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Research Article

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Positive solutions for nonlinear nonhomogeneous parametric Robin problems

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Abstract: We study a parametric Robin problem driven by a nonlinear nonhomogeneous differential operator and with a superlinear Carathéodory reaction term. We prove a bifurcation-type theorem for small values of the parameter. Also, we show that as the parameter $\lambda > 0$ approaches zero, we can find positive solutions with arbitrarily big and arbitrarily small Sobolev norm. Finally, we show that for every admissible parameter value, there is a smallest positive solution u_λ^* of the problem, and we investigate the properties of the map $\lambda \mapsto u_\lambda^*$.

Keywords: Robin boundary condition, nonlinear nonhomogeneous differential operator, nonlinear regularity, nonlinear maximum principle, bifurcation-type result, extremal positive solution

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear, nonhomogeneous parametric Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $u > 0$, $\lambda > 0$ and $1 < p < \infty$. In this problem, the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is monotone continuous (hence maximal monotone, too) and satisfies certain other regularity and growth conditions, listed in Hypotheses 2.3 below. These conditions on $a(\cdot)$ are general enough to incorporate in our framework many differential operators of interest such as the p-Laplacian ($1 < p < \infty$) and the (p, q)-Laplacian ($1 < q < p < \infty$). The differential operator in (1.1) is not in general $(p-1)$-homogeneous and this is a source of technical difficulties in the analysis of problem (1.1). Also $\xi \in L^\infty(\Omega)$ and $\xi \geqslant 0$. In the reaction term (right-hand side of the equation) $\lambda > 0$ is a parameter and $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, the mapping $x \mapsto f(z, x)$ is continuous) which exhibits $(p-1)$-superlinear growth in the x-variable near $+\infty$, but without satisfying the usual for superlinear problems Ambrosetti–Rabinowitz condition (AR-condition for short). Instead we use a more general condition, which permits the consideration

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of $(p-1)$-superlinear functions with "slower" growth near $+\infty$ which fail to satisfy the AR-condition (see the examples below). Also near 0^+, the nonlinearity $f(z,\cdot)$ has a concave term (that is, a $(p-1)$-sublinear term).

In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the generalized normal derivative (the conormal derivative) of u, defined by the extension of

$$\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This kind of directional derivative on the boundary $\partial\Omega$ is dictated by the nonlinear Green's identity (see [15, p. 210]), and is also used by Lieberman [23]. For the boundary coefficient $\beta(z)$, we assume that

$$\beta \in C^{0,\alpha}(\partial\Omega) \quad \text{for some } \alpha \in (0,1), \qquad \beta(z) \geqslant 0 \quad \text{for all } z \in \partial\Omega.$$

We assume also that

$$\xi \neq 0 \quad \text{or} \quad \beta \neq 0.$$

If $\beta = 0$, then we recover the Neumann problem.

Our aim in this paper is to study the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. In this direction, we prove a bifurcation-type theorem for small values of the parameter, that is, we show that there exists a critical parameter value $\lambda^* \in (0, +\infty)$ such that
- for all $\lambda \in (0, \lambda^*)$, problem (1.1) admits at least two positive solutions;
- for $\lambda = \lambda^*$, problem (1.1) has at least one positive solution;
- for all $\lambda > \lambda^*$, problem (1.1) has no positive solutions.

Moreover, we show that if $\lambda_n \to 0^+$, then we can find pairs $\{u_{\lambda_n}, \hat{u}_{\lambda_n}\}_{n\in\mathbb{N}}$ of positive solutions such that

$$\|u_{\lambda_n}\| \to 0 \quad \text{and} \quad \|\hat{u}_{\lambda_n}\| \to +\infty \quad \text{as } n \to \infty.$$

Here $\|\cdot\|$ denotes the norm of the Sobolev space $W^{1,p}(\Omega)$.

Finally, if $\lambda \in (0, \lambda^*)$, then we show that problem (1.1) has a smallest positive solution u_λ^* and we investigate the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$.

Parametric problems with competing nonlinearities ("concave-convex" problems), were first investigated by Ambrosetti, Brezis and Cerami [4] for semilinear Dirichlet problems driven by the Laplacian (that is, $p = 2$) and with zero potential (that is, $\xi \equiv 0$). Their work was extended to Dirichlet problems driven by the p-Laplacian ($1 < p < \infty$) by García Azorero, Peral Alonso and Manfredi [14], Guo and Zhang [19], and Hu and Papageorgiou [21]. All the aforementioned papers consider "concave-convex" reaction terms modeled after the function

$$\lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geqslant 0,$$

with $q < p < r < p^*$. So, in their equations the concave and convex inputs in the reaction are decoupled and the parameter $\lambda > 0$ multiplies only the concave term.

Closer to problem (1.1) are the works of Gasinski and Papageorgiou [17], Papageorgiou and Rădulescu [30], and Aizicovici, Papageorgiou and Staicu [3]. All three papers deal with equations driven by the p-Laplacian and have a reaction term of the form $\lambda f(z,x)$ (as is the case here). In [17] the problem is Dirichlet, and bifurcation-type results for small and big values of the parameter $\lambda > 0$ are proved. In [30] the problem is Robin (with $\xi \equiv 0$ and $\beta \neq 0$), and a bifurcation-type result for large values of the parameter is proved. Finally, we mention also the related recent work of Papageorgiou and Smyrlis [39], who deal with singular Dirichlet problems, and of Papageorgiou and Rădulescu [31], dealing with p-Laplacian Robin problems with competing nonlinearities.

We denote by $\|\cdot\|_p$ the usual L^p-norm in $L^p(\Omega)$ and by $|\cdot|$ the Euclidean norm on $\mathbb{R}^N$. Throughout this paper, the symbol $\xrightarrow{w}$ is used for the weak convergence.

2 Mathematical background – auxiliary results

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). If $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

- Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and
$$(1+\|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n\to\infty$$
admits a strongly convergent subsequence.

This compactness-type condition on the functional φ leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. Central in that theory is the well-known "mountain pass theorem" due to Ambrosetti and Rabinowitz [5], stated here in a slightly more general form (see [15, p. 648]).

Theorem 2.1. *Assume that X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$,*
$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u-u_0\| = \rho\} = m_\rho$$
and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geqslant m_\rho$, and c is a critical value of φ.

Remark 2.2. The result is in fact more generally true in Banach–Finsler manifolds.

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, defined by
$$\|u\| = [\|u\|_p^p + \|Du\|_p^p]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

In addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$ and certain closed subspaces of it, and the "boundary" Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant \infty)$. The space $C^1(\overline{\Omega})$ is an ordered Banach space, with a positive (order) cone given by
$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

The cone has a nonempty interior containing
$$D_+ = \{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 \colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that
$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

We know that
$$\operatorname{im}\gamma_0 = W^{1/p',p}(\partial\Omega)\left(\frac{1}{p} + \frac{1}{p'} = 1\right) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

The trace map γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p})$ if $N > p$, and into $L^q(\partial\Omega)$ for all $q \geqslant 1$ if $p \geqslant N$. In the sequel, for the sake of notational simplicity, we will drop the use of the map γ_0. The restrictions of all Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $\vartheta \in C^1(0,+\infty)$ with $\vartheta(t) > 0$ for all $t > 0$, and assume that
$$0 < \hat{c} \leqslant \frac{\vartheta'(t)t}{\vartheta(t)} \leqslant c_0 \quad \text{and} \quad c_1 t^{p-1} \leqslant \vartheta(t) \leqslant c_2(1+t^{p-1}) \quad \text{for all } t > 0, \tag{2.1}$$
for some $c_1, c_2 > 0$.

Our hypotheses on the map $a(\cdot)$ are the following.

Hypotheses 2.3. Assume that $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$, and that the following hold:
(i) $a_0 \in C^1(0,\infty)$, $t \mapsto a_0(t)t$, is strictly increasing on $(0,+\infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$, and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1.$$

(ii) There exists $c_3 > 0$ such that

$$|\nabla a(y)| \leqslant c_3 \frac{\vartheta(|y|)}{|y|} \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

(iii) We have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \frac{\vartheta(|y|)}{|y|}|\xi|^2 \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\} \text{ and all } \xi \in \mathbb{R}^N.$$

(iv) If $G_0(t) = \int_0^t a_0(s)s\,ds$, then there exist $1 < q < p < r_0 < p^*$ (recall that $p^* = \frac{Np}{N-p}$ if $N > p$, and $p^* = +\infty$ if $N \leqslant p$) such that

$$\limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} \leqslant c^*,$$

$t \mapsto G_0(t^{1/q})$ is convex and

$$r_0 G_0(t) - a_0(t)t^2 \geqslant \bar{c}t^p, \quad pG_0(t) - a_0(t)t^2 \geqslant -\bar{c}_0 \quad \text{for all } t > 0,$$

for some $\bar{c}, \bar{c}_0 > 0$.

Remark 2.4. Hypotheses 2.3 (i)–(iii) are motivated by the nonlinear regularity theory of Lieberman [25] and the nonlinear maximum principle of Pucci and Serrin [41]. Hypothesis 2.3 (iv) serves the particular needs of our problem, but it is not restrictive and it is satisfied in many cases of interest as the examples below illustrate. Similar conditions were also used in recent works of the authors, see [33, 34, 37].

Hypotheses 2.3 (i)–(iii) imply that $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. So, $G(\cdot)$ is convex, $G(0) = 0$ and

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}, \qquad \nabla G(0) = 0.$$

Therefore, $G(\cdot)$ is the primitive of $a(\cdot)$. From the convexity of $G(\cdot)$ and since $G(0) = 0$, we have

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of the map $a(\cdot)$, which we will use in the sequel. These properties are straightforward consequences of Hypotheses 2.3 (i)–(iii) and of (2.1).

Lemma 2.5. *Under Hypotheses 2.3* (i)–(iii)*, the following hold:*
(a) $y \mapsto a(y)$ *is continuous and strictly monotone (hence maximal monotone, too),*
(b) $|a(y)| \leqslant c_4(1 + |y|^{p-1})$ *for all* $y \in \mathbb{R}^N$, *for some* $c_4 > 0$,
(c) $(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

This lemma and (2.2) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.6. *If Hypotheses 2.3* (i)–(iii) *hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1 + |y|^p) \quad \textit{for all } y \in \mathbb{R}^N,$$

for some $c_5 > 0$.

