**Proposition 6.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold,  $\lambda \in \mathcal{L}$  and  $\vartheta < \lambda$ , then  $\vartheta \in \mathcal{L}$ .*

*Proof.* By hypothesis,  $\lambda \in \mathcal{L}$ . So, we can find  $u_\lambda \in S_\lambda \subseteq D_+$ . With  $\eta > \|\xi\|_\infty$  as before, we introduce the following truncation-perturbation of the reaction in problem  $(P_\lambda)$ :

$$k_\vartheta(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ (\vartheta + \eta) x^{p-1} + f(z, x) & \text{if } 0 \leq x \leq u_\lambda(z) \\ (\vartheta + \eta) u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{13}$$

This is a Carath\u00e9odory function. We set  $K_\vartheta(z, x) = \int_0^x k_\vartheta(z, s) ds$  and consider the  $C^1$ -functional  $\hat{\varphi}_\vartheta : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\hat{\varphi}_\vartheta(u) = \frac{1}{p} \mu(u) + \frac{\eta}{p} \|u\|_p^p - \int_\Omega K_\vartheta(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Clearly,  $\hat{\varphi}_\vartheta(\cdot)$  is coercive (see (13)) and sequentially weakly lower semicontinuous. So, we can find  $u_\vartheta \in W^{1,p}(\Omega)$  such that

$$\hat{\varphi}_\vartheta(u_\vartheta) = \inf \{ \hat{\varphi}_\vartheta(u) : u \in W^{1,p}(\Omega) \}. \tag{14}$$

As in the proof of Proposition 5, using hypotheses  $H(a)(iv)$  and  $H(f)(iii)$ , we show that  $\hat{\varphi}_\vartheta(u_\vartheta) < 0 = \hat{\varphi}_\vartheta(0)$ , hence  $u_\vartheta \neq 0$ . From (14) we have

$$\begin{aligned} & \hat{\varphi}'_\vartheta(u_\vartheta) = 0, \\ \Rightarrow \quad & \langle A(u_\vartheta), h \rangle + \int_\Omega (\xi(z) + \eta) |u_\vartheta|^{p-2} u_\vartheta h dz + \int_{\partial\Omega} \beta(z) |u_\vartheta|^{p-2} u_\vartheta h d\sigma \tag{15} \\ & = \int_\Omega k_\vartheta(z, u_\vartheta) h dz \text{ for all } h \in W^{1,p}(\Omega) \end{aligned}$$

In (15) we first choose  $h = -u_\vartheta^- \in W^{1,p}(\Omega)$ . Then using Lemma 2 and (13) we obtain

$$\begin{aligned} & \frac{c_1}{p-1} \|Du_\vartheta^-\|_p^p + \int_\Omega [\xi(z) + \eta] (u_\vartheta^-)^p dz \leq 0 \text{ (see hypothesis } H(\beta)), \\ \Rightarrow \quad & u_\vartheta \geq 0, \quad u_\vartheta \neq 0. \end{aligned}$$

Next, in (15) we choose  $h = (u_\vartheta - u_\lambda)^+ \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} & \langle A(u_\vartheta), (u_\vartheta - u_\lambda)^+ \rangle + \int_\Omega (\xi(z) + \eta) u_\vartheta^{p-1} (u_\vartheta - u_\lambda)^+ dz \\ & + \int_{\partial\Omega} \beta(z) u_\vartheta^{p-1} (u_\vartheta - u_\lambda)^+ d\sigma \\ = & \int_\Omega ((\vartheta + \eta) u_\lambda^{p-1} + f(z, u_\lambda)) (u_\vartheta - u_\lambda)^+ dz \text{ (see (13))} \\ \leq & \int_\Omega ((\lambda + \eta) u_\lambda^{p-1} + f(z, u_\lambda)) (u_\vartheta - u_\lambda)^+ dz \text{ (recall that } \vartheta < \lambda) \\ = & \langle A(u_\lambda), (u_\vartheta - u_\lambda)^+ \rangle + \int_\Omega (\xi(z) + \eta) u_\lambda^{p-1} (u_\vartheta - u_\lambda)^+ dz \\ & + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} (u_\vartheta - u_\lambda)^+ d\sigma \\ & \text{(recall that } u_\lambda \in S_\lambda) \\ \Rightarrow \quad & u_\vartheta \leq u_\lambda. \end{aligned}$$

We have proved that

$$u_\vartheta \in [0, u_\lambda], \quad u_\vartheta \neq 0. \tag{16}$$

It follows from (13), (15), (16) that  $\vartheta \in \mathcal{L}$  and  $u_\vartheta \in S_\vartheta \subseteq D_+$ . □

Let  $\lambda^* = \sup \mathcal{L}$ .

**Proposition 7.** *If hypotheses  $h(a), H(\xi), H(\beta), H(f)_1$  hold, then  $\lambda^* < +\infty$ .*

*Proof.* Hypotheses  $H(\xi), H(f)_1$  imply that for large enough  $\tilde{\lambda} > 0$  we have

$$(\tilde{\lambda} - \xi(z))x^{p-1} + f(z, x) \geq x^{p-1} \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{17}$$

Let  $\lambda > \tilde{\lambda}$  and suppose that  $\lambda \in \mathcal{L}$ . Then by Proposition 5 we can find  $u \in S_\lambda \subseteq D_+$ . We set

$$m = \min_\Omega u > 0 \quad \text{(since } u \in D_+). \tag{18}$$

For  $\delta > 0$  we set  $m_\delta = m + \delta > 0$ . Also, let  $\rho = \|u\|_\infty$  and let  $\hat{\xi}_\rho > 0$  be as postulated by hypothesis  $H(f)_1(iv)$ . We can always take  $\hat{\xi}_\rho > \max\{\lambda, \|\xi\|_\infty\}$ .

We have that for almost all  $z \in \Omega$  the function  $x \mapsto (\lambda + \hat{\xi}_\rho)x^{p-1} + f(z, x)$  is nondecreasing on  $[0, \rho]$ . We have

$$\begin{aligned} & -\operatorname{div} a(Dm_\delta) + (\xi(z) + \hat{\xi}_\rho)m_\delta^{p-1} \\ \leq & (\xi(z) + \hat{\xi}_\rho)m^{p-1} + \gamma(\delta) \text{ with } \gamma(\delta) \rightarrow 0^+ \text{ as } \delta \rightarrow 0^+ \\ \leq & (\tilde{\lambda} + \hat{\xi}_\rho)m^{p-1} + f(z, m) + \gamma(\delta) \text{ (see (17))} \\ = & (\lambda + \hat{\xi}_\rho)m^{p-1} + f(z, m) - (\lambda - \tilde{\lambda})m^{p-1} + \gamma(\delta) \\ \leq & (\lambda + \hat{\xi}_\rho)m^{p-1} + f(z, m) \text{ for small enough } \delta > 0 \text{ (so that } \gamma(\delta) < (\lambda - \tilde{\lambda})m^{p-1}\text{)} \\ \leq & (\lambda + \hat{\xi}_\rho)u^{p-1} + f(z, u) \text{ (see (18))} \\ = & -\operatorname{div} a(Du) + (\xi(z) + \hat{\xi}_\rho)u^{p-1}. \end{aligned}$$

Let

$$\begin{aligned} h_1(z) &= (\xi(z) + \hat{\xi}_\rho)m^{p-1} + \gamma(\delta) \\ h_2(z) &= (\lambda + \hat{\xi}_\rho)u^{p-1} + f(z, u). \end{aligned}$$

Evidently,  $h_1, h_2 \in L^\infty(\Omega)$  and for  $\delta > 0$  small we have

$$0 < \tilde{\gamma} \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega.$$

So, by Proposition 2 for small enough  $\delta > 0$  we have

$$u - m_\delta \in \operatorname{int} C_+,$$

a contradiction to (18). Therefore  $\lambda \notin \mathcal{L}$  and so  $\lambda^* \leq \tilde{\lambda} < +\infty$ . □

Fix  $\lambda < \lambda^*$ . Then by Proposition 6 we have  $\lambda \in \mathcal{L}$ . We will show that  $S_\lambda \subseteq D_+$  admits a smallest element. Let  $r \in (p, p^*)$ . On account of hypotheses  $H(f)_1$  we can find  $c_9 > 0$  and  $c_{10} = c_{10}(\lambda) > 0$  both large enough such that

$$\lambda x^{p-1} + f(z, x) \geq c_9 x^{q-1} - c_{10} x^{r-1} \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{19}$$

Motivated by this one-sided growth condition on the reaction of problem  $(P_\lambda)$ , we consider the following auxiliary nonlinear nonhomogeneous Robin problem

$$\left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) + |\xi(z)|u^{p-1} = c_9 u(z)^{q-1} - c_{10} u(z)^{r-1} \text{ in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \ u > 0. \end{array} \right\} \tag{20}$$

**Proposition 8.** *If hypotheses  $H(a), H(\xi), H(\beta)$  hold and  $c_9, c_{10} > 0$  are both large enough, then problem (20) admits a unique solution  $u_*^\lambda \in D_+$*

*Proof.* Let  $\Psi : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  be the  $C^1$ -functional defined by

$$\begin{aligned} \Psi(u) &= \int_\Omega G(Du)dz + \frac{1}{p} \int_\Omega |\xi(z)||u|^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p d\sigma + \frac{1}{p} \|u^-\|_p^p \\ &\quad + \frac{c_{10}}{r} \|u^+\|_r^r - \frac{c_9}{q} \|u^+\|_q^q \text{ for all } u \in W^{1,p}(\Omega). \end{aligned}$$

By Corollary 1 and the fact that  $q \leq p < r$ , by taking  $c_{10} > 0$  large enough (see (19)), we see that  $\Psi_\lambda(\cdot)$  is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find  $u_*^\lambda \in W^{1,p}(\Omega)$  such that

$$\Psi(u_*^\lambda) = \inf \{ \Psi(u) : u \in W^{1,p}(\Omega) \}. \tag{21}$$

On account of hypothesis  $H(f)(iii)$ , we can choose  $c_9 > 0$  large enough so that

$$\begin{aligned} \Psi(u_*^\lambda) &< 0 = \Psi(0) \text{ (recall that } q \leq p < r\text{)} \\ \Rightarrow \quad u_*^\lambda &\neq 0. \end{aligned}$$