The examples which follow confirm the generality of Hypotheses 2.3.

Examples. The following maps satisfy Hypotheses 2.3:

(a) $a(y) = |y|^{p-2}y$, with $1 < p < \infty$. The corresponding differential operator is the p-Laplacian defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$, with $1 < q < p < \infty$, and the corresponding differential operator is the (p,q)-Laplacian defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics, see [7] (quantum physics) and [9] (plasma physics). Recently, there have been some existence and multiplicity results for such equations. We mention the papers [1, 2, 10, 27, 29, 32, 35, 38, 40, 42].

(c) $a(y) = (1 + |y|^2)^{(p-2)/2}y$, with $1 < p < \infty$. The corresponding differential operator is the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |Du|^2)^{(p-2)/2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$, with $1 < p < \infty$. The corresponding differential operator is defined by

$$\Delta_p u + \operatorname{div}\Big(\frac{|Du|^{p-2}Du}{1 + |Du|^p}\Big) \quad \text{for all } u \in W^{1,p}(\Omega).$$

This operator arises in problems of plasticity (see [13]).

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

The next proposition is a particular case of a more general result due to Gasinski and Papageorgiou [16].

Proposition 2.7. *If Hypotheses 2.3* (i)–(iii) *hold, then the map $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0 \implies u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

We introduce the following conditions on the coefficient functions $\xi(\cdot)$ and $\beta(\cdot)$:

(C1) $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for almost all $z \in \Omega$.

(C2) $\beta \in C^{0,\alpha}(\partial\Omega)$, with $\alpha \in (0,1)$, and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

(C3) $\xi \neq 0$ or $\beta \neq 0$.

Lemma 2.8. *If $\hat{\xi} \in L^\infty(\Omega)$ and $\hat{\xi}(z) \geqslant 0$ for almost all $z \in \Omega$, $\hat{\xi} \neq 0$, then there exists $c_6 > 0$ such that*

$$\|Du\|_p^p + \int_\Omega \hat{\xi}(z)|u|^p\,dz \geqslant c_6\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proof. Let $\psi\colon W^{1,p}(\Omega) \to \mathbb{R}_+$ be the C^1-functional defined by

$$\psi(u) = \|Du\|_p^p + \int_\Omega \hat{\xi}(z)|u|^p\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Arguing by contradiction, suppose that the lemma is not true. Since $\psi(\cdot)$ is p-homogeneous, we can find $\{u_n\}_{n\geqslant1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| = 1 \quad \text{for all } n \in \mathbb{N}, \qquad \psi(u_n) \to 0^+ \quad \text{as } n \to \infty. \tag{2.3}$$

Since $\{u_n\}_{n\geqslant1} \subseteq W^{1,p}(\Omega)$ is bounded, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega), \qquad u_n \to u \quad \text{in } L^p(\Omega). \tag{2.4}$$

The functional $\psi(\cdot)$ is sequentially weakly lower semicontinuous. So, from (2.3) and (2.4), we obtain $\psi(u) \leqslant 0$, which implies

$$\|Du\|_p^p \leqslant -\int_\Omega \hat{\xi}(z)|u|^p\,dz \leqslant 0, \tag{2.5}$$

hence $u = \eta \in \mathbb{R}$. If $\eta = 0$, then from (2.4) we see that $\|Du_n\|_p \to 0$, and thus

$$u_n \to 0 \quad \text{in } W^{1,p}(\Omega),$$

a contradiction to the fact that $\|u_n\| = 1$ for all $n \in \mathbb{N}$. If $\eta \neq 0$, then from (2.5) we have

$$0 \leqslant -|\eta|^p \int_\Omega \hat{\xi}(z)\,dz < 0,$$

a contradiction. □

Lemma 2.9. *If $\hat{\beta} \in L^\infty(\partial\Omega)$, $\hat{\beta}(z) \geqslant 0$ for σ-almost all $z \in \partial\Omega$, $\hat{\beta} \neq 0$, then there exists $c_7 > 0$ such that*

$$\|Du\|_p^p + \int_{\partial\Omega} \hat{\beta}(z)|u|^p\,d\sigma \geqslant c_7\|u\|^p \quad \textit{for all } u \in W^{1,p}(\Omega).$$

Proof. Let $\psi_0 \colon W^{1,p}(\Omega) \to \mathbb{R}_+$ be the C^1-functional defined by

$$\psi_0(u) = \|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

We claim that we can find $\hat{c}_0 > 0$ such that

$$\|u\|_p^p \leqslant \hat{c}_0\psi_0(u) \quad \text{for all } u \in W^{1,p}(\Omega). \tag{2.6}$$

Arguing by contradiction, suppose that (2.6) is not true. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\|_p^p > n\psi_0(u_n) \quad \text{for all } n \in \mathbb{N}.$$

Since ψ_0 is p-homogeneous, we normalize in $L^p(\Omega)$ and have

$$\psi_0(u_n) < \frac{1}{n} \quad \text{and} \quad \|u_n\|_p = 1 \quad \text{for all } n \in \mathbb{N}, \tag{2.7}$$

and thus $\psi_0(u_n) \to 0^+$ as $n \to \infty$. From (2.7), it follows that $\|Du_n\|_p \to 0$ as $n \to \infty$, hence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega), \qquad u_n \to u \quad \text{in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{2.8}$$

From (2.7), (2.8) and the sequential weak lower semicontinuity of $\psi_0(\cdot)$, we have $\psi_0(u) \leqslant 0$, hence

$$\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p\,d\sigma \leqslant 0. \tag{2.9}$$

Therefore, $u = \eta_0 \in \mathbb{R}$. If $\eta_0 = 0$, then from (2.8) we have $u_n \to 0$ in $L^p(\Omega)$, a contradiction with the fact that $\|u_n\|_p = 1$ for all $n \in \mathbb{N}$. If $\eta_0 \neq 0$, then from (2.9) we have

$$0 < |\eta_0|^p \int_{\partial\Omega} \beta(z)\,d\sigma \leqslant 0,$$

again a contradiction. Therefore, (2.6) holds and from this it follows that we can find $c_7 > 0$ such that

$$c_7\|u\|^p \leqslant \psi_0(u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

The proof is completed. □

Next we prove a strong comparison result which will be useful in what follows. This proposition was inspired by analogous comparison results for Dirichlet problems with the p-Laplacian as established by Guedda and Véron (see [18, Proposition 2.2]), and Arcoya and Ruiz (see [6, Proposition 2.6]).

Proposition 2.10. *Assume that Hypotheses 2.3* (i)–(iii) *hold,* $\hat{\xi} \in L^\infty(\Omega)$, $\hat{\xi}(z) \geqslant 0$ *for almost all* $z \in \Omega$, *and* $h_1, h_2 \in L^\infty(\Omega)$ *are such that*

$$0 < c_8 \leqslant h_2(z) - h_1(z) \quad \textit{for almost all } z \in \Omega.$$

Let $u, v \in C^1(\overline{\Omega}) \setminus \{0\}$, $u \leqslant v$, *satisfy*

$$\begin{aligned} -\operatorname{div} a(Du(z)) + \hat{\xi}(z)|u(z)|^{p-2}u(z) &= h_1(z) \quad \textit{for almost all } z \in \Omega,\\ -\operatorname{div} a(Dv(z)) + \hat{\xi}(z)|v(z)|^{p-2}v(z) &= h_2(z) \quad \textit{for almost all } z \in \Omega. \end{aligned}$$

Then $(v-u)(z) > 0$ *for all* $z \in \Omega$, *and if* $\Sigma_0 = \{z \in \partial\Omega : u(z) = v(z)\}$, *then*

$$\frac{\partial(v-u)}{\partial n}\Big|_{\Sigma_0} < 0.$$

Proof. We have

$$-\operatorname{div}(a(Dv(z)) - a(Du(z))) = h_2(z) - h_1(z) - \hat{\xi}(z)(|v(z)|^{p-2}v(z) - |u(z)|^{p-2}u(z)) \quad \text{for almost all } z \in \Omega. \tag{2.10}$$

Let $a = (a_k)_{k=1}^N$ with $a_k : \mathbb{R}^N \to \mathbb{R}$ being the kth component function, $k \in \{1, \dots, N\}$. From the mean value theorem, we have

$$a_k(y) - a_k(y') = \sum_{i=1}^N \int_0^1 \frac{\partial a_k}{\partial y_i}(y' + t(y-y'))(y_i - y_i')\,dt$$

for all $y = (y_i)_{i=1}^N \in \mathbb{R}^N$, $y' = (y_i')_{i=1}^N \in \mathbb{R}^N$ and all $k \in \{1, \dots, N\}$.

Consider the functions

$$\tilde{c}_{k,i}(z) = \int_0^1 \frac{\partial a_k}{\partial y_i}(Du(z) + t(Dv(z) - Du(z)))(D_i v(z) - D_i u(z))\,dt, \quad z \in \Omega,\ k \in \{1, \dots, N\}.$$

Then $\tilde{c}_{k,i} \in C(\overline{\Omega})$, and using these functions we introduce the following linear differential operator in divergence form:

$$L(w) = -\operatorname{div}\Big(\sum_{i=1}^N \tilde{c}_{k,i}(z)\frac{\partial w}{\partial z_i}\Big) = -\sum_{k,i=1}^N \frac{\partial}{\partial z_k}\Big(\tilde{c}_{k,i}(z)\frac{\partial w}{\partial z_i}\Big), \quad w \in H^1(\Omega).$$

We set $y = v - u \in C_+ \setminus \{0\}$. From (2.10) we have

$$L(y) = h_2(z) - h_1(z) - \hat{\xi}(z)(|v(z)|^{p-2}v(z) - |u(z)|^{p-2}u(z)) \quad \text{for almost all } z \in \Omega. \tag{2.11}$$

Suppose that at $z_0 \in \Omega$, we have $u(z_0) = v(z_0)$. Exploiting the uniform continuity of the map $x \mapsto |x|^{p-2}x$ and the fact that $\hat{\xi} \in L^\infty(\Omega)$, from (2.11) we see that for $\delta > 0$ sufficiently small, we have

$$L(y) \geqslant \frac{c_8}{2} > 0 \quad \text{for almost all } z \in B_\delta(z_0).$$