From (21) we have

$$\begin{aligned} & \Psi'(u_*^\lambda) = 0, \\ \Rightarrow & \langle A(u_*^\lambda), h \rangle + \int_{\Omega} |\xi(z)| |u_*^\lambda|^{p-2} u_*^\lambda h dz + \int_{\partial\Omega} \beta(z) |u_*^\lambda|^{p-2} u_*^\lambda h d\sigma \\ & - \int_{\Omega} ((u_*^\lambda)^-)^{p-1} h dz \\ = & c_9 \int_{\Omega} ((u_*^\lambda)^+)^{q-1} h dz - c_{10} \int_{\Omega} ((u_*^\lambda)^+)^{r-1} h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{22}$$

In (22) we choose  $h = -(u_*^\lambda)^- \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} & \frac{c_1}{p-1} \|D(u_*^\lambda)^-\|_p^p + \int_{\Omega} (|\xi(z)| + 1) ((u_*^\lambda)^-)^p dz \leq 0 \\ & \text{(see Lemma 2.2 and hypothesis } H(\beta)) \\ \Rightarrow & u_*^\lambda \geq 0, \quad u_*^\lambda \neq 0. \end{aligned}$$

It follows from (22) that  $u_*^\lambda$  is a positive solution of (20). The nonlinear regularity theory implies that  $u_*^\lambda \in C_+ \setminus \{0\}$ . Moreover, we have

$$\operatorname{div} a(Du_*^\lambda(z)) \leq (\|\xi\|_\infty + c_{10} \|u_*^\lambda\|_\infty^{r-p}) u_*^\lambda(z) \text{ for almost all } z \in \Omega.$$

Next, we show that this positive solution of (20) is unique. For this purpose we introduce the functional  $l : L^1(\Omega) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  defined by

$$l(u) = \begin{cases} \int_{\Omega} G(Du^{\frac{1}{q}}) dz + \frac{1}{p} \int_{\Omega} |\xi(z)| u^{\frac{p}{q}} dz + \frac{1}{p} \int_{\partial\Omega} \beta(z) u^{\frac{p}{q}} d\sigma & \text{if } u \geq 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Here,  $q \leq p$  is as in hypothesis  $H(a)(iv)$ . Let  $\operatorname{dom} l = \{u \in L^1(\Omega) : l(u) < +\infty\}$  (the effective domain of  $l(\cdot)$ ). Let  $u_1, u_2 \in \operatorname{dom} l$  and consider  $u = [(1-t)u_1 + tu_2]^{\frac{1}{q}}$  with  $t \in [0, 1]$ . From Lemma 1 of Diaz & Saa [6], we have

$$\begin{aligned} & |Du(z)| \leq [(1-t)|Du_1(z)^{\frac{1}{q}}|^q + t|Du_2(z)^{\frac{1}{q}}|^q]^{\frac{1}{q}} \text{ for almost all } z \in \Omega \\ \Rightarrow & G_0(|Du(z)|) \leq G_0([(1-t)|Du_1(z)^{\frac{1}{q}}|^q + t|Du_2(z)^{\frac{1}{q}}|^q]^{\frac{1}{q}}) \\ & \text{(since } G_0(\cdot) \text{ is increasing)} \\ & \leq (1-t)G_0(|Du_1(z)^{\frac{1}{q}}|) + tG_0(|Du_2(z)^{\frac{1}{q}}|) \text{ for almost all } z \in \Omega \\ & \text{(see hypotheses } H(a)(iv)) \\ \Rightarrow & G(Du(z)) \leq (1-t)G(Du_1(z)^{\frac{1}{q}}) + tG(Du_2(z)^{\frac{1}{q}}) \text{ for almost all } z \in \Omega, \\ \Rightarrow & \operatorname{dom} l \ni u \mapsto \int_{\Omega} G(Du^{\frac{1}{q}}) dz \text{ is convex.} \end{aligned}$$

Since  $q \leq p$  and  $\beta \geq 0$  (see hypotheses  $H(\beta)$ ), we deduce that the mapping

$$\operatorname{dom} l \ni u \mapsto \int_{\Omega} |\xi(z)| u^{\frac{p}{q}} dz + \int_{\partial\Omega} \beta(z) u^{\frac{p}{q}} d\sigma \text{ is convex.}$$

Therefore, we conclude that  $l(\cdot)$  is convex and by Fatou's lemma it is also lower semicontinuous.

Assume that  $v_*^\lambda$  is another positive solution for problem (20). Again, we can show that  $v_*^\lambda \in D_+$ . Let  $h \in C^1(\overline{\Omega})$ . For  $|t| < 1$  small enough we have

$$(u_*^\lambda)^q + th \in \operatorname{dom} l \text{ and } (v_*^\lambda)^q + th \in \operatorname{dom} l.$$

It is easily seen that  $l(\cdot)$  is Gâteaux differentiable at  $(u_*^\lambda)^q$  and at  $(v_*^\lambda)^q$  in the direction  $h$ . Using the chain rule and the nonlinear Green identity (see Gasinski &

Papageorgiou [8, p. 210]), we obtain

$$\begin{aligned}
 l'((u_*^\lambda)^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\operatorname{div} a(Du_*^\lambda) + |\xi(z)|(u_*^\lambda)^{p-1}}{(u_*^\lambda)^{q-1}} h dz \\
 l'((v_*^\lambda)^q)(h) &= \frac{1}{q} \int_{\Omega} \frac{-\operatorname{div} a(Dv_*^\lambda) + |\xi(z)|(v_*^\lambda)^{p-1}}{(v_*^\lambda)^{q-1}} h dz.
 \end{aligned}$$

The convexity of  $l(\cdot)$  implies the monotonicity of  $l'(\cdot)$ . Therefore

$$\begin{aligned}
 0 &\leq \int_{\Omega} \left[ \frac{-\operatorname{div} a(Du_*^\lambda) + |\xi(z)|(u_*^\lambda)^{p-1}}{(u_*^\lambda)^{q-1}} - \frac{-\operatorname{div} a(Dv_*^\lambda) + |\xi(z)|(v_*^\lambda)^{p-1}}{(v_*^\lambda)^{q-1}} \right] \\
 &\quad \left( (u_*^\lambda)^q - (v_*^\lambda)^q \right) dz \\
 &= \int_{\Omega} c_{10} [(u_*^\lambda)^{r-q} - (v_*^\lambda)^{r-q}] (u_*^\lambda)^q - (v_*^\lambda)^q dz \text{ (see (20))} \\
 \Rightarrow u_*^\lambda &= v_*^\lambda \text{ (recall that } q \leq p < r \text{)}.
 \end{aligned}$$

So, the positive solution  $u_*^\lambda \in D_+$  of problem (20) is unique. □

**Proposition 9.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold and  $\lambda < \lambda^*$ , then  $u_*^\lambda \leq u$  for all  $u \in S_\lambda$ .*

*Proof.* From Proposition 6 we know that  $\lambda \in \mathcal{L}$ . Let  $u \in S_\lambda \subseteq D_+$  (see Proposition 5). Again we fix  $\eta > \|\xi\|_\infty$  and consider the Carathéodory function  $\vartheta : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\vartheta(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ c_9 x^{q-1} - c_{10} x^{r-1} + \eta x^{p-1} & \text{if } 0 \leq x \leq u(z) \\ k_9 u(z)^{q-1} - c_{10} u(z)^{r-1} + \eta u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{23}$$

We set  $\Theta(z, x) = \int_0^x \vartheta(z, s) ds$  and consider the  $C^1$ -functional  $\zeta : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\zeta(u) = \frac{1}{p} \mu(u) + \frac{\eta}{p} \|u\|_p^p - \int_{\Omega} \Theta(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

As before,  $\zeta(\cdot)$  is coercive and sequentially weakly lower semicontinuous. So, we can find  $\tilde{u}_*^\lambda \in W^{1,p}(\Omega)$  such that

$$\zeta(\tilde{u}_*^\lambda) = \inf \{ \zeta(u) : u \in W^{1,p}(\Omega) \}. \tag{24}$$

Since  $q \leq p < r$ , for  $c_9, c_{10} > 0$  large enough as in Proposition 7, we have

$$\begin{aligned}
 \zeta(\tilde{u}_*^\lambda) &< 0 = \zeta(0), \\
 \Rightarrow \tilde{u}_*^\lambda &\neq 0.
 \end{aligned}$$

From (24) we have

$$\begin{aligned}
 \zeta'(\tilde{u}_*^\lambda) &= 0, \\
 \Rightarrow \langle A(\tilde{u}_*^\lambda), h \rangle + \int_{\Omega} [\xi(z) + \eta] |\tilde{u}_*^\lambda|^{p-2} \tilde{u}_*^\lambda h dz + \int_{\partial\Omega} \beta(z) |\tilde{u}_*^\lambda|^{p-2} \tilde{u}_*^\lambda h d\sigma &\tag{25} \\
 &= \int_{\Omega} \vartheta(z, \tilde{u}_*^\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega).
 \end{aligned}$$

Let  $h = -(\tilde{u}_*^\lambda)^- \in W^{1,p}(\Omega)$  in (25). Then

$$\begin{aligned}
 \frac{c_1}{p-1} \|D(\tilde{u}_*^\lambda)^-\|_p^p + \int_{\Omega} [\xi(z) + \eta] ((\tilde{u}_*^\lambda)^-)^p dz &\leq 0 \\
 \text{(see Lemma 2.2, hypothesis } H(\beta), \text{ and (23))} & \\
 \Rightarrow \tilde{u}_*^\lambda \geq 0, \tilde{u}_*^\lambda \neq 0. &
 \end{aligned}$$