Then invoking Harnack's inequality (see [26, p. 212]) or alternatively using the tangency principle of Pucci and Serrin [41, p. 35], we have

$$(v-u)(z) > 0 \quad \text{for all } z \in B_\delta(z_0),$$

a contradiction since $u(z_0) = v(z_0)$. Therefore, we must have that

$$(v-u)(z) > 0 \quad \text{for all } z \in \Omega.$$

Next suppose that $\hat{z}_0 \in \Sigma_0$. Since $\partial\Omega$ is a C^2-manifold, for $\rho > 0$ small, there exists a ρ-ball B_ρ such that

$$B_\rho \subseteq \Omega \quad \text{and} \quad \hat{z}_0 \in \partial\Omega \cap \partial B_\rho.$$

Choosing $\rho > 0$ small, from (2.11) and since $u(\hat{z}_0) = v(\hat{z}_0)$ (recall that $\hat{z}_0 \in \Sigma_0$), we see that $L(\cdot)$ is strictly elliptic. Then, from Hopf's theorem (see [26, p. 217] and [41, p. 120]), we have

$$\frac{\partial y}{\partial n}(z_0) = \frac{\partial(v-u)}{\partial n}(z_0) < 0,$$

and hence

$$\frac{\partial(v-u)}{\partial n}\Big|_{\Sigma_0} < 0. \qquad \square$$

Remark 2.11. With $\Sigma_0 = \{z \in \partial\Omega : u(z) = v(z)\}$, we introduce the following Banach spaces:

$$C^1_*(\overline{\Omega}) = \{h \in C^1(\overline{\Omega}) : h|_{\Sigma_0} = 0\}, \quad W^{1,p}_*(\Omega) = \overline{C^1_*(\overline{\Omega})}^{\|\cdot\|}$$

(recall that $\|\cdot\|$ is the norm of $W^{1,p}(\Omega)$).

From Proposition 2.10, we have

$$\frac{\partial(v-u)}{\partial n}\Big|_{\Sigma_0} \leqslant -\eta < 0.$$

Let U be a neighborhood of Σ_0 in $\overline{\Omega}$ such that

$$\frac{\partial(v-u)}{\partial n}\Big|_{U} \leqslant -\frac{\eta}{2} < 0.$$

Then we can find $\epsilon > 0$ small such that $h \in C^1_*(\overline{\Omega})$ and $\|h\|_{C^1(\overline{\Omega})} \leqslant \epsilon$. Therefore,

$$\frac{\partial(v-(u+h))}{\partial h} \leqslant -\frac{\eta}{4} < 0 \tag{2.12}$$

and

$$(v-(u_0+h))|_{\overline{\Omega}\setminus U} \geqslant \hat{\eta} > 0. \tag{2.13}$$

From (2.12) we see that for $\epsilon > 0$ small, we have

$$v(z) - (u+h)(z) \geqslant 0 \quad \text{for all } z \in U \text{ and all } h \in C^1_*(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \leqslant \epsilon.$$

Comparing this with (2.13), we see that $u + B^c_\epsilon \in v - C^*_+(\Sigma_0)$, with B^c_ϵ being the ϵ-ball centered at zero in $C^1_*(\overline{\Omega})$, and $C^*_+(\Sigma_0)$ is the positive cone of $C^1_*(\overline{\Omega})$. This cone has a nonempty interior given by

$$\operatorname{int} C^*_+(\Sigma_0) = \Big\{h \in C^*_+ : h(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial h}{\partial n}\Big|_{\Sigma_0} < 0\Big\}.$$

If $\Sigma_0 = \emptyset$, then $v - u \in D_+$.

The next result is an outgrowth of the nonlinear regularity theory of Lieberman [25] and can be found in [28] (subcritical case) and in [36] (critical case).

So, let V and X be two Banach subspaces of $C^1(\overline{\Omega})$ and $W^{1,p}(\Omega)$, respectively, such that V is dense in X. Suppose that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f_0(z,x)| \leqslant a_0(z)(1+|x|^{r-1}) \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$, $1 < r \leqslant p^*$. We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$, and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega F_0(z,u)\,dz, \quad u \in W^{1,p}(\Omega).$$

Proposition 2.12. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local V-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \text{for all } h \in V,\ \|h\|_{C^1(\overline{\Omega})} \leqslant \rho_0.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is also a local X-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \text{for all } h \in X,\ \|h\| \leqslant \rho_1.$$

We conclude this section with some notation that we will use throughout this work. For every $x \in \mathbb{R}$, let $x^{\pm} = \max\{\pm x, 0\}$. Then, for $u \in W^{1,p}(\Omega)$, we set $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u = u^+ - u^-, \quad |u| = u^+ + u^-, \quad u^+, u^- \in W^{1,p}(\Omega).$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, if X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

3 Bifurcation-type theorem

In this section we prove a bifurcation-type theorem for problem (1.1) for small values of the parameter $\lambda > 0$.

We introduce the following conditions on the reaction term $f(z,x)$.

Hypotheses 3.1. Assume that $f : \Omega \times \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$, and that the following hold:

(i) $f(z,x) \leqslant a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$ and all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$.

(ii) If $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$\lim_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) If $e(z,x) = f(z,x)x - pF(z,x)$, then there exists $d \in L^1(\Omega)$ such that

$$e(z,x) \leqslant e(z,y) + d(z) \quad \text{for almost all } z \in \Omega \text{ and all } 0 \leqslant x \leqslant y.$$

(iv) For every $s > 0$, we can find $\eta_s > 0$ such that

$$\eta_s \leqslant \inf\{f(z,x) : x \geqslant s\} \quad \text{for almost all } z \in \Omega,$$

and there exist $\delta_0 > 0$, $\hat{\eta}, \hat{\eta}_0 > 0$ and $\tau \in (1,q)$ (see Hypothesis 2.3 (iv)) such that

$$\hat{\eta}_0 x^{\tau-1} \leqslant f(z,x) \leqslant \hat{\eta} x^{\tau-1} \quad \text{for almost all } z \in \Omega \text{ and all } 0 \leqslant x \leqslant \delta_0.$$

(v) For every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$, the function $x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 3.2. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis, without any loss of generality, we may assume that $f(z,x) = 0$ for almost all $z \in \Omega$ and all $x \leqslant 0$. Hypotheses 3.1 (ii)–(iii) imply that

$$\lim_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \quad \text{uniformly for almost all } z \in \Omega.$$

So, the reaction term $f(z,\cdot)$ is $(p-1)$-superlinear. However, we stress that we do not use the usual AR-condition for "superlinear" problems. We recall that the AR-condition (unilateral version, since we deal only with the positive semiaxis) says that there exist $\vartheta > p$ and $M > 0$ such that

$$0 < \vartheta F(z,x) \leqslant f(z,x)x \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant M, \tag{3.1}$$

and (see [5])

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{3.2}$$

Integrating (3.1) and using (3.2), we obtain the weaker condition

$$c_9 x^{\vartheta} \leqslant F(z,x) \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant M, \tag{3.3}$$

for some $c_9 > 0$. Therefore, the AR-condition implies that $f(z,\cdot)$ has at least $(\vartheta-1)$-polynomial growth near $+\infty$. This excludes from consideration $(p-1)$-superlinear nonlinearities with "slower" growth near $+\infty$ (see the examples below). For this reason, in this work we use the less restrictive Hypothesis 3.1 (iii). This is a quasimonotonicity condition on the function $e(z,\cdot)$. This is a slightly more general version of a condition used by Li and Yang [24]. If there exists $M > 0$ such that for almost all $z \in \Omega$, the function $x \mapsto \frac{f(z,x)}{x^{p-1}}$ is nondecreasing on $[M,+\infty)$, then Hypothesis 3.1 (iii) is satisfied (see [24]). Evidently, this property is weaker than condition (3.3).

Examples. The following functions satisfy Hypotheses 3.1; for the sake of simplicity, we drop the z-dependence:

$$f_1(x) = \begin{cases} x^{\tau-1} & \text{if } x \in [0,1], \\ x^{r-1} & \text{if } 1 \leqslant x, \end{cases} \qquad \text{with } 1 < \tau < q < p < r < p^*,$$

$$f_2(x) = \begin{cases} x^{\tau-1} - x^{s-1} & \text{if } x \in [0,1], \\ x^{p-1}\ln x & \text{if } 1 \leqslant x, \end{cases} \qquad \text{with } 1 < \tau < p, s.$$

Note that $f_2(\cdot)$ does not satisfy the AR-condition.

Hypotheses 3.1 (i), (iv) imply that

$$0 \leqslant f(z,x) \leqslant \hat{\eta} x^{\tau-1} + c_{10} x^{r-1} \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0, \tag{3.4}$$

for some $c_{10} > 0$. This growth estimate on $f(z,\cdot)$ leads to the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda(\hat{\eta}u(z)^{\tau-1} + c_{10}u(z)^{r-1}) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

with $u > 0$ and $\lambda > 0$.

Proposition 3.3. *If Hypotheses 2.3 and conditions* (C1)–(C3) *hold, and* $1 < \tau < q < p < r < p^*$, *then for* $\lambda > 0$ *small, problem* (3.5) *admits a positive solution* $\tilde{u}_\lambda \in D_+$.

Proof. For $\lambda > 0$, we consider the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \frac{\lambda\hat{\eta}}{\tau}\|u^+\|_\tau^\tau - \frac{\lambda c_{10}}{r}\|u^+\|_r^r, \quad u \in W^{1,p}(\Omega).$$

Claim 1. For every $\lambda > 0$, the functional ψ_λ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ such that

$$|\psi_\lambda(u_n)| \leqslant M_1 \quad \text{for all } n \in \mathbb{N} \text{ (for some } M_1 > 0), \tag{3.6}$$

$$(1 + \|u_n\|)\psi_\lambda'(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{3.7}$$

From (3.7) we have

$$\Big|\langle A(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\,dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\,d\sigma - \lambda\hat{\eta}\int_\Omega (u_n^+)^{\tau-1}h\,dz - -\lambda c_{10}\int_\Omega (u_n^+)^{r-1}h\,dz\Big|$$
$$\leqslant \frac{\epsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.8}$$

In (3.8) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\int_\Omega (a(-Du_n^-), -Du_n^-)_{\mathbb{R}^N}\,dz + \int_\Omega \xi(z)(u_n^-)^p\,dz + \int_{\partial\Omega} \beta(z)(u_n^-)^p\,d\sigma \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N},$$

and thus (see Lemma 2.5)

$$\frac{c_1}{p-1}\|Du_n^-\|_p^p + \int_\Omega \xi(z)(u_n^-)^p\,dz + \int_{\partial\Omega} \beta(z)(u_n^-)^p\,d\sigma \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N}.$$

Hence, $c_{11}\|u_n^-\|^p \leqslant \epsilon_n$ for all $n \in \mathbb{N}$, for some $c_{11} > 0$ (see (C3) and Lemmata 2.8 and 2.9), and therefore

$$u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega). \tag{3.9}$$

We can always assume that $r_0 \leqslant r < p^*$ (see Hypotheses 2.3 (iv) and 3.1 (i)). From (3.6) and (3.9), we have that

$$\int_\Omega rG(Du_n^+)\,dz + \frac{r}{p}\int_\Omega \xi(z)(u_n^+)^p + \frac{r}{p}\int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma - \frac{\lambda\hat{\eta}r}{\tau}\|u_n^+\|_\tau^\tau - \lambda c_{10}\|u_n^+\|_r^r \leqslant M_2 \quad \text{for all } n \in \mathbb{N}, \tag{3.10}$$

for some $M_2 > 0$. In (3.8) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}\,dz - \int_\Omega \xi(z)(u_n^+)^p\,dz - \int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma + \lambda\hat{\eta}\|u_n^+\|_\tau^\tau + \lambda c_{10}\|u_n^+\|_r^r \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

We add (3.10) and (3.11), and obtain

$$\begin{aligned}\int_\Omega [rG(Du_n^+) - (a(Du_n^+), Du_n^+)_{\mathbb{R}^N}]\,dz + \Big(\frac{r}{p} - 1\Big)\Big[\int_\Omega \xi(z)(u_n^+)^p\,dz + \int_{\partial\Omega} \beta(z)(u_n^+)^p\,d\sigma\Big] \\ \leqslant M_3(1 + \lambda\|u_n^+\|_\tau^\tau) \quad \text{for all } n \in \mathbb{N},\end{aligned}$$

for some $M_3 > 0$. Therefore, by Hypothesis 2.3 (iv), condition (C3), Lemmata 2.8 and 2.9, and the fact that $r > p$, we have

$$c_{12}\|u_n^+\|^p \leqslant M_3(1 + \lambda\|u_n^+\|^\tau) \quad \text{for all } n \in \mathbb{N}, \tag{3.12}$$

for some $c_{12} > 0$. Since $\tau < p$, from (3.12) it follows that $\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded, which implies that $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded (see (3.9)). So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W^{1,p}(\Omega), \qquad u_n \to u \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.13}$$

In (3.8) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.13). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

hence (see (3.13) and Proposition 2.7)

$$u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

Therefore, for every $\lambda > 0$, ψ_λ satisfies the C-condition.

This proves claim 1.

Claim 2. There exist $\rho > 0$ and $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, we have

$$\inf\{\psi_\lambda(u) : \|u\| = \rho\} = m_\lambda > 0 = \psi_\lambda(0).$$

For every $u \in W^{1,p}(\Omega)$, we have (see Corollary 2.6, Lemmata 2.8 and 2.9, and (C3))

$$\psi_\lambda(u) \geqslant c_{13}\|u\|^p - \lambda c_{14}(\|u\|^\tau + \|u\|^r) = [c_{13} - \lambda c_{14}(\|u\|^{\tau-p} + \|u\|^{r-p})]\|u\|^p, \tag{3.14}$$

for some $c_{13}, c_{14} > 0$. Let $\mathbb{J}(t) = t^{\tau-p} + t^{r-p}$, $t > 0$. Since $\tau < p < r$, we have

$$\mathbb{J}(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore, we can find $t_0 \in (0, +\infty)$ such that $\mathbb{J}(t_0) = \inf_{t>0} \mathbb{J}$.

From (3.14) we see that

$$\psi_\lambda(u) \geqslant [c_{13} - \lambda c_{14}\mathbb{J}\|u\|]\|u\|^p \quad \text{for all } u \in W^{1,p}(\Omega).$$

If $\|u\| = t_0$, then we set $\lambda_0 = \frac{c_{13}}{c_{14}\mathbb{J}(t_0)} > 0$ and for all $\lambda \in (0, \lambda_0)$, we see that

$$\inf\{\psi_\lambda(u) : \|u\| = \rho = t_0\} = m_\lambda > 0 = \psi_\lambda(0).$$

This proves claim 2.

Since $r > p$, if $u \in D_+$, then

$$\psi_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.15}$$

Claims 1 and 2 and (3.15) permit the use of Theorem 2.1 (the mountain pass theorem). So, for every $\lambda \in (0, \lambda_0)$, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\tilde{u}_\lambda \in K_{\psi_\lambda} \quad \text{and} \quad m_\lambda \leqslant \psi_\lambda(\tilde{u}_\lambda). \tag{3.16}$$

From (3.16) and claim 2, it follows that $\tilde{u}_\lambda \neq 0$ and $\psi'_\lambda(\tilde{u}_\lambda) = 0$. Therefore,

$$\begin{aligned}\langle A(\tilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda h\,d\sigma \\ = \lambda\hat{\eta}\int_\Omega (\tilde{u}_\lambda^+)^{\tau-1}h\,dz + \lambda c_{10}\int_\Omega (\tilde{u}_\lambda^+)^{r-1}h\,dz \quad \text{for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{3.17}$$

In (3.17) we choose $h = -\tilde{u}_\lambda^- \in W^{1,p}(\Omega)$. Then, by Lemma 2.5,

$$\frac{c_1}{p-1}\|D\tilde{u}_\lambda^-\|_p^p + \int_\Omega \xi(z)(\tilde{u}_\lambda^-)^p\,dz + \int_{\partial\Omega} \beta(z)(\tilde{u}_\lambda^-)^p\,d\sigma \leqslant 0.$$

Hence, $c_{15}\|\tilde{u}_\lambda^-\|^p \leqslant 0$ for some $c_{15} > 0$ (see (C3) and Lemmata 2.8 and 2.9), and thus $\tilde{u}_\lambda \geqslant 0$, $\tilde{u}_\lambda \neq 0$. Then (3.17) becomes

$$\langle A(\tilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)\tilde{u}_\lambda^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_\lambda^{p-1}h\,d\sigma = \int_\Omega [\lambda\hat{\eta}\tilde{u}_\lambda^{\tau-1} + \lambda c_{10}\tilde{u}_\lambda^{r-1}]h\,dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

which implies (see [28])

$$\begin{cases} -\operatorname{div} a(D\tilde{u}_\lambda(z)) + \xi(z)\tilde{u}_\lambda(z)^{p-1} = \lambda[\hat{\eta}\tilde{u}_\lambda(z)^{\tau-1} + c_{10}\tilde{u}_\lambda(z)^{r-1}] & \text{for almost all } z \in \Omega, \\ \dfrac{\partial\tilde{u}_\lambda}{\partial n_a} + \beta(z)\tilde{u}_\lambda^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

From (3.18), [22] (subcritical case), and [36] (critical case), we have $\tilde{u}_\lambda \in L^\infty(\Omega)$. Then, from [25], we infer that $\tilde{u}_\lambda \in C_+ \setminus \{0\}$. From (3.18), (C1) and (C2), we have

$$\operatorname{div} a(D\tilde{u}_\lambda(z)) \leqslant \|\xi\|_\infty \tilde{u}_\lambda(z)^{p-1} \quad \text{for almost all } z \in \Omega.$$

Therefore, $\tilde{u}_\lambda \in D_+$ (see [41, pp. 111, 120]). □

In fact, we can show that for every $\lambda \in (0, \lambda_0)$, problem (3.5) admits a smallest positive solution.

Let $\tilde{S}_+^\lambda$ be the set of positive solutions of problem (3.5). We have seen in Proposition 3.3 and its proof that

$$\emptyset \neq \tilde{S}_+^\lambda \subseteq D_+ \quad \text{for all } \lambda \in (0, \lambda_0).$$

Moreover, as in [12], we have that $\tilde{S}_+^\lambda$ is downward directed (that is, if $\tilde{u}_1, \tilde{u}_2 \in \tilde{S}_+^\lambda$, then we can find $\tilde{u} \in \tilde{S}_+^\lambda$ such that $\tilde{u} \leqslant \tilde{u}_1$ and $\tilde{u} \leqslant \tilde{u}_2$).

Proposition 3.4. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, and* $\lambda \in (0, \lambda_0)$, *then problem* (3.5) *admits a smallest positive solution* $\tilde{u}_\lambda \in \tilde{S}_+^\lambda \subseteq D_+$ *(that is,* $\tilde{u}_\lambda \leqslant u$ *for all* $u \in \tilde{S}_+^\lambda$*).*

Proof. We consider the following Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda\hat{\eta}u(z)^{\tau-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.19}$$

with $u > 0$, $\lambda > 0$.

Since $\tau < p$, a straightforward application of the direct method of the calculus of variations reveals that for every $\lambda > 0$, problem (3.19) admits a positive solution $\overline{u}_\lambda \in D_+$ (nonlinear regularity theory and the nonlinear maximum principle).

Claim 3. $\overline{u}_\lambda \in D_+$ is the unique positive solution of problem (3.19).

Consider the integral functional $j\colon L^1(\Omega) \to \overline{R} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega G(Du^{1/q})\,dz + \frac{1}{p}\int_\Omega \xi(z)(u^{p/q})\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^{p/q})\,d\sigma & \text{if } u \geqslant 0,\ w^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of the functional $j(\cdot)$) and set

$$u = ((1-t)u_1 + tu_2)^{1/q}, \quad t \in [0,1].$$

Using [11, Lemma 1], we have

$$|Du(z)| \leqslant \big[(1-t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q\big] \quad \text{for almost all } z \in \Omega.$$

Then, by Hypothesis 2.3 (iv) and since $G_0(\cdot)$ is increasing, we have

$$\begin{aligned} G_0(|Du(z)|) &\leqslant G_0\big((1-t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q\big) \\ &\leqslant (1-t)G_0(|Du_1(z)^{1/q}|) + tG_0(|Du_2(z)|^{1/q}) \quad \text{for almost all } z \in \Omega. \end{aligned}$$

Therefore,

$$G(Du(z)) \leqslant (1-t)G(Du_1(z))^{1/q} + tG(Du_2(z)^{1/q}) \quad \text{for almost all } z \in \Omega,$$

and thus $j(\cdot)$ is convex (recall that $q < p$ and see (C1)–(C2)).