Also, let  $h = (\tilde{u}_*^\lambda - u)^+ \in W^{1,p}(\Omega)$  in (25). Then

$$\begin{aligned} & \langle A(\tilde{u}_*^\lambda), (\tilde{u}_*^\lambda - u)^+ \rangle + \int_{\Omega} [\xi(z) + \eta](\tilde{u}_*^\lambda)^{p-1}(\tilde{u}_*^\lambda - u)^+ dz \\ & + \int_{\partial\Omega} \beta(z)(\tilde{u}_*^\lambda)^{p-1}(\tilde{u}_*^\lambda - u)^+ d\sigma \\ & = \int_{\Omega} (c_9 u^{q-1} - c_{10} u^{r-1} + \eta u^{p-1})(\tilde{u}_*^\lambda - u)^+ dz \text{ (see (23))} \\ & \leq \int_{\Omega} [(\lambda + \eta)u^{p-1} + f(z, u)](\tilde{u}_*^\lambda - u)^+ dz \text{ (see (19))} \\ & = \langle A(u), (\tilde{u}_*^\lambda - u)^+ \rangle + \int_{\Omega} [\xi(z) + \eta]u^{p-1}(\tilde{u}_*^\lambda - u) dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\tilde{u}_*^\lambda - u)^+ d\sigma \\ & \text{(since } u \in S_\lambda) \\ & \Rightarrow \tilde{u}_*^\lambda \leq u. \end{aligned}$$

So, we have proved that

$$\tilde{u}_*^\lambda \in [0, u], \tilde{u}_*^\lambda \neq 0. \tag{26}$$

Then from (23), (25), (26) we infer that

$$\begin{aligned} & \tilde{u}_*^\lambda \text{ is a positive solution of (20),} \\ & \Rightarrow \tilde{u}_*^\lambda = u_*^\lambda \in D_+ \text{ (see Proposition 8),} \\ & \Rightarrow u_*^\lambda \leq u \text{ for all } u \in S_\lambda \text{ (see (26)).} \end{aligned}$$

The proof is complete. □

From Papageorgiou, Rădulescu & Repovš [21] (proof of Proposition 4), we know that the set  $S_\lambda$  is downward directed (that is, if  $u_1, u_2 \in S_\lambda$ , then we can find  $u \in S_\lambda$  such that  $u \leq u_1, u \leq u_2$ ).

**Proposition 10.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold and  $\lambda < \lambda^*$ , then  $S_\lambda$  admits a smallest element  $\bar{u}_\lambda \in D_+$ , that is,*

$$\bar{u}_\lambda \leq u \text{ for all } u \in S_\lambda.$$

*Proof.* According to Lemma 3.10 of Hu & Papageorgiou [11, p. 178], we can find  $\{u_n\}_{n \geq 1} \subseteq S_\lambda$  such that

$$\inf S_\lambda = \inf_{n \geq 1} u_n.$$

Moreover, since  $S_\lambda$  is downward directed, we can choose  $\{u_n\}_{n \geq 1} \subseteq S_\lambda$  to be decreasing. We have

$$\begin{aligned} & \langle A(u_n), h \rangle + \int_{\Omega} \xi(z)u_n^{p-1}h dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h d\sigma = \lambda \int_{\Omega} u_n^{p-1}h dz + \\ & \int_{\Omega} f(z, u_n)h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N} \end{aligned} \tag{27}$$

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{28}$$

From (27), (28), we infer that  $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$  is bounded. So, we may assume that

$$u_n \xrightarrow{w} \bar{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow \bar{u}_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{29}$$

In (27) we choose  $h = u_n - \bar{u}_\lambda \in W^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$  and use (29). Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \langle A(u_n), u_n - \bar{u}_\lambda \rangle = 0, \\ & \Rightarrow u_n \rightarrow \bar{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned} \tag{30}$$

In (27) we pass to the limit as  $n \rightarrow \infty$  and use (30). Then

$$\begin{aligned} \langle A(\bar{u}_\lambda), h \rangle + \int_\Omega \xi(z) \bar{u}_\lambda^{p-1} h dz + \int_{\partial\Omega} \beta(z) \bar{u}_\lambda^{p-1} h d\sigma &= \lambda \int_\Omega \bar{u}_\lambda^{p-1} h dz \\ + \int_\Omega f(z, \bar{u}_\lambda) h dz &\text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{31}$$

Moreover, from Proposition 8 we have

$$\begin{aligned} u_*^\lambda &\leq u_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow u_*^\lambda &\leq \bar{u}_\lambda, \text{ hence } \bar{u}_\lambda \neq 0, \\ \Rightarrow \bar{u}_\lambda &\in S_\lambda \text{ (see (31)) and } \bar{u}_\lambda = \inf S_\lambda. \end{aligned}$$

The proof is now complete. □

In the next proposition we establish the monotonicity and continuity properties of the map  $\mathcal{L} \ni \lambda \mapsto u_\lambda \in C^1(\bar{\Omega})$ .

**Proposition 11.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold, then the map  $\lambda \mapsto \bar{u}_\lambda$  from  $\mathcal{L}$  into  $C^1(\bar{\Omega})$  is:*

(a) *strictly increasing in the sense that*

$$\vartheta < \lambda \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+;$$

(b) *left continuous, that is, if  $\lambda_n \rightarrow \lambda^-$  with  $\lambda \in \mathcal{L}$ , then  $\bar{u}_{\lambda_n} \rightarrow \bar{u}_\lambda$  in  $C^1(\bar{\Omega})$ .*

*Proof.* (a) Let  $\vartheta < \lambda \in \mathcal{L}$ . Let  $\bar{u}_{\lambda_n} \in S_\lambda \subseteq D_+$  be the minimal solution of  $(P_\lambda)$  (see Proposition 9). From Proposition 6 and its proof we know that  $\vartheta \in \mathcal{L}$  and we can find  $u_\vartheta \in S_\vartheta \subseteq D_+$  such that  $u_\vartheta \leq \bar{u}_\lambda$  (see (16)). Therefore  $\bar{u}_\vartheta \leq \bar{u}_\lambda$ .

Let  $\rho = \|\bar{u}_\lambda\|_\infty$  and let  $\hat{\xi}_\rho > 0$  be as postulated by hypothesis  $H(f)_1(iv)$ . We can always take  $\hat{\xi}_\rho > \|\xi\|_\infty$ . Then

$$\begin{aligned} &-\text{div } a(D\bar{u}_\vartheta) + [\xi(z) + \hat{\xi}_\rho] \bar{u}_\vartheta^{p-1} \\ &= \vartheta \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1} \\ &= \lambda \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1} - (\lambda - \vartheta) \bar{u}_\vartheta^{p-1} \\ &\leq \lambda \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1} - (\lambda - \vartheta) m_\vartheta^{p-1} \text{ with } m_\vartheta = \min_{\bar{\Omega}} \bar{u}_\vartheta > 0 \\ &\text{(recall that } \bar{u}_\vartheta \in D_+) \\ &< \lambda \bar{u}_\lambda^{p-1} + f(z, \bar{u}_\lambda) + \hat{\xi}_\rho \bar{u}_\lambda^{p-1} \text{ (since } \bar{u}_\vartheta \leq \bar{u}_\lambda) \\ &= -\text{div } a(D\bar{u}_\lambda) + [\xi(z) + \hat{\xi}_\rho] \bar{u}_\lambda^{p-1} \text{ for almost all } z \in \Omega. \end{aligned}$$

Let

$$h_1(z) = \vartheta \bar{u}_\lambda^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1}$$

and

$$h_2(z) = \lambda \bar{u}_\lambda^{p-1} + f(z, \bar{\lambda}) + \hat{\xi}_\rho \bar{u}_\lambda^{p-1}.$$

Evidently,  $h_1, h_2 \in L^\infty(\Omega)$  (see hypothesis  $H(f)_1(i)$ ) and

$$0 < (\lambda - \vartheta) m_\vartheta^{p-1} \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega.$$

So, we can apply Proposition 2 and conclude that  $\bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$ . This proves that the mapping  $\lambda \mapsto \bar{u}_\lambda$  is strictly increasing.

(b) Let  $\lambda_n \rightarrow \lambda^-$  with  $\lambda \in \mathcal{L}$ . We set  $\bar{u}_n = \bar{u}_{\lambda_n} \in S_{\lambda_n} \subseteq D_+$  for all  $n \in \mathbb{N}$ . Evidently,  $\{\bar{u}_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$  is bounded. So, we may assume that

$$\bar{u}_n \rightharpoonup u_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } \bar{u}_n \rightarrow u_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{32}$$

We have

$$\langle A(\bar{u}_n), h \rangle + \int_{\Omega} \xi(z) \bar{u}_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) \bar{u}_n^{p-1} h d\sigma = \int_{\Omega} [\lambda_n \bar{u}_n^{p-1} + f(z, \bar{u}_n)] h dz \tag{33}$$

for all  $h \in W^{1,p}(\Omega)$ ,  $n \in \mathbb{N}$ .

In (33) we choose  $h = \bar{u}_n - u_\lambda \in W^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$  and use (32). Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \langle A(\bar{u}_n), \bar{u}_n - u_\lambda \rangle = 0, \\ \Rightarrow & \bar{u}_n \rightarrow u_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned} \tag{34}$$

So, if in (32) we pass to the limit as  $n \rightarrow \infty$  and use (34), then we can infer that  $u_\lambda \in S_\lambda \subseteq D_+$ . On account of (34) and Proposition 7 of Papageorgiou & Rădulescu [20], we can find  $c_{11} > 0$  such that

$$\|\bar{u}_n\|_\infty \leq c_{11} \text{ for all } n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [13] implies that there exist  $\tau \in (0, 1)$  and  $c_{12} > 0$  such that

$$\bar{u}_n \in C^{1,\tau}(\bar{\Omega}) \text{ and } \|\bar{u}_n\|_{C^{1,\tau}(\bar{\Omega})} \leq c_{12} \text{ for all } n \in \mathbb{N}.$$

The existence of a compact embedding of  $C^{1,\tau}(\bar{\Omega})$  into  $C^1(\bar{\Omega})$  and (34), imply that

$$\bar{u}_n \rightarrow u_\lambda \text{ in } C^1(\bar{\Omega}) \text{ as } n \rightarrow \infty. \tag{35}$$

We show that  $u_\lambda = \bar{u}_\lambda$ . Arguing by contradiction, suppose that  $u_\lambda \neq \bar{u}_\lambda$ . Then we can find  $z_0 \in \bar{\Omega}$  such that

$$\begin{aligned} & \bar{u}_\lambda(z_0) < u_\lambda(z_0), \\ \Rightarrow & \bar{u}_\lambda(z_0) < \bar{u}_n(z_0) \text{ for all } n \geq n_0 \text{ (see (35))}, \end{aligned}$$

which contradicts (a). Therefore the mapping  $\lambda \mapsto \bar{u}_\lambda$  is left continuous. □

Now we ready to show the non-admissibility of  $\lambda^*$ .

**Proposition 12.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold, then  $\lambda^* \notin \mathcal{L}$ .*

*Proof.* Arguing by contradiction, suppose that  $\lambda^* \in \mathcal{L}$ . According to Proposition 10, problem  $(P_{\lambda^*})$  admits a smallest positive solution  $\bar{u}_* = \bar{u}_{\lambda^*} \in D_+$ . Let  $\lambda > \lambda^*, \eta > \|\xi\|_\infty$  and consider the Carathéodory function

$$\gamma_\lambda(z, x) = \begin{cases} (\lambda + \eta) \bar{u}_*(z)^{p-1} + f(z, \bar{u}_*(z)) & \text{if } x \leq \bar{u}_*(z) \\ (\lambda + \eta) x^{p-1} + f(z, x) & \text{if } \bar{u}_*(z) < x. \end{cases} \tag{36}$$

We set  $\Gamma_\lambda(z, x) = \int_0^x \gamma_\lambda(z, s) ds$  and consider the  $C^1$ -functional  $\tilde{\varphi}_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\tilde{\varphi}_\lambda(u) = \frac{1}{p} \mu(u) + \frac{\eta}{p} \|u\|_p^p - \int_{\Omega} \Gamma_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

As in the proof of Proposition 5, using hypothesis  $H(f)_1(ii)$ , we show that  $\tilde{\varphi}_\lambda(\cdot)$  is coercive. Moreover,  $\tilde{\varphi}_\lambda(\cdot)$  is sequentially weakly lower semicontinuous. So, we



can find  $u_\lambda \in W^{1,p}(\Omega)$  such that

$$\begin{aligned} & \tilde{\varphi}_\lambda(u_\lambda) = \inf\{\tilde{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\}, \\ \Rightarrow & \tilde{\varphi}'_\lambda(u_\lambda) = 0, \\ \Rightarrow & \langle A(u_\lambda), h \rangle + \int_\Omega [\xi(z) + \eta]|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\ = & \int_\Omega \gamma_\lambda(z, u_\lambda) h dz \tag{37} \\ & \text{for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In (37) we choose  $h = (\bar{u}_* - u_\lambda)^+ \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} & \langle A(u_\lambda), (\bar{u}_* - u_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \eta]|u_\lambda|^{p-2}u_\lambda(\bar{u}_* - u_\lambda)^+ dz \\ & \quad + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda(\bar{u}_* - u_\lambda)^+ d\sigma \\ = & \int_\Omega [(\lambda + \eta)\bar{u}_*^{p-1} + f(z, \bar{u}_*)](\bar{u}_* - u_\lambda)^+ dz \text{ (see (36))} \\ \geq & \int_\Omega [(\lambda^* + \eta)\bar{u}_*^{p-1} + f(z, \bar{u}_*)](\bar{u}_* - u_\lambda)^+ dz \text{ (since } \lambda > \lambda^*) \\ = & \langle A(\bar{u}_*), (\bar{u}_* - u_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \eta]\bar{u}_*^{p-1}(\bar{u}_* - u_\lambda)^+ dz \\ & \quad + \int_{\partial\Omega} \beta(z)\bar{u}_*^{p-1}(\bar{u}_* - u_\lambda)^+ d\sigma \\ & \text{(since } \bar{u}_* \in S_{\lambda^*}), \\ \Rightarrow & \bar{u}_* \leq u_\lambda. \end{aligned}$$

Then from (36) and (37) it follows that  $\bar{u}_\lambda \in S_\lambda$  and so  $\lambda \in \mathcal{L}$ , a contradiction. This proves that  $\lambda^* \notin \mathcal{L}$ .  $\square$

So, summarizing the situation for problem  $(P_\lambda)$  when the perturbation  $f(z, \cdot)$  is  $(p - 1)$ -sublinear, we can state the following theorem.

**Theorem 3.1.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_1$  hold, then there exists  $\lambda^* < +\infty$  such that*

- (a) *for every  $\lambda \geq \lambda^*$ , problem  $(P_\lambda)$  has no positive solutions;*
- (b) *for every  $\lambda < \lambda^*$ , problem  $(P_\lambda)$  has at least one positive solution  $u_\lambda \in D_+$ ;*
- (c) *for every  $\lambda < \lambda^*$ , problem  $(P_\lambda)$  has a smallest positive solution  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $(-\infty, \lambda^*)$  into  $C^1(\bar{\Omega})$  is*
  - *strictly increasing, that is, if  $\vartheta < \lambda < \lambda^*$ , then*

$$\bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+;$$

- *left continuous, that is, if  $\lambda_n \rightarrow \lambda^-$  and  $\lambda < \lambda^*$ , then  $\bar{u}_{\lambda_n} \rightarrow \bar{u}_\lambda$  in  $C^1(\bar{\Omega})$ .*

In the special case of the  $p$ -Laplacian (that is,  $a(y) = |y|^{p-2}y$  for all  $y \in \mathbb{R}^N$  with  $1 < p < \infty$ ), we can identify  $\lambda^*$  as  $\hat{\lambda}_1(p, \xi, \beta)$ , when  $f(z, x) > 0$  for almost all  $z \in \Omega$ , and for all  $x > 0$ .

So, we consider the following nonlinear Robin problem:

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{p-1} + f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, u > 0, \lambda \in \mathbb{R}, 1 < p < \infty. \end{array} \right\} \tag{PL}_\lambda$$

**Proposition 13.** *Assume that hypotheses  $H(\xi), H(\beta)$  hold and let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that*

- *for almost all  $z \in \Omega$ ,  $f(z, 0) = 0$  and  $f(z, x) > 0$  for all  $x > 0$ ;*
- *$f(z, x) \leq a(z)(1 + x^{p^*-1})$  for almost all  $z \in \Omega$ , and for all  $x \geq 0$ , with  $a \in L^\infty(\Omega)$ .*

*Then for all  $\lambda \geq \hat{\lambda}_1(p, \xi, \beta)$ ,  $S_\lambda = \emptyset$ .*

*Proof.* Arguing by contradiction, suppose that  $S_\lambda \neq \emptyset$  and let  $u \in S_\lambda$ . The nonlinear regularity theory implies that  $u \in D_+$ . Let  $\hat{u}_1 = \hat{u}_1(p, \xi, \beta) \in D_+$  (see Proposition 4). We consider the function

$$R(\hat{u}_1, u)(z) = |D\hat{u}_1(z)|^p - |Du(z)|^{p-2}(Du(z), D\left(\frac{\hat{u}_1^p}{u^{p-1}}\right)(z))_{\mathbb{R}^N}.$$

The nonlinear Picone identity of Allegretto & Huang [2] implies that

$$\begin{aligned} 0 &\leq R(\hat{u}_1, u)(z) \text{ for almost all } z \in \Omega, \\ \Rightarrow 0 &\leq \int_\Omega R(\hat{u}_1, u) dz \\ &= \|D\hat{u}_1\|_p^p - \int_\Omega |Du|^{p-2} \left( Du, D\left(\frac{\hat{u}_1^p}{u^{p-1}}\right) \right)_{\mathbb{R}^N} dz \\ &= \|D\hat{u}_1\|_p^p - \int_\Omega (-\Delta_p u) \frac{\hat{u}_1^p}{u^{p-1}} dz + \int_{\partial\Omega} \beta(z) u^{p-1} \frac{\hat{u}_1^p}{u^{p-1}} d\sigma \end{aligned}$$

(using the nonlinear Green identity, see Gasinski & Papageorgiou [8, p. 211])

$$= \|D\hat{u}_1\|_p^p + \int_\Omega \xi(z) \hat{u}_1^{p-1} dz + \int_{\partial\Omega} \beta(z) \hat{u}_1^p d\sigma - \lambda - \int_\Omega f(z, u) \frac{\hat{u}_1^p}{u^{p-1}}$$

(see  $(PL_\lambda)$  and recall that  $\|\hat{u}_1\|_p = 1$ )

$$< \mu(\hat{u}_1) - \lambda \text{ (recall that } f(z, x) > 0 \text{ for almost all } z \in \Omega, \text{ and for all } x > 0)$$

$$= \hat{\lambda}_1 - \lambda < 0,$$

a contradiction. Therefore  $S_\lambda = \emptyset$  and so  $\lambda \notin \mathcal{L}$  for all  $\lambda \geq \hat{\lambda}_1(p, \xi, \beta)$ . □

Moreover, reasoning as in the proof of Proposition 5, via the direct method of the calculus of variations, we obtain the following result. Note that now in hypothesis  $H(a)(iv)$  we take  $q = p$ .

**Proposition 14.** *If hypotheses  $H(\xi), H(\beta), H(f)_1$  hold and  $\lambda < \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$ , then  $\lambda \in \mathcal{L}$ .*

We introduce the following stronger version of hypotheses  $H(f)_1$ .