By Fatou's lemma, we see that $j(\cdot)$ is also lower semicontinuous.

Let $\overline{v}_\lambda \in W^{1,p}(\Omega)$ be another positive solution of problem (3.19). Again we have $\overline{v}_\lambda \in D_+$. If $h \in C^1(\overline{\Omega})$, then for $t > 0$ small, we have $\overline{u}_\lambda^q + th \in \operatorname{dom} j$ and $\overline{v}_\lambda^q + th \in \operatorname{dom} j$. Then we can easily show that $j(\cdot)$ is Gâteaux differentiable at $\overline{u}_\lambda^q$ and at $\overline{v}_\lambda^q$ in the direction h. Moreover, via the chain rule and the nonlinear Green's theorem (see [15, p. 210]), we have

$$\begin{aligned} j'(\overline{u}_\lambda^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\overline{u}_\lambda) + \xi(z)\overline{u}_\lambda^{p-1}}{\overline{u}_\lambda^{q-1}} h\,dz, \\ j'(\overline{v}_\lambda^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\overline{v}_\lambda) + \xi(z)\overline{v}_\lambda^{p-1}}{\overline{v}_\lambda^{q-1}} h\,dz \end{aligned}$$

for all $h \in W^{1,p}(\Omega)$. The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So (see problem (3.19))

$$\begin{aligned} 0 &\leqslant \int_\Omega \Big(\frac{-\operatorname{div} a(D\overline{u}_\lambda) + \xi(z)\overline{u}_\lambda^{p-1}}{\overline{u}_\lambda^{q-1}} - \frac{-\operatorname{div} a(D\overline{v}_\lambda) + \xi(z)\overline{v}_\lambda^{p-1}}{\overline{v}_\lambda^{q-1}}\Big)(\overline{u}_\lambda^q - \overline{v}_\lambda^q)\,dz \\ &\leqslant \lambda\hat{\eta}\int_\Omega \Big[\frac{1}{\overline{u}_\lambda^{\tau-q}} - \frac{1}{\overline{v}_\lambda^{\tau-q}}\Big](\overline{u}_\lambda^q - \overline{v}_\lambda^q)\,dz, \end{aligned}$$

and hence $\overline{u}_\lambda = \overline{v}_\lambda$ (since $\tau < q$).

This proves claim 3.

Claim 4. $\overline{u}_\lambda \leqslant u$ for all $u \in \tilde{S}_+^\lambda$.

Let $u \in \tilde{S}_+^\lambda$. We introduce the following Carathéodory function:

$$k_\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \lambda\hat{\eta}x^{\tau-1} & \text{if } 0 \leqslant x \leqslant u(z), \\ \lambda\hat{\eta}u(z)^{\tau-1} & \text{if } u(z) < x, \end{cases} \tag{3.20}$$

for $(z,x) \in \Omega \times \mathbb{R}$.

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$ and consider the C^1-functional $\overline{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\overline{\psi}_\lambda(y) = \int_\Omega G(Dy)\,dz + \frac{1}{p}\int_\Omega \xi(z)|y|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|y|^p\,d\sigma - \int_\Omega K_\lambda(z,y)\,dz, \quad y \in W^{1,p}(\Omega).$$

From (3.20), Lemma 2.5 and (C3), together with Lemmata 2.8 and 2.9, we see that the functional $\overline{\psi}_\lambda$ is coercive. Also, the Sobolev embedding theorem and the compactness of the trace map, imply that $\overline{\psi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\overline{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\overline{\psi}_\lambda(\overline{u}_\lambda^*) = \inf\{\overline{\psi}_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{3.21}$$

Hypothesis 2.3 (iv) and Corollary 2.6 imply that

$$G(y) \leqslant c_{16}(|y|^q + |y|^p) \quad \text{for all } y \in \mathbb{R}^N, \tag{3.22}$$

for some $c_{16} > 0$. Since $\tau < q < p$, if $v \in D_+$, then for $t \in [0,1]$ small (such that $tv \leqslant u$, recall that $u \in D_+$), we have

$$\overline{\psi}(tv) \leqslant c_{16}t^q(\|Dv\|_q^q + \|Dv\|_p^p) + \frac{t^p}{p}\Big[\int_\Omega \xi(z)v^p\,dz + \int_{\partial\Omega} \beta(z)v^p\,d\sigma\Big] - \frac{\lambda\hat{\eta}t^\tau}{\tau}\|v\|_\tau^\tau < 0.$$

Therefore, by (3.21), $\overline{\psi}_\lambda(\overline{u}_\lambda^*) < 0 = \overline{\psi}_\lambda(0)$, and hence $\overline{u}_\lambda^* \neq 0$.

From (3.21) we have $\overline{\psi}_\lambda'(\overline{u}_\lambda^*) = 0$. Thus,

$$\langle A(\overline{u}_\lambda^*), h\rangle + \int_\Omega \xi(z)|\overline{u}_\lambda^*|^{p-2}\overline{u}_\lambda^* h\,dz + \int_{\partial\Omega} \beta(z)|\overline{u}_\lambda^*|^{p-2}\overline{u}_\lambda^* h\,d\sigma = \int_\Omega k_\lambda(z,\overline{u}_\lambda^*)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.23}$$

In (3.23) we first choose $-(\overline{u}_\lambda^*)^- \in W^{1,p}(\Omega)$. Then, by (3.21), Lemmata 2.5, 2.8 and 2.9, and (C3), we have

$$c_{17}\|(\overline{u}_\lambda^*)^-\|^p \leqslant 0 \quad \text{for some } c_{17} > 0,$$

hence $\overline{u}_\lambda^* \geqslant 0$, $\overline{u}_\lambda^* \neq 0$.

Next, in (3.23) we choose $h = (\overline{u}_\lambda^* - u)^+ \in W^{1,p}(\Omega)$. Then, by (3.20) and since $u \in \tilde{S}_+^\lambda$, we have

$$\begin{aligned}
\langle A(\overline{u}_\lambda^*), (\overline{u}_\lambda^* - u)^+\rangle &+ \int_\Omega \xi(z)(\overline{u}_\lambda^*)^{p-1}(\overline{u}_\lambda^* - u)^+\,dz + \int_{\partial\Omega} \beta(z)(\overline{u}_\lambda^*)^{p-1}(\overline{u}_\lambda^* - u)^+\,d\sigma \\
&= \int_\Omega \lambda\hat{\eta}u^{\tau-1}(\overline{u}_\lambda^* - u)^+\,dz \\
&\leqslant \int_\Omega [\lambda\hat{\eta}u^{\tau-1} + \lambda c_{10}u^{r-1}](\overline{u}_\lambda^* - u)^+\,dz \\
&= \langle A(u), (\overline{u}_\lambda^* - u)^+\rangle + \int_\Omega \xi(z)u^{p-1}(\overline{u}_\lambda^* - u)^+\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\overline{u}_\lambda^* - u)^+\,d\sigma.
\end{aligned}$$

Therefore, $\overline{u}_\lambda^* \leqslant u$.

So, we have proved that

$$\overline{u}_\lambda^* \in [0,u] = \{y \in W^{1,p}(\Omega) : 0 \leqslant y(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}, \quad \overline{u}_\lambda^* \neq 0,$$

that is, $\overline{u}_\lambda^*$ is a positive solution of (3.19), and hence, by claim 3, $\overline{u}_\lambda^* = \overline{u}_\lambda$. Therefore, $\overline{u}_\lambda \leqslant u$ for all $u \in \tilde{S}_+^\lambda$, which proves claim 4.

Invoking [20, Lemma 3.10], we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq \tilde{S}_+^\lambda$ such that

$$\inf \tilde{S}_+^\lambda = \inf_{n\geqslant 1} u_n.$$

Evidently, $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded, and so we may assume that

$$u_n \xrightarrow{w} \tilde{u}_\lambda^* \quad \text{in } W^{1,p}(\Omega), \qquad u_n \to \tilde{u}_\lambda^* \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.24}$$

In (3.23) we choose $h = u_n - \tilde{u}_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.24). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - \tilde{u}_\lambda^* \rangle = 0,$$

and therefore (see (3.24) and Proposition 2.7)

$$u_n \to \tilde{u}_\lambda^* \quad \text{in } W^{1,p}(\Omega). \tag{3.25}$$

So, if in (3.23) we pass to the limit as $n \to \infty$ and use (3.25), then

$$\langle A(\tilde{u}_\lambda^*), h\rangle + \int_\Omega \xi(z)(\tilde{u}_\lambda^*)^{p-1} h\, dz + \int_{\partial\Omega} \beta(z)(\tilde{u}_\lambda^*)^{p-1} h\, d\sigma = \int_\Omega [\lambda\hat{\eta}(\tilde{u}_\lambda^*)^{\tau-1} + \lambda c_{10}(\tilde{u}_\lambda^*)^{r-1}] h\, dz \tag{3.26}$$

for all $h \in W^{1,p}(\Omega)$. Also, from claim 4, we have $\overline{u}_\lambda \leqslant u_n$ for all $n \in \mathbb{N}$, and thus

$$\overline{u}_\lambda \leqslant \tilde{u}_\lambda^*. \tag{3.27}$$

From (3.26) and (3.27) it follows that $\tilde{u}_\lambda^* \in \tilde{S}_+^\lambda$ and $\tilde{u}_\lambda^* = \inf \tilde{S}_+^\lambda$. □

Let

$$\mathcal{L} = \{\lambda > 0 : \text{problem (1.1) admits a positive solution}\}.$$

Proposition 3.5. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, then* $\mathcal{L} \neq \emptyset$.