$H(f)'_1 : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that for almost all  $z \in \Omega$ ,  $f(z, 0) = 0$ ,  $f(z, x) > 0$  for all  $x > 0$  and hypotheses  $H(f)'_1(i), (ii), (iii), (iv)$  are the same as the corresponding hypotheses  $H(f)_1(i), (ii), (iii), (iv)$ .

Using this stronger version of  $H(f)_1$  and combining Propositions 13 and 14 we have the following theorem concerning the positive solutions of  $(PL_\lambda)$  as the parameter  $\lambda \in \mathbb{R}$  varies.

**Theorem 3.2.** *If hypotheses  $H(\xi), H(\beta), H(f)'_1$  hold, then*

- (a) *for every  $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$ , problem  $(PL_\lambda)$  has no positive solutions;*
- (b) *for every  $\lambda < \hat{\lambda}_1$ , problem  $(PL_\lambda)$  has at least one positive solution  $u_\lambda \in D_+$ ;*

- (c) for every  $\lambda < \hat{\lambda}$ , problem  $(PL_\lambda)$  has a smallest positive solution  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $(-\infty, \hat{\lambda}_1)$  into  $C^1(\bar{\Omega})$  is
- strictly increasing (that is,  $\vartheta < \lambda < \hat{\lambda}_1 \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$ );
  - left continuous.

If we further restrict the conditions on the perturbation  $f(z, x)$ , we can have uniqueness for the positive solution.

The new hypotheses on  $f(z, x)$  are the following:

$H(f)_1'' : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that for almost all  $z \in \Omega$ ,  $f(z, 0) = 0, f(z, x) > 0$  for all  $x > 0$ , hypotheses  $H(f)_1''(i), (ii), (iii), (iv)$  are the same as the corresponding hypotheses  $H(f)_1(i), (ii), (iii), (iv)$  and

- (v) if  $x - y \geq m > 0$ , then  $\frac{f(z,y)}{y^{p-1}} - \frac{f(z,x)}{x^{p-1}} \geq c_m > 0$  for almost all  $z \in \Omega$ .

**Proposition 15.** *If hypotheses  $H(\xi), H(\beta), H(f)_1''$  hold and  $\lambda < \hat{\lambda}_1$ , then problem  $(P_\lambda)$  admits a unique solution  $\bar{u}_\lambda \in D_+$ .*

*Proof.* By Theorem 3.2 we already have a positive solution  $\bar{u}_\lambda \in D_+$ . Suppose that  $\bar{u}_\lambda$  is another positive solution of  $(P_\lambda)$ . Again we have that  $\bar{v}_\lambda \in D_+$ . By Proposition 2.1 of Marano & Papageorgiou [15], we can find  $t > 0$  such that

$$t\bar{v}_\lambda \leq \bar{u}_\lambda. \tag{38}$$

Let  $t > 0$  be the biggest real for which (38) holds. Suppose that  $t < 1$ . Also, let  $\rho = \|\bar{u}_\lambda\|_\infty$  and let  $\hat{\xi}_\rho > 0$  be as postulated by hypothesis  $H(f)_1''(iv)$ . We can always assume that  $\hat{\xi}_\rho > \|\xi\|_\infty$ . Also let  $\bar{m}_\lambda = \min_{\bar{\Omega}} \bar{v}_\lambda > 0$ . We have

$$\begin{aligned} & -\Delta_p(t\bar{v}_\lambda) + [\xi(z) + \hat{\xi}_\rho](t\bar{v}_\lambda)^{p-1} \\ &= t^{p-1}(-\Delta_p\bar{v}_\lambda + [\xi(z) + \hat{\xi}_\rho]\bar{v}_\lambda^{p-1}) \\ &= t^{p-1}(\lambda\bar{v}_\lambda^{p-1} + f(z, \bar{v}_\lambda) + \hat{\xi}_\rho\bar{v}_\lambda^{p-1}) \text{ (since } \bar{v}_\lambda \in S_\lambda) \\ &\leq \lambda(t\bar{v}_\lambda)^{p-1} + f(z, t\bar{v}_\lambda) + \hat{\xi}_\rho(t\bar{v}_\lambda)^{p-1} - (1-t)\bar{m}_\lambda^{p-1} \\ &\text{(see hypothesis } H(f)_1''(v) \text{ and recall that } t < 1) \\ &< \lambda\bar{u}_\lambda^{p-1} + f(z, \bar{u}_\lambda) + \hat{\xi}_\rho\bar{u}_\lambda^{p-1} \text{ (see (38) and hypothesis } H(f)_1''(iv)) \\ &= -\Delta_p\bar{u}_\lambda + [\xi(z) + \hat{\xi}_\rho]\bar{u}_\lambda^{p-1} \text{ for almost all } z \in \Omega \text{ (since } \bar{u}_\lambda \in S_\lambda), \\ \Rightarrow & \bar{u}_\lambda - t\bar{v}_\lambda \in \text{int } C_+ \text{ (see Proposition 2),} \end{aligned}$$

which contradicts the maximality of  $t > 0$ . Therefore  $t \geq 1$  and we have

$$\bar{v}_\lambda \leq \bar{u}_\lambda \text{ (see (38)).}$$

Interchanging the roles of  $\bar{u}_\lambda$  and  $\bar{v}_\lambda$  in the above argument, we obtain

$$\begin{aligned} & \bar{u}_\lambda \leq \bar{v}_\lambda, \\ \Rightarrow & \bar{u}_\lambda = \bar{v}_\lambda. \end{aligned}$$

This proves the uniqueness of the positive solution of problem  $(PL_\lambda)$ . □

So, we can state the following existence and uniqueness theorem for problem  $(PL_\lambda)$ .

**Theorem 3.3.** *If hypotheses  $H(\xi), H(\beta), H(f)_1''$  hold, then*

- (a) for every  $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$ , problem  $(PL_\lambda)$  has no positive solutions;

- (b) for every  $\lambda < \hat{\lambda}_1$ , problem  $(PL_\lambda)$  has a unique positive solution  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $(-\infty, \hat{\lambda}_1)$  into  $C^1(\bar{\Omega})$  is
- strictly increasing (that is,  $\vartheta < \lambda < \hat{\lambda}_1 \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$ );
  - left continuous.

For the general nonhomogeneous problem, to have uniqueness, we need to set  $\xi \equiv 0$ . So, we consider the problem:

$$\left\{ \begin{array}{l} -\text{div } a(Du(z)) = \lambda u(z)^{p-1} + f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, u > 0, \lambda \in \mathbb{R}, 1 < p < \infty. \end{array} \right\} \quad (P'_\lambda)$$

Then reasoning as in the proof of Proposition 8 (see also Fragnelli, Mugnai & Papageorgiou [7, Theorem 7]), we have uniqueness of the positive solution and we can formulate the following theorem.

**Theorem 3.4.** *If hypotheses  $H(a), H(\beta), H(f)_1''$  hold, then there exists  $\lambda^* \in \mathbb{R}$  such that*

- (a) for every  $\lambda \geq \lambda^*$ , problem  $(P'_\lambda)$  has no positive solutions;
- (b) for every  $\lambda < \lambda^*$ , problem  $(P'_\lambda)$  has a unique positive solution  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $(-\infty, \lambda^*)$  into  $C^1(\bar{\Omega})$  is
  - strictly increasing (that is,  $\vartheta < \lambda < \lambda^* \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$ );
  - left continuous;

4.  **$(p - 1)$ -superlinear perturbation.** In this section we examine what happens in problem  $(P_\lambda)$  when the perturbation  $f(z, \cdot)$  is  $(p - 1)$ -superlinear. We do not assume that  $f(z, \cdot)$  satisfies the usual (for “superlinear” problems) “Ambrosetti-Rabinowitz condition” (the “AR-condition” for short). Instead, we employ a less restrictive condition involving the function

$$d(z, x) = f(z, x)x - pF(z, x) \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

In this way we incorporate in our framework  $(p - 1)$ -superlinear functions with “slower” growth near  $+\infty$ , which fail to satisfy the AR-condition.

So, we introduce the following condition on the perturbation  $f(z, x)$ .

$H(f)_2 : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that  $f(z, 0) = 0$  for almost all  $z \in \Omega$  and

- (i)  $|f(z, x)| \leq a(z)(1 + x^{r-1})$  for almost all  $z \in \Omega$ , and for all  $x \geq 0$ , with  $a \in L^\infty(\Omega)$ ,  $p < r < p^*$ ;
- (ii) if  $F(z, x) = \int_0^x f(z, s)ds$ , then  $\lim_{x \rightarrow +\infty} \frac{F(z, x)}{x^p} = +\infty$  uniformly for almost all  $z \in \Omega$ ;
- (iii) if  $d(z, x) = f(z, x)x - pF(z, x)$ , then  $d(z, x) \leq d(z, y) + \nu(z)$  for almost all  $z \in \Omega$ , and for all  $0 \leq x \leq y$  with  $\nu(\cdot) \in L^1(\Omega)$ ;
- (iv)  $\lim_{x \rightarrow 0^+} \frac{f(z, x)}{x^{\tau-1}} = 0$  uniformly for almost all  $z \in \Omega$ , with  $1 < \tau < q$ ,  $q \leq p$  as in  $H(a)(iv)$  and there exist  $s \in (\tau, q)$ ,  $\delta_0 > 0$  such that  $\tilde{c}_0 x^{s-1} \leq f(z, x)$  for almost all  $z \in \Omega$ ,  $x \in [0, \delta_0]$  with  $\tilde{c}_0 > 0$ ;
- (v) for every  $\rho > 0$ , there exists  $\hat{\xi}_\rho > 0$  such that for almost all  $z \in \Omega$  the mapping  $x \mapsto f(z, x) + \hat{\xi}_\rho x^{p-1}$  is nondecreasing on  $[0, \rho]$ .

**Remark 4.** Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis  $\mathbb{R}_+ = [0, +\infty)$ , we may assume without any loss of

generality, as we did in the sublinear case, that  $f(z, x) = 0$  for almost all  $z \in \Omega$ , all  $x \leq 0$ . Hypotheses  $H(f)_2(ii), (iii)$  imply that

$$\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \text{ uniformly for almost all } z \in \Omega. \tag{39}$$

So, the perturbation  $f(z, \cdot)$  is  $(p - 1)$ -superlinear. Usually for such problems, the superlinearity is expressed through the AR-condition, which says that there exist  $\tau > p$  and  $M > 0$  such that

$$\begin{aligned} 0 < \tau F(z, x) &\leq f(z, x)x \text{ for almost all } z \in \Omega, \\ \text{and for all } x \geq M &\text{ whereess } \inf_{\Omega} F(\cdot, M) > 0. \end{aligned} \tag{40}$$

Here we have a unilateral version of the AR-condition, since  $f(z, \cdot)_{(-\infty, 0]} = 0$ . Integrating (39) we obtain the more general condition

$$c_{13}x^\tau \leq F(z, x) \text{ for almost all } z \in \Omega, \text{ for all } x \geq M, \text{ and for some } c_{13} > 0. \tag{41}$$

Evidently, (40) and (41) imply that (39) holds. Using the AR-condition (40) we can easily verify the C-condition for the energy functional. However, from (41) we see that the AR-condition is rather restrictive. It excludes from consideration superlinear functions with slower growth near  $+\infty$  (see the examples below). We have replaced the AR-condition by hypotheses  $H(f)_2(ii), (iii)$ , which incorporate in our framework such functions. Hypothesis  $H(f)_2(iii)$  is a quasi-monotonicity condition on  $d(z, \cdot)$  on  $\mathbb{R}_+$ . This hypothesis is a slightly more general version of a condition used by Li & Yang [12], who compared this condition with other superlinearity conditions that can be found in the literature.

*Examples.* The following functions satisfy hypotheses  $H(f)_2$ . For the sake of simplicity we drop the  $z$ -dependence.

$$f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{\tau-1} - 2x^{\vartheta-1} & \text{if } 0 \leq x \leq 1 \\ x^{\tau-1} - 2x^{p-1} & \text{if } 1 < x \end{cases}$$

with  $1 < \tau < \vartheta < p < r$

$$f_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{\tau-1} - 2x^{\vartheta-1} & \text{if } 0 \leq x \leq 1 \\ x^{p-1}(\ln x - 1) & \text{if } 1 < x \end{cases}$$

with  $1 < \tau < \vartheta < p$ . Note that  $f_1$  satisfies the AR-condition, whereas  $f_2$  does not. Also, both functions may be sign-changing.

As before, we denote

$$\begin{aligned} \mathcal{L} &= \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution}\}, \\ S_\lambda &= \text{the set of all positive solutions of problem } (P_\lambda). \end{aligned}$$

**Proposition 16.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_2$  hold, then  $\mathcal{L} \neq \emptyset$  and  $S_\lambda \subseteq D_+$ .*

*Proof.* Let  $\eta > \|\xi\|_\infty$  and consider the functional  $\psi_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\psi_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}\|u^-\|_p^p - \frac{\lambda}{p}\|u^+\|_p^p - \int_{\Omega} F(z, u^+)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses  $H(f)_2(i), (iv)$  imply that given  $\epsilon > 0$ , we can find  $c_{14} = c_{14}(\epsilon) > 0$  such that

$$F(z, x) \leq \frac{\epsilon}{p}x^\tau + c_{14}x^r \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{42}$$

Then for  $\lambda < 0$ , with  $|\lambda| > \|\xi\|_\infty$ , we have

$$\begin{aligned} \psi_\lambda(u) &\geq \frac{1}{p}\mu(u^-) + \frac{\eta}{p}\|u^-\|_p^p + \frac{1}{p}\mu(u^+) + \frac{|\lambda|}{p}\|u^+\|_p^p - \epsilon c_{15}\|u\|^\tau - c_{16}\|u\|^\tau \\ &\quad \text{with } c_{15}, c_{16} > 0 \text{ (see (42))} \\ &\geq [c_{17} - (\epsilon c_{15}\|u\|^{\tau-p} + c_{16}\|u\|^{r-p})] \|u\|^p \text{ for some } c_{17} > 0. \end{aligned} \tag{43}$$

Let  $k_0(t) = \epsilon c_{15}t^{\tau-p} + c_{16}t^{r-p}$   $t \geq 0$ . Since  $1 < \tau < p < r$ , we see that  $k_0(t) \rightarrow +\infty$  as  $t \rightarrow 0^+$  and as  $t \rightarrow +\infty$ . So, we can find  $t_0 > 0$  such that  $k_0(t_0) = \min_{t>0} k_0$ . We have

$$k'_0(t_0) = 0, \quad \Rightarrow \quad t_0 = \left[ \frac{\epsilon c_{15}(p - \tau)}{c_{16}(r - p)} \right]^{\frac{1}{r-\tau}}.$$

Then  $k_0(t_0) \rightarrow 0^+$  as  $\epsilon \rightarrow 0^+$ . So, it follows from (43) that we can find small enough  $\rho \in (0, 1)$  such that

$$0 < \inf\{\psi_\lambda(u) : \|u\| = \rho\} = m_\rho^\lambda. \tag{44}$$

Hypothesis  $H(f)_2(ii)$  implies that if  $u \in D_+$ , then

$$\psi_\lambda(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \tag{45}$$

**Claim 1.** For every  $\lambda \in \mathbb{R}$ ,  $\psi_\lambda(\cdot)$  satisfies the C-condition.

Consider a sequence  $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$  such that

$$|\psi_\lambda(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ and for all } n \in \mathbb{N}, \tag{46}$$

$$(1 + \|u_n\|)\psi'_\lambda(u_n) \rightarrow 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty. \tag{47}$$

From (47) we have

$$\begin{aligned} |\langle A(u_n), h \rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}h d\sigma - \eta \int_\Omega (u_n^-)^{p-1}h dz \\ - \int_\Omega \lambda(u_n^+)^{p-1}h dz - \int_\Omega f(z, u_n^+)h dz| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \end{aligned} \tag{48}$$

for all  $h \in W^{1,p}(\Omega)$ , with  $\epsilon_n \rightarrow 0^+$ .

In (48) we choose  $h = -u_n^- \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} \mu(u_n^-) + \eta\|u_n^-\|_p^p \leq \epsilon_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow c_{18}\|u_n^-\|_p^p \leq \epsilon_n \text{ for some } c_{18} > 0, \text{ and for all } n \in \mathbb{N} \text{ (recall that } \eta > \|\xi\|_\infty), \\ \Rightarrow u_n^- \rightarrow 0 \text{ in } W^{1,p}(\Omega) \text{ as } n \rightarrow \infty. \end{aligned} \tag{49}$$

Next, in (48) we choose  $h = u_n^+ \in W^{1,p}(\Omega)$ . Then

$$-\mu(u_n^+) + \lambda\|u_n^+\|_p^p + \int_\Omega f(z, u_n^+)u_n^+ dz \leq \epsilon_n \text{ for all } n \in \mathbb{N}. \tag{50}$$

From (46) and (49) we have

$$\mu(u_n^+) - \lambda\|u_n^+\|_p^p - \int_\Omega pF(z, u_n^+)dz \leq M_2 \text{ for some } M_2 > 0, \text{ and for all } n \in \mathbb{N}. \tag{51}$$

We add (50), (51) and obtain

$$\int_\Omega d(z, u_n^+)dz \leq M_3 \text{ for some } M_3 > 0, \text{ and for all } n \in \mathbb{N}. \tag{52}$$

We will use (52) to show that  $\{u_n^+\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$  is bounded. Arguing by contradiction, suppose that  $\|u_n^+\| \rightarrow \infty$ . We set  $y_n = \frac{u_n^+}{\|u_n^+\|}$  for all  $n \in \mathbb{N}$ . We have  $\|y_n\| = 1$  for all  $n \in \mathbb{N}$  and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \rightarrow \infty. \tag{53}$$

First, we assume that  $y \neq 0$ . Let  $\Omega_0 = \{z \in \Omega : y(z) = 0\}$ . Then  $|\Omega \setminus \Omega_0|_N > 0$  (by  $|\cdot|_N$  we denote the Lebesgue measure on  $\mathbb{R}^N$ ) and  $u_n^+(z) \rightarrow +\infty$  for almost all  $z \in \Omega \setminus \Omega_0$  as  $n \rightarrow \infty$ . Hence hypothesis  $H(f)_2(ii)$  implies that

$$\begin{aligned} \frac{F(z, u_n^+(z))}{\|u_n^+\|^p} &= \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \rightarrow +\infty \text{ for almost all } z \in \Omega \setminus \Omega_0 \text{ as } n \rightarrow \infty, \\ \Rightarrow \int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz &\rightarrow +\infty \text{ as } n \rightarrow +\infty \text{ (by Fatou's lemma)}. \end{aligned} \tag{54}$$

Corollary 1 and hypothesis  $H(a)(iv)$  imply that

$$G(y) \leq c_{19}(|y|^q + |y|^p) \text{ for some } c_{19} > 0, \text{ and for all } y \in \mathbb{R}^N. \tag{55}$$

From (46) and (49), we have

$$\begin{aligned} & - \int_{\Omega} G(Du_n^+) dz - \frac{1}{p} \int_{\Omega} \xi(z)(u_n^+)^p dz - \frac{1}{p} \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma + \frac{\lambda}{p} \|u_n^+\|_p^p + \\ & \int_{\Omega} F(z, u_n^+) dz \leq M_4 \text{ for some } M_4 > 0, \text{ and for all } n \in \mathbb{N} \\ \Rightarrow \int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|^p} dz &\leq M_5 \text{ for some } M_5 > 0, \text{ and for all } n \in \mathbb{N} \end{aligned} \tag{56}$$

(see (53), (55) and hypotheses  $H(\xi), H(\beta)$ ).

Comparing (54) and (56), we get a contradiction.