Proof. Let $\tilde{u}_\lambda^* \in \tilde{S}_+^\lambda \subseteq D_+$ be the minimal positive solution of problem (3.5) ($\lambda \in (0, \lambda_0)$), see Proposition 3.4. We introduce the following truncation of the reaction term in problem (1.1):

$$\gamma_\lambda(z,x) = \begin{cases} \lambda f(z,x) & \text{if } x \leqslant \tilde{u}_\lambda^*(z), \\ \lambda f(z, \tilde{u}_\lambda^*(z)) & \text{if } \tilde{u}_\lambda^*(z) < x. \end{cases} \tag{3.28}$$

This is a Carathéodory function. We set $\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du)\, dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\, d\sigma - \int_\Omega \Gamma_\lambda(z,u)\, dz, \quad u \in W^{1,p}(\Omega).$$

From (3.28), Corollary 2.6, (C3), and Lemmata 2.8 and 2.9, we see that $\hat{\varphi}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda(u_\lambda) = \inf\{\hat{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{3.29}$$

Let $\delta_0 > 0$ be as postulated by Hypothesis 3.1 (iv). Given $u \in D_+$, we can find $t \in (0,1)$ small such that

$$tu(z) \in (0, \delta_0] \quad \text{for all } z \in \overline{\Omega}.$$

Then Hypothesis 3.1 (iv) implies that

$$F(z, tu(z)) \geqslant \frac{\hat{\eta}_0}{\tau}(tu(z))^\tau \quad \text{for almost all } z \in \Omega.$$

We have (see (3.22) and recall that $t \in (0,1)$)

$$\hat{\varphi}_\lambda(tu) \leqslant c_{16}t^q(\|Du\|_q^q + \|Du\|_p^p) + \frac{t^p}{p}\int_\Omega \xi(z)u^p\,dz + \frac{t^p}{p}\int_{\partial\Omega}\beta(z)u^p\,d\sigma - \frac{\lambda\hat{\eta}_0}{\tau}t^\tau\|u\|_\tau^\tau \leqslant c_{18}t^q - \lambda c_{19}t^p \tag{3.30}$$

for some $c_{18}, c_{19} > 0$. Since $\tau < q < p$, from (3.30) it follows that by choosing $t \in (0,1)$ even smaller if necessary, we have $\hat{\varphi}(tu) < 0$, and thus, by (3.29), $\hat{\varphi}_\lambda(u_\lambda) < 0 = \hat{\varphi}_\lambda(0)$. Therefore, $u_\lambda \neq 0$.

From (3.29) we have $\hat{\varphi}_\lambda'(u_\lambda) = 0$, hence

$$\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h\,dz + \int_{\partial\Omega}\beta(z)|u_\lambda|^{p-2}u_\lambda h\,d\sigma = \int_\Omega \gamma_\lambda(z,u_\lambda)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.31}$$

In (3.31) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then, as before,

$$c_{20}\|u_\lambda^-\|^p \leqslant 0 \quad \text{for some } c_{20} > 0,$$

and thus $u_\lambda \geqslant 0$, $u_\lambda \neq 0$.

Also, in (3.31) we choose $h = (u_\lambda - \tilde{u}_\lambda^*)^+ \in W^{1,p}(\Omega)$. Then, by (3.28), (3.4) and since $\tilde{u}_\lambda^* \in \tilde{S}_+^\lambda$, we have

$$\begin{aligned}
\langle A(u_\lambda), (u_\lambda - \tilde{u}_\lambda^*)^+\rangle &+ \int_\Omega \xi(z)u_\lambda^{p-1}(u_\lambda - \tilde{u}_\lambda^*)^+\,dz + \int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\lambda - \tilde{u}_\lambda^*)^+\,d\sigma \\
&= \int_\Omega \lambda f(z,\tilde{u}_\lambda^*)(u_\lambda - \tilde{u}_\lambda^*)^+\,dz \\
&\leqslant \int_\Omega \lambda\left[\hat{\eta}(\tilde{u}_\lambda^*)^{\tau-1} + c_{10}(\tilde{u}_\lambda^*)^{r-1}\right](u_\lambda - \tilde{u}_\lambda^*)^+\,dz \\
&= \langle A(\tilde{u}_\lambda^*), (u_\lambda - \tilde{u}_\lambda^*)^+\rangle + \int_\Omega \xi(z)(\tilde{u}_\lambda^*)^{p-1}(u_\lambda - \tilde{u}_\lambda^*)^+\,dz + \int_{\partial\Omega}\beta(z)(\tilde{u}_\lambda^*)^{p-1}(u_\lambda - \tilde{u}_\lambda^*)^+\,d\sigma.
\end{aligned}$$

Therefore, $u_\lambda \leqslant \tilde{u}_\lambda^*$.

So, we have proved that $u_\lambda \in [0, \tilde{u}_\lambda^*]$, $u_\lambda \neq 0$, and hence u_λ is a positive solution of problem (1.1) (see (3.28)). As before, the nonlinear regularity theory implies that $u_\lambda \in C_+ \setminus \{0\}$.

Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by Hypothesis 3.1 (v). Then

$$-\operatorname{div} a(Du_\lambda(z)) + (\xi(z) + \hat{\xi}_\rho)u_\lambda(z)^{p-1} \geqslant 0 \quad \text{for almost all } z \in \Omega$$

and thus, by (C1), we have

$$\operatorname{div} a(Du_\lambda(z)) \leqslant [\|\xi\|_\infty + \hat{\xi}_\rho]u_\lambda(z)^{p-1} \quad \text{for almost all } z \in \Omega.$$

Hence, $u_\lambda \in D_+$ (see [41, pp. 111, 120]).

Therefore, we infer that $(0, \lambda_0) \subseteq \mathcal{L}$, and so $\mathcal{L} \neq \emptyset$. □

Let S_+^λ be the set of positive solutions of problem (1.1). A byproduct of the proof of Proposition 3.5 is the following corollary.

Corollary 3.6. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, then* $S_+^\lambda \subseteq D_+$.

The next proposition reveals a basic property of the set $\mathcal{L}$ of admissible parameter values.

Proposition 3.7. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold,* $\lambda \in \mathcal{L}$ *and* $\alpha \in (0, \lambda)$, *then* $\alpha \in \mathcal{L}$.

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_+^\lambda \subseteq D_+$ (see Corollary 3.6). We introduce the Carathéodory function $\mu_\alpha\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\mu_\alpha(z,x) = \begin{cases} \alpha f(z,x) & \text{if } x \leqslant u_\lambda(z), \\ \alpha f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.32}$$

We set $M_\alpha(z,x) = \int_0^x \mu_\alpha(z,s)\,ds$ and consider the C^1-functional $w_\alpha\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$w_\alpha(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega M_\alpha(z,u)\,dz, \quad u \in W^{1,p}(\Omega).$$

Clearly, $w_\alpha(\cdot)$ is coercive (see (3.32)) and sequentially weakly lower semicontinuous. So, we can find $u_\alpha \in W^{1,p}(\Omega)$ such that

$$w_\alpha(u_\alpha) = \inf\{w_\alpha(u) : u \in W^{1,p}(\Omega)\}. \tag{3.33}$$

As before (see the proof of Proposition 3.4), using Hypothesis 3.1 (iv), we have $w_\alpha(u_\alpha) < 0 = w_\alpha(0)$, hence $u_\alpha \neq 0$.

From (3.33), we have $w_\alpha'(u_\alpha) = 0$. Thus,

$$\langle A(u_\alpha), h\rangle + \int_\Omega \xi(z)|u_\alpha|^{p-2}u_\alpha h\,dz + \int_{\partial\Omega} \beta(z)|u_\alpha|^{p-2}u_\alpha h\,d\sigma = \int_\Omega \mu_\alpha(z,u_\alpha)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.34}$$

In (3.34), we first choose $h = -u_\alpha^- \in W^{1,p}(\Omega)$. Then we obtain $0 \leqslant u_\alpha$, $u_\alpha \neq 0$.

Next we choose $h = (u_\alpha - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then, by (3.32) and since $f \geqslant 0$, $\alpha \leqslant \lambda$ and $u_\lambda \in S_+^\lambda$, we have

$$\begin{aligned}
\langle A(u_\alpha), (u_\alpha - u_\lambda)^+\rangle &+ \int_\Omega \xi(z)u_\alpha^{p-1}(u_\alpha - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)u_\alpha^{p-1}(u_\alpha - u_\lambda)^+\,d\sigma \\
&= \int_\Omega \alpha f(z,u_\lambda)(u_\alpha - u_\lambda)^+\,dz \\
&\leqslant \int_\Omega \lambda f(z,u_\lambda)(u_\alpha - u_\lambda)^+\,dz \\
&= \langle A(u_\lambda), (u_\alpha - u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(u_\alpha - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(u_\alpha - u_\lambda)^+\,d\sigma.
\end{aligned}$$

Therefore, $u_\alpha \leqslant u_\lambda$.

So, we have proved that $u_\alpha \in [0, u_\lambda]$, $u_\alpha \neq 0$, hence $u_\alpha \in S_+^\alpha \subseteq D_+$ (see (3.32)), and so $\alpha \in \mathcal{L}$. □

Remark 3.8. Proposition 3.7 implies that $\mathcal{L}$ is an interval.

Corollary 3.9. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold,* $\lambda \in \mathcal{L}$, $\alpha \in (0,\lambda)$ *and* $u_\lambda \in S_+^\lambda \subseteq D_+$, *then we can find* $u_\alpha \in S_+^\alpha$ *such that*

$$u_\lambda - u_\alpha \in \operatorname{int} C_+^*(\Sigma_0), \quad \textit{with } \Sigma_0 = \{z \in \partial\Omega : u_\lambda(z) = u_\alpha(z)\}.$$

Proof. From the proof of Proposition 3.7, we know that we can find $u_\alpha \in S_+^\alpha$ such that

$$u_\lambda - u_\alpha \in C_+ \setminus \{0\}.$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by Hypothesis 3.1 (v). Then, by Hypotheses 3.1 (iv)–(v) and the fact that $u_\alpha \leqslant u_\lambda$ and $u_\lambda \in D_+$, we have

$$\begin{aligned}
-\operatorname{div} a(Du_\alpha) + (\xi(z) + \alpha\hat{\xi}_\rho)u_\alpha^{p-1} &= \alpha f(z,u_\alpha) + \alpha\hat{\xi}_\rho u_\alpha^{p-1} \\
&\leqslant \alpha f(z,u_\lambda) + \alpha\hat{\xi}_\rho u_\lambda^{p-1} \\
&= \lambda f(z,u_\lambda) - (\lambda-\alpha)f(z,u_\lambda) + \alpha\hat{\xi}_\rho u_\lambda^{p-1} \\
&\leqslant \lambda f(z,u_\lambda) - (\lambda-\alpha)\eta_s + \alpha\hat{\xi}_\rho u_\lambda^{p-1} \\
&< -\operatorname{div} a(Du_\lambda) + \alpha\hat{\xi}_\rho u_\lambda^{p-1} \quad \text{for almost all } z \in \Omega,
\end{aligned}$$

with $s = \min_{\overline{\Omega}} u_\lambda > 0$. It follows that (see Proposition 2.10)

$$u_\lambda - u_\alpha \in \operatorname{int} C_+^*(\Sigma_0), \quad \text{with } \Sigma_0 = \{z \in \partial\Omega : u_\lambda(z) = u_\alpha(z)\}.$$

The proof is now complete. □

Now let $\lambda^* = \sup \mathcal{L}$.