So, we assume that  $y = 0$ . We consider the  $C^1$ -functional  $\hat{\psi}_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \hat{\psi}_\lambda(u) &= \frac{c_1}{p(p-1)} \|Du\|_p^p + \frac{1}{p} \int_{\Omega} \xi(z)|u|^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p d\sigma + \frac{\eta}{p} \|u^-\|_p^p \\ & - \frac{\lambda}{p} \|u^+\|_p^p - \int_{\Omega} F(z, u^+) dz \text{ for all } u \in W^{1,p}(\Omega). \end{aligned}$$

Evidently,  $\hat{\psi}_\lambda \leq \psi_\lambda$  (see Corollary 1).

We define  $\vartheta_n(t) = \hat{\psi}_\lambda(tu_n^+)$  for all  $t \in [0, 1]$ , and for all  $n \in \mathbb{N}$ . Let  $t_n \in [0, 1]$  be such that

$$\vartheta_n(t_n) = \max_{0 \leq t \leq 1} \vartheta_n(t) = \max_{0 \leq t \leq 1} \hat{\psi}_\lambda(tu_n^+) \text{ for all } n \in \mathbb{N}. \tag{57}$$

For  $\gamma > 0$ , let  $v_n = (2\gamma)^{1/p} y_n \in W^{1,p}(\Omega)$ . Evidently,  $v_n \rightarrow 0$  in  $L^r(\Omega)$  (see (53) and recall that we have assumed that  $y = 0$ ). Then

$$\int_{\Omega} F(z, v_n) dz \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{58}$$

Since  $\|u_n^+\| \rightarrow \infty$ , we can find  $n_0 \in \mathbb{N}$  such that

$$(2\gamma)^{1/p} \frac{1}{\|u_n^+\|} \in (0, 1) \text{ for all } n \geq n_0. \tag{59}$$

Then (57) and (59) imply that

$$\begin{aligned} \vartheta_n(t_n) &\geq \vartheta_n\left(\frac{(2\gamma)^{1/p}}{\|u_n^+\|}\right) \text{ for all } n \geq n_0 \\ \Rightarrow \hat{\psi}_\lambda(t_n u_n^+) &\geq \hat{\psi}_\lambda((2\gamma)^{1/p} y_n) = \hat{\psi}_\lambda(v_n) \text{ for all } n \geq n_0 \\ &\geq \frac{2\gamma c_1}{p(p-1)} \left( \|Dy_n\|_p^p + \frac{p-1}{c_1} \int_\Omega [\xi(z) + \eta - \lambda] y_n^p dz \right) \\ &\quad - \left( \int_\Omega F(z, v_n) dz + \frac{\eta}{p} \|v_n\|_p^p \right) \\ &\geq \frac{2\gamma c_{20}}{p(p-1)} - \left( \int_\Omega F(z, v_n) dz + \frac{\eta}{p} \|v_n\|_p^p \right) \\ &\quad \text{for some } c_{20} > 0, \text{ and for all } n \geq n_0. \end{aligned} \tag{60}$$

Recall that  $v_n \rightarrow 0$  in  $L^p(\Omega)$ . Using this fact and (58) in (60), we see that

$$\hat{\psi}_\lambda(t_n u_n^+) \geq \frac{\gamma c_{20}}{p(p-1)} \text{ for some } n \geq n_1 \geq n_0.$$

But recall that  $\gamma > 0$  is arbitrary. So, it follows that

$$\hat{\psi}_\lambda(t_n u_n^+) \rightarrow +\infty \text{ as } n \rightarrow \infty. \tag{61}$$

We have  $0 \leq t_n u_n^+ \leq u_n^+$  for all  $n \in \mathbb{N}$ . So, on account of hypothesis  $H(f)_2(iii)$ , we have

$$\int_\Omega d(z, t_n u_n^+) dz \leq \int_\Omega d(z, u_n^+) dz + \|\nu\|_1 \leq M_6 \tag{62}$$

for some  $M_6 > 0$ , and for all  $n \in \mathbb{N}$  (see (52)).

We know that

$$\begin{aligned} \hat{\psi}_\lambda(0) &= 0 \text{ and } \hat{\psi}_\lambda(u_n^+) \leq M_7 \text{ for some } M_7 > 0 \\ &\text{(see (46), (52) and recall that } \hat{\psi}_\lambda \leq \psi_\lambda \text{)}. \end{aligned} \tag{63}$$

From (61) and (63) we infer that  $t_n \in (0, 1)$  for all  $n \geq n_2$ . Hence we have

$$0 = t_n \frac{d}{dt} \hat{\psi}_\lambda(tu_n^+) |_{t=t_n} = \langle \psi'_\lambda(t_n u_n^+), t_n u_n^+ \rangle \text{ for all } n \geq n_2 \text{ (see (57))}. \tag{64}$$

Combining (62) and (64) we see that

$$p \hat{\psi}_\lambda(t_n u_n^+) \leq M_6 \text{ for all } n \geq n_2. \tag{65}$$

Comparing (61) and (65) we have a contradiction. Therefore

$$\begin{aligned} \{u_n^+\}_{n \geq 1} &\subseteq W^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow \{u_n\}_{n \geq 1} &\subseteq W^{1,p}(\Omega) \text{ is bounded (see (49)).} \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{66}$$

We return to (48) and choose  $h = u_n - u \in W^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$  and use (66). Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle &= 0, \\ \Rightarrow u_n &\rightarrow u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned}$$

Therefore  $\psi_\lambda$  satisfies the C-condition and this proves the claim.



Then (44), (45) and the claim, permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find  $u_\lambda \in W^{1,p}(\Omega)$  ( $\lambda < 0, |\lambda| > \|\xi\|_\infty$ ) such that

$$u_\lambda \in K_{\psi_\lambda} \text{ and } m_\rho^\lambda \leq \psi_\lambda(u_\lambda). \tag{67}$$

It follows from (67) that  $u_\lambda \neq 0$  (see (44)) and

$$\begin{aligned} \langle A(u_\lambda), h \rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\ - \eta \int_\Omega (u_\lambda^-)^{p-1} h dz = \lambda \int_\Omega (u_\lambda^+)^{p-1} h dz + \int_\Omega f(z, u_\lambda^+) h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{68}$$

In (68) we choose  $h = -u_\lambda^- \in W^{1,p}(\Omega)$ . Then

$$\begin{aligned} \frac{c_1}{p-1} \|Du_\lambda^-\|_p^p + \int_\Omega [\xi(z) + \eta]|u_\lambda^-|^p dz \leq 0 \text{ (see Lemma 2.2),} \\ \Rightarrow u_\lambda \geq 0, u_\lambda \neq 0. \end{aligned}$$

It follows from (68) that  $u_\lambda$  is a positive solution of  $(P_\lambda)$ , hence  $\lambda \in \mathcal{L}$  and so  $\mathcal{L} \neq \emptyset$ . Moreover, from the nonlinear regularity theory (see [13]) and the nonlinear maximum principle (see [25]), we can deduce that  $S_\lambda \subseteq D_+$ .  $\square$

In the present setting, on account of hypotheses  $H(f)_2(i), (iv)$ , we have that

$$\lambda x^{p-1} + f(z, x) \geq \tilde{c}_0 x^{s-1} - c_{21} x^{r-1} \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0, \tag{69}$$

for some big enough  $c_{21} = c_{21}(\lambda) > 0$ . An inspection of the proofs of Propositions 6–11 reveals that their conclusions remain valid in the present setting. Now, instead of (19) we use (69). So, we can state the following proposition summarizing these conclusions.

**Proposition 17.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_2$  hold, then*

- (a) *if  $\lambda \in \mathcal{L}$  and  $\vartheta < \lambda$ , then  $\vartheta \in \mathcal{L}$ ;*
- (b)  *$\lambda^* = \sup \mathcal{L} < +\infty$ ;*
- (c) *for every  $\lambda \in \mathcal{L}$ , problem  $(P_\lambda)$  admits a smallest element  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $\mathcal{L}$  into  $C^1(\bar{\Omega})$  is*
  - *strictly increasing (that is,  $\vartheta < \lambda \in \mathcal{L} \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$ );*
  - *left continuous.*

Again we show that the critical parameter  $\lambda^*$  is not admissible, hence

$$\mathcal{L} = (-\infty, \lambda^*).$$

**Proposition 18.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_2$  hold, then  $\lambda^* \notin \mathcal{L}$ .*

*Proof.* Arguing by contradiction, suppose that  $\lambda^* \in \mathcal{L}$ . Then according to Proposition 17 problem  $(P_{\lambda^*})$  admits a smallest positive solution  $\bar{u}_* = \bar{u}_{\lambda^*} \in D_+$ .

Consider  $\lambda > \lambda^*$  and, as always, let  $\eta > \|\xi\|_\infty$ . We introduce the Carathéodory function  $\hat{\gamma}_\lambda(z, x)$  define by

$$\hat{\gamma}_\lambda(z, x) = \begin{cases} (\lambda + \eta)\bar{u}_*(z)^{p-1} + f(z, \bar{u}_*(z)) & \text{if } x \leq \bar{u}_*(z) \\ (\lambda + \eta)x^{p-1} + f(z, x) & \text{if } \bar{u}_*(z) < x. \end{cases} \tag{70}$$

We set  $\hat{\Gamma}_\lambda(z, x) = \int_0^x \hat{\gamma}_\lambda(z, s) ds$  and consider the  $C^1$ -functional  $\tilde{\psi}_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\tilde{\psi}_\lambda(u) = \frac{1}{p} \mu(u) + \frac{\eta}{p} \|u\|_p^p - \int_\Omega \hat{\Gamma}_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (70) we can easily verify that

$$K_{\tilde{\psi}_\lambda} \subseteq [\bar{u}_*] \cap C^1(\bar{\Omega}). \tag{71}$$

As in the proof of Proposition 17 of [21] (see the Claim), we may assume that

$$\bar{u}_* \text{ is a local minimizer of } \tilde{\psi}_\lambda(\cdot). \tag{72}$$

Without any loss of generality, we assume that  $K_{\psi_\lambda}$  is finite (see (70), (71)). Then on account of (??) we can find small enough  $\rho_1 \in (0, 1)$  such that

$$\inf\{\tilde{\psi}_\lambda(u) : \|u - \bar{u}_*\| = \rho_1\} = \tilde{m}_1 > \tilde{\psi}_\lambda(\bar{u}_*) \text{ (see [1]).} \tag{73}$$

Also, hypothesis  $H(f)_2(ii)$  implies that if  $u \in D_+$ , then

$$\tilde{\psi}_\lambda(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \tag{74}$$

Moreover, on account of (70), reasoning as in the proof of Proposition 16 (see the claim), we can show that

$$\tilde{\psi}_\lambda(\cdot) \text{ satisfies the C-condition.} \tag{75}$$

Then (73), (74), (75) permit the use of Theorem 2.1 (the mountain pass theorem). Hence we can find  $u_\lambda \in W^{1,p}(\Omega)$  such that

$$u_\lambda \in K_{\tilde{\psi}_\lambda} \subseteq [\bar{u}_*] \cap C^1(\bar{\Omega}) \text{ (see (71)), } \tilde{\psi}_\lambda(u_\lambda) \geq \tilde{m}_1.$$

It follows that  $u_\lambda \in S_\lambda$  and so  $\lambda \in \mathcal{L}$ , a contradiction. Therefore  $\lambda^* \notin \mathcal{L}$ . The proof is now complete.  $\square$

In this case for  $\lambda \in \mathcal{L} = (-\infty, \lambda^*)$  we have a multiplicity result for problem  $(P_\lambda)$ .

**Proposition 19.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_2$  hold and  $\lambda \in \mathcal{L} = (-\infty, \lambda^*)$ , then problem  $(P_\lambda)$  admits at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in D_+, \quad u_\lambda \leq \hat{u}_\lambda, \quad u_\lambda \neq \hat{u}_\lambda.$$

*Proof.* Since  $\lambda \in \mathcal{L}$  we can find  $u_\lambda \in S_\lambda \subseteq D_+$  (see Proposition 16). We may assume that  $u_\lambda$  is the minimal positive solution of  $(P_\lambda)$  produced in Proposition 17 (that is,  $u_\lambda = \bar{u}_\lambda$ ). With  $\eta > \|\xi\|_\infty$ , we introduce the Carathéodory function  $k_\lambda(z, x)$  defined by

$$k_\lambda(z, x) = \begin{cases} (\lambda + \eta)u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } x \leq u_\lambda(z) \\ (\lambda + \eta)x^{p-1} + f(z, x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{76}$$

Let  $K_\lambda(z, x) = \int_0^x k_\lambda(z, s)ds$  and consider the  $C^1$ -functional  $j_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$j_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}\|u\|_p^p - \int_\Omega K_\lambda(z, u)dz.$$

Working with  $j_\lambda(\cdot)$  as in the proof of Proposition 18 and using (76), we produce  $\hat{u}_\lambda \in W^{1,p}(\Omega)$  such that

$$\hat{u}_\lambda \in K_{j_\lambda} \subseteq [u_\lambda] \cap C^1(\bar{\Omega}), \quad \hat{u}_\lambda \notin \{0, u_\lambda\}. \tag{77}$$

It follows from (76) and (77) that  $\hat{u}_\lambda \in D_+$  is the second positive solution of  $(P_\lambda)$ .  $\square$

Summarizing the situation for the “superlinear” case, we can state the following result.

**Theorem 4.1.** *If hypotheses  $H(a), H(\xi), H(\beta), H(f)_2$  hold, then there exists  $\lambda^* < +\infty$  such that*

- (a) for every  $\lambda \geq \lambda^*$ , problem  $(P_\lambda)$  has no positive solutions;
- (b) for every  $\lambda < \lambda^*$ , problem  $(P_\lambda)$  has at least two positive solutions  $u_\lambda, \hat{u}_\lambda \in D_+$ ,  $u_\lambda \leq \hat{u}_\lambda, u_\lambda \neq \hat{u}_\lambda$ ;
- (c) for every  $\lambda < \lambda^*$ , problem  $(P_\lambda)$  has a smallest positive solution  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $\mathcal{L} = (-\infty, \hat{\lambda}_1)$  into  $C^1(\bar{\Omega})$  is
  - strictly increasing (that is,  $\vartheta < \lambda \in \mathcal{L} \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int}C_+$ );
  - left continuous.

Again, in the special case of the  $p$ -Laplacian, see problem  $(PL_\lambda)$  ( $a(y) = |y|^{p-2}y$  for all  $y \in \mathbb{R}^N$ ), we can identify  $\lambda^*$  as  $\hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$ , provided that  $f(z, x) > 0$  for almost all  $z \in \Omega$ , and for all  $x > 0$  and we restrict the condition near zero (that is,  $H(f)_2(iv)$ ).

So, the new conditions on the perturbation  $f(z, x)$  are the following:

$H(f)'_2 : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that for almost all  $z \in \Omega$ ,  $f(z, 0) = 0, f(z, x) > 0$  for all  $x > 0$ , hypotheses  $H(f)'_2(i), (ii), (iii), (v)$  are the same as the corresponding hypotheses  $H(f)'_2(i), (ii), (iii), (v)$  and

(iv)  $\lim_{x \rightarrow 0^+} \frac{f(z, x)}{x^{p-1}} = 0$  uniformly for almost all  $z \in \Omega$ .

From Proposition 13, we already know that for  $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$  problem  $(PL_\lambda)$  has no positive solutions.

**Proposition 20.** *If hypotheses  $H(\xi), H(\beta), H(f)'_2$  hold and  $\lambda < \hat{\lambda}_1$ , then  $\lambda \in \mathcal{L}$ .*

*Proof.* Let  $\lambda \in (-\infty, \hat{\lambda}_1)$  and consider the Carathéodory function  $\hat{\vartheta}_\lambda(z, x)$  defined by

$$\hat{\vartheta}_\lambda(z, x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \lambda x^{p-1} + f(z, x) & \text{if } 0 < x. \end{cases} \tag{78}$$

We set  $\hat{\Theta}_\lambda(z, x) = \int_0^x \hat{\vartheta}_\lambda(z, s) ds$  and with  $\eta > \|\xi\|_\infty$ , we consider the  $C^1$ -functional  $w_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$w_\lambda(u) = \frac{1}{p} \mu(u) + \frac{\eta}{p} \|u^-\|_p^p - \int_\Omega \hat{\Theta}_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses  $H(f)'_2(i), (iv)$  imply that given  $\epsilon > 0$ , we can find  $c_{22} > 0$  such that

$$F(z, x) \leq \frac{\epsilon}{p} x^p + c_{22} x^r \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{79}$$

Then from (78) and (79), we have

$$w_\lambda(u) \geq \frac{1}{p} [\mu(u^-) + \eta \|u^-\|_p^p] + \frac{1}{p} [\mu(u^+) - (\lambda + \epsilon) \|u^+\|_p^p] - c_{22} \|u^+\|_r^r \text{ (see (79)).}$$

Choosing  $\epsilon \in (0, \hat{\lambda}_1 - \lambda)$ , we have

$$\begin{aligned} w_\lambda(u) &\geq c_{23} \|u\|^p - c_{24} \|u\|^r \text{ for some } c_{23}, c_{24} > 0, \\ \Rightarrow u &= 0 \text{ local minimizer of } w_\lambda(\cdot) \text{ (recall that } r > p). \end{aligned}$$

So, we can find  $\rho \in (0, 1)$  small such that

$$w_\lambda(0) = 0 < \inf\{w_\lambda(u) : \|u\| = \rho\} = m_\lambda \tag{80}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 29).

Also, hypothesis  $H(f)'_2(ii)$  implies that if  $u \in D_+$ , then

$$w_\lambda(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \tag{81}$$

Finally, from the proof of Proposition 16 (see the claim), we know that

$$w_\lambda(\cdot) \text{ satisfies the C-condition.} \quad (82)$$

Then (80), (81), (82) permit the use of Theorem 2.1 (the mountain pass theorem) and produce  $u_\lambda \in W^{1,p}(\Omega)$  such that

$$\begin{aligned} u_\lambda &\in K_{w_\lambda} \subseteq D_+ \cup \{0\} \text{ (see the proof of Proposition 5)} \\ w_\lambda(0) &= 0 < m_\lambda \leq w_\lambda(u_\lambda). \end{aligned}$$

Therefore  $u_\lambda \in D_+$  is a positive solution of  $(PL_\lambda)$ , hence  $\lambda \in \mathcal{L}$ .  $\square$

So, for problem  $(PL_\lambda)$  we can state the following theorem covering the case of a  $(p-1)$ -superlinear perturbation.

**Theorem 4.2.** *If hypotheses  $H(\xi), H(\beta), H(f)'_2$  hold, then*

- (a) *for every  $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$ , problem  $(PL_\lambda)$  has no positive solutions;*
- (b) *for every  $\lambda < \hat{\lambda}_1$  problem  $(PL_\lambda)$  has at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in D_+, \quad u_\lambda \leq \hat{u}_\lambda, \quad u_\lambda \neq \hat{u}_\lambda;$$

- (c) *for every  $\lambda < \hat{\lambda}_1$  problem  $(PL_\lambda)$  has a smallest positive solution  $\bar{u}_\lambda \in D_+$  and the map  $\lambda \mapsto \bar{u}_\lambda$  from  $\mathcal{L} = (-\infty, \hat{\lambda}_1)$  into  $C^1(\bar{\Omega})$  is*
  - *strictly increasing (that is,  $\vartheta < \lambda < \hat{\lambda}_1 \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$ );*
  - *left continuous.*

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E-mail address: [npapg@math.ntua.gr](mailto:npapg@math.ntua.gr)

E-mail address: [vicentiu.radulescu@imar.ro](mailto:vicentiu.radulescu@imar.ro)

E-mail address: [dusan.repovs@guest.arnes.si](mailto:dusan.repovs@guest.arnes.si)