Proposition 3.10. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, then* $\lambda^* < +\infty$.

Proof. Hypotheses 3.1 (i), (iv) and (C1) imply that we can find $\overline{\lambda} > 0$ big such that

$$\overline{\lambda} f(z, x) - \xi(z)x^{p-1} \geqslant x^{p-1} \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{3.35}$$

Let $\lambda > \overline{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_+^\lambda \subseteq D_+$. So, we have

$$m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0.$$

For $\delta > 0$, we set $m_\lambda^\delta = m_\lambda + \delta \in D_+$. Also, for $\rho = \|u_\lambda\|_\infty$, let $\hat{\xi}_\rho > 0$ be as postulated by Hypothesis 3.1 (v). Then, by Hypotheses 3.1 (iv), (v), (3.35) and the fact that $\lambda > \overline{\lambda}$, we have

$$\begin{aligned}
&-\operatorname{div} a(Dm_\lambda^\delta) + (\xi(z) + \lambda\hat{\xi}_\rho)(m_\lambda^\delta)^{p-1} \\
&\quad \leqslant (\xi(z) + \lambda\hat{\xi}_\rho)m_\lambda^{p-1} + \chi(\delta) && (\text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+) \\
&\quad \leqslant \xi(z)m_\lambda^{p-1} + (1 + \lambda\hat{\xi}_\rho)m_\lambda^{p-1} + \chi(\delta) \\
&\quad \leqslant \overline{\lambda} f(z, m_\lambda) + \lambda\hat{\xi}_\rho m_\lambda^{p-1} + \chi(\delta) \\
&\quad < \lambda f(z, m_\lambda) - (\lambda - \overline{\lambda})f(z, u_\lambda) + \lambda\hat{\xi}_\rho m_\lambda^{p-1} + \chi(\delta) \\
&\quad \leqslant \lambda f(z, m_\lambda) + \lambda\hat{\xi}_\rho m_\lambda^{p-1} - (\lambda - \overline{\lambda})\eta_s + \chi(\delta) && (\text{with } s = m_\lambda > 0) \\
&\quad \leqslant \lambda f(z, m_\lambda) + \lambda\hat{\xi}_\rho m_\lambda(\vartheta) - \vartheta && (\text{for some } \vartheta > 0 \text{ and all } \delta > 0 \text{ small}) \\
&\quad \leqslant \lambda f(z, u_\lambda) + \lambda\hat{\xi}_\rho u_\lambda - \vartheta \\
&\quad < \lambda f(z, u_\lambda) + \lambda\hat{\xi}_\rho u_\lambda \\
&\quad = -\operatorname{div} a(Du_\lambda) + (\xi(z) + \lambda\hat{\xi}_\rho)u_\lambda^{p-1} && \text{for almost all } z \in \Omega.
\end{aligned} \tag{3.36}$$

If $\beta = 0$ (Neumann problem), then by acting on (3.36) with $(m_\lambda^\delta - u_\lambda)^+ \in W^{1,p}(\Omega)$, we obtain

$$m_\lambda^\delta \leqslant u_\lambda \quad \text{for } \delta > 0 \text{ small},$$

a contradiction to the definition of m_λ.

If $\beta \neq 0$, then from the boundary condition we infer that $\Sigma_0 = \{z \in \partial\Omega : u_\lambda(z) = m_\lambda\} \neq \partial\Omega$. Then, from (3.36) and Proposition 2.10, we have

$$u_\lambda - m_\lambda \in \operatorname{int} C_+^*(\Sigma_0),$$

which again contradicts the definition of m_λ.

So, it follows that $\lambda \notin \mathcal{L}$, and we have $\lambda^* = \sup \mathcal{L} \leqslant \overline{\lambda} < \infty$. ☐

In what follows, for every $\lambda > 0$, $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-energy (Euler) functional for problem (1.1), defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \lambda\int_\Omega F(z, u)\,dz, \quad u \in W^{1,p}(\Omega).$$

Proposition 3.11. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, then* $\lambda^* \in \mathcal{L}$.

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ be an increasing sequence such that $\lambda_n \to \lambda^-$. We can find $u_n \in S_+^{\lambda_n}$ $(n \in \mathbb{N})$ such that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N} \tag{3.37}$$

(see the proof of Proposition 3.7).

Also, we have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\,d\sigma = \lambda_n\int_\Omega f(z, u_n)h\,dz \tag{3.38}$$

for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$.

Claim. $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

Arguing by contradiction, suppose that the claim is not true. Then we may assume that $\|u_n\| \to +\infty$.

From (3.37) we have

$$\int_\Omega pG(Du_n)\,dz + \int_\Omega \xi(z)u_n^p\,dz + \int_{\partial\Omega} \beta(z)u_n^p\,d\sigma - \lambda_n \int_\Omega pF(z,u_n)\,dz < 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

On the other hand, if in (3.38) we choose $h = u_n \in W^{1,p}(\Omega)$, then

$$-\int_\Omega (a(Du_n), Du_n)_{\mathbb{R}^N}\,dz - \int_\Omega \xi(z)u_n^p\,dz - \int_{\partial\Omega} \beta(z)u_n^p\,d\sigma + \lambda_n \int_\Omega f(z,u_n)u_n\,dz = 0. \tag{3.40}$$

We add (3.39) and (3.40), and obtain

$$\int_\Omega [pG(Du_n) - (a(Du_n), Du_n)_{\mathbb{R}^N}]\,dz + \lambda_n \int_\Omega e(z,u_n)\,dz < 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence,

$$\lambda_n \int_\Omega e(z,u_n)\,dz \leqslant c_{21} \quad \text{for all } n \in \mathbb{N},$$

for some $c_{21} > 0$.

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega), \qquad y_n \to y \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega),\ y \geqslant 0. \tag{3.41}$$

First assume that $y \neq 0$, and let $E = \{z \in \Omega : y(z) \neq 0\}$. We have $|E|_N > 0$, and so

$$u_n(z) \to +\infty \quad \text{for almost all } z \in E.$$

Hypothesis 3.1 (ii) implies that

$$\frac{F(z,u_n)}{\|u_n\|^p} = \frac{F(z,u_n)}{u_n^p} y_n^p \to +\infty \quad \text{for almost all } z \in E. \tag{3.42}$$

From (3.42) and Fatou's lemma (Hypothesis 3.1 (ii) permits its use), we have

$$\frac{1}{\|u_n\|^p} \int_E F(z,u_n)\,dz \to +\infty. \tag{3.43}$$

Then, since $F \geqslant 0$,

$$\int_\Omega \frac{F(z,u_n)}{\|u_n\|^p}\,dz = \int_E \frac{F(z,u_n)}{\|u_n\|^p}\,dz + \int_{\Omega\setminus E} \frac{F(z,u_n)}{\|u_n\|^p}\,dz \geqslant \int_E \frac{F(z,u_n)}{\|u_n\|^p}\,dz \quad \text{for almost all } n \in \mathbb{N},$$

and so (see (3.43))

$$\int_\Omega \frac{F(z,u_n)}{\|u_n\|^p}\,dz \to +\infty \quad \text{as } n \to \infty. \tag{3.44}$$

Hypothesis 3.1 (iii) implies that

$$0 \leqslant e(z,x) + d(z) \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0,$$

and thus

$$pF(z,x) - d(z) \leqslant f(z,x)x \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{3.45}$$

From (3.40), (3.45) and Hypothesis 2.3 (iv), we have

$$\lambda_n \int_\Omega pF(z,u_n)\,dz \leqslant \int_\Omega pG(Du_n)\,dz + \int_\Omega \xi(z)u_n^p\,dz + \int_{\partial\Omega} \beta(z)u_n^p\,d\sigma + c_{22} \quad \text{for all } n \in \mathbb{N},$$

for some $c_{22} > 0$. Hence,

$$\begin{aligned}\lambda_n \int_\Omega \frac{pF(zu_n)}{\|u_n\|^p}\,dz &\leqslant \int_\Omega \frac{pG(Du_n)}{\|u_n\|^p}\,dz + \int_\Omega \xi(z)y_n^p\,dz + \int_{\partial\Omega} \beta(z)y_n^p\,d\sigma + \frac{c_{22}}{\|u_n\|^p}\\ &\leqslant pc_5\Big(\frac{1}{\|u_n\|^p} + \|Dy_n\|_p^p\Big) + \int_\Omega \xi(z)y_n^p\,dz + \int_{\partial\Omega} \beta(z)y_n^p\,d\sigma + \frac{c_{22}}{\|u_n\|^p}\\ &\leqslant c_{23} \quad \text{for all } n \in \mathbb{N},\end{aligned} \tag{3.46}$$

for some $c_{23} > 0$. Comparing (3.44) and (3.46), we have a contradiction.

Next assume that $y = 0$. For $\mu > 0$, we set

$$v_n = (p\mu)^{1/p} y_n \in W^{1,p}(\Omega) \quad \text{for all } n \in \mathbb{N}.$$

Note that (see (3.41) and recall that $y = 0$) $v_n \to 0$ in $L^r(\Omega)$. Hence, by Hypothesis 3.1 (i),

$$\int_\Omega F(z, v_n)\,dz \to 0. \tag{3.47}$$

Since $\|u_n\| \to \infty$, we can find $n_0 \in \mathbb{N}$ such that

$$(p\mu)^{1/p}\frac{1}{\|u_n\|} \leqslant 1 \quad \text{for all } n \geqslant n_0. \tag{3.48}$$

Consider the C^1-functional $\tilde{\psi}_{\lambda_n} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\psi}_{\lambda_n}(u) = \frac{c_1}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \lambda_n \int_\Omega F(z,u)\,dz, \quad u \in W^{1,p}(\Omega).$$

Let $t_n \in [0, 1]$ be such that

$$\tilde{\psi}_{\lambda_n}(t_n u_n) = \max\{\tilde{\psi}_{\lambda_n}(tu_n) : 0 \leqslant t \leqslant 1\}. \tag{3.49}$$

From (3.47)–(3.49), it follows that (see (C3), Lemmata 2.8 and 2.9, and recall that $F \geqslant 0$, $\lambda_n \leqslant \lambda^*$)

$$\begin{aligned}\tilde{\psi}_{\lambda_n}(t_n u_n) &\geqslant \tilde{\psi}_{\lambda_n}(v_n)\\ &= \mu\Big[\frac{c_1}{p-1}\|Dy_n\|_p^p + \int_\Omega \xi(z)y_n^p\,dz + \int_{\partial\Omega} \beta(z)y_n^p\,d\sigma\Big] - \lambda_n \int_\Omega F(z, v_n)\,dz\\ &\geqslant \mu c_{24} - \lambda^* \int_\Omega F(z, v_n)\,dz\\ &\geqslant \mu\frac{c_{24}}{2} > 0 \quad \text{for all } n \geqslant n_1 \geqslant n_0,\end{aligned} \tag{3.50}$$

for some $c_{24} > 0$. But $\mu > 0$ is arbitrary. So, from (3.50) we infer that

$$\tilde{\psi}_{\lambda_n}(t_n u_n) \to +\infty \quad \text{as } n \to +\infty. \tag{3.51}$$

Note that

$$\tilde{\psi}_{\lambda_n}(0) = 0 \quad \text{and} \quad \tilde{\psi}_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}, \tag{3.52}$$

by (3.37), Corollary 2.6, and the fact that $\tilde{\psi}_{\lambda_n} \leqslant \varphi_{\lambda_n}$ for all $n \in \mathbb{N}$. Then, from (3.51) and (3.52), it follows that

$$t_n \in (0, 1) \quad \text{for all } n \geqslant n_2. \tag{3.53}$$

So, from (3.49) and (3.53), we have

$$\frac{d}{dt}\tilde{\psi}_{\lambda_n}(tu_n)|_{t=t_n} = 0 \quad \text{for all } n \geqslant n_2,$$

and by the chain rule,

$$\langle \tilde{\psi}'_{\lambda_n}(t_n u_n), t_n u_n \rangle = 0 \quad \text{for all } n \geqslant n_2.$$

Therefore,

$$\frac{c_1}{p-1}\|D(t_n u_n)\|_p^p + \int_\Omega \xi(z)(t_n u_n)^p \, dz + \int_{\partial\Omega} \beta(z)(t_n u_n)^p \, d\sigma = \lambda_n \int_\Omega f(z, t_n u_n)(t_n u_n) \, dz \quad \text{for all } n \geqslant n_2,$$

and thus

$$p\tilde{\psi}_{\lambda_n}(t_n u_n) + \lambda_n \int_\Omega pF(z, t_n u_n) \, dz = \lambda_n \int_\Omega f(z, t_n u_n)(t_n u_n) \, dz \quad \text{for all } n \geqslant n_2.$$

By (3.53), Hypothesis 3.1 (iii) and since $\lambda_n \leqslant \lambda^*$ for all $n \in \mathbb{N}$ and $e \geqslant 0$, it follows that

$$\begin{aligned} p\tilde{\psi}_{\lambda_n}(t_n u_n) &\leqslant \lambda_n \int_\Omega e(z, t_n u_n) \, dz \\ &\leqslant \lambda^* \int_\Omega e(z, t_n u_n) \, dz \\ &\leqslant \lambda^* \int_\Omega e(z, u_n) \, dz + \lambda^* \|d\|_1 \\ &\leqslant M_4 \quad \text{for all } n \geqslant n_2, \end{aligned} \tag{3.54}$$

for some $M_4 > 0$. Comparing (3.51) and (3.54) again we have a contradiction.

This proves the claim.

On account of the claim, we may assume that

$$u_n \xrightarrow{w} u^* \quad \text{in } W^{1,p}(\Omega), \qquad u_n \to u^* \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.55}$$

In (3.38) we choose $h = u_n - u^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.55). Then we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u^* \rangle = 0,$$

hence (see Proposition 2.7)

$$u_n \to u^* \quad \text{in } W^{1,p}(\Omega). \tag{3.56}$$

So, if in (3.38) we pass to the limit as $n \to \infty$ and use (3.56), then

$$\langle A(u^*), h \rangle + \int_\Omega \xi(z)(u^*)^{p-1} h \, dz + \int_{\partial\Omega} \beta(z)(u^*)^{p-1} h \, d\sigma = \lambda^* \int_\Omega f(z, u^*) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

Therefore,

$$-\operatorname{div} a(Du^*(z)) + \xi(z) u^*(z)^{p-1} = \lambda^* f(z, u^*(z)) \quad \text{for almost all } z \in \Omega,$$

and (see [28])

$$\frac{\partial u^*}{\partial n_a} + \beta(z)(u^*)^{p-1} = 0 \quad \text{on } \partial\Omega. \tag{3.57}$$

We know that

$$\overline{u}_{\lambda_1} \leqslant u_n \quad \text{for all } n \in \mathbb{N}$$

(see claim 4 in the proof of Proposition 3.4 and use the fact that $\lambda \mapsto \overline{u}_\lambda$ is nondecreasing from $(0, +\infty)$ into $C^1(\overline{\Omega})$). Hence, as $n \to \infty$, we obtain $\overline{u}_{\lambda_1} \leqslant u^*$, thus $u^* \in S_+^{\lambda^*}$ (see (3.57)), and so $\lambda^* \in \mathcal{L}$. ☐

Proposition 3.12. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, and* $\lambda \in (0, \lambda^*)$, *then problem* (1.1) *has at least two positive solutions*

$$v_\lambda, \hat{u}_\lambda \in D_+, \quad \textit{with } \hat{u}_\lambda - u_\lambda \in C_+ \setminus \{0\}.$$

Proof. From Proposition 3.11 we know that $\lambda^* \in \mathcal{L}$. So, we can find $u^* \in S_+^{\lambda^*} \subseteq D_+$. Invoking Corollary 3.9, we can find $u_\lambda \in S_+^\lambda \subseteq D_+$ such that

$$u^* - u_\lambda \in \operatorname{int} C_+^*(\Sigma_0), \quad \text{with } \Sigma_0 = \{z \in \partial\Omega : u_\lambda(z) = u^*(z)\}. \tag{3.58}$$

Moreover, from the proof of Proposition 3.7, we know that u_λ is a global minimizer of the functional w_λ.

Using the fact that $u_\lambda \in S_+^\lambda \subseteq D_+$, we introduce the following truncation of the reaction term in problem (1.1):

$$\vartheta_\lambda(z,x) = \begin{cases} \lambda f(z, u_\lambda(z)) & \text{if } x \leqslant u_\lambda(z), \\ \lambda f(z,x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.59}$$

This is a Carathéodory function. Now we set $\Theta_\lambda(z,x) = \int_0^x \vartheta_\lambda(z,s)\,ds$, and we consider the C^1-functional $\hat{\varphi}_\lambda \colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega \Theta_\lambda(z,u)\,dz, \quad u \in W^{1,p}(\Omega).$$

From (3.59) it is clear $\vartheta_\lambda(z,\cdot)$ has the same asymptotic behavior for $x \to +\infty$ as $f(z,\cdot)$. So, reasoning as in the claim in the proof of Proposition 3.11, we show that

$$\hat{\varphi}_\lambda \text{ satisfies the C-condition.} \tag{3.60}$$

Claim. $K_{\hat{\varphi}_\lambda} \subseteq [u_\lambda) \cap D_+ = \{u \in D_+ : u_\lambda(z) \leqslant u(z) \text{ for all } z \in \overline{\Omega}\}$.

Let $u \in K_{\hat{\varphi}_\lambda}$. Then

$$\langle A(u), h\rangle + \int_\Omega \xi(z)|u|^{p-2}uh\,dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\,d\sigma = \int_\Omega \vartheta_\lambda(z,u)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.61}$$

In (3.61) we choose $h = (u_\lambda - u)^+ \in W^{1,p}(\Omega)$. Then, by (3.59) and since $u_\lambda \in S_+^\lambda$, we have

$$\begin{aligned} &\langle A(u), (u_\lambda - u)^+\rangle + \int_\Omega \xi(z)|u|^{p-2}u(u_\lambda - u)^+\,dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}u(u_\lambda - u)^+\,d\sigma \\ &\quad = \int_\Omega \lambda f(z,u_\lambda)(u_\lambda - u)^+\,dz \\ &\quad = \langle A(u_\lambda), (u_\lambda - u)^+\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(u_\lambda - u)^+\,dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(u_\lambda - u)^+\,d\sigma, \end{aligned}$$

hence $u_\lambda \leqslant u$. As before, the nonlinear regularity theory implies that $u \in D_+$. This proves the claim.

The claim allows us to assume that

$$K_{\hat{\varphi}_\lambda} \cap [u_\lambda, u^*] = \{u_\lambda\}. \tag{3.62}$$

Indeed, otherwise we already have a second positive smooth (due to nonlinear regularity) solution of problem (1.1), which is bigger than u_λ, and so we are done.

We consider the following truncation of $\vartheta_\lambda(z,\cdot)$:

$$\tilde{\vartheta}_\lambda(z,x) = \begin{cases} \vartheta_\lambda(z,x) & \text{if } x \leqslant u^*(z), \\ \vartheta_\lambda(z,u^*(z)) & \text{if } u^*(z) < x. \end{cases} \tag{3.63}$$

This is a Carathéodory function. Now we set $\tilde{\Theta}_\lambda(z,x) = \int_0^x \tilde{\vartheta}_\lambda(z,s)\,ds$, and we consider the C^1-functional $\tilde{\varphi}_\lambda \colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \int_\Omega \tilde{\Theta}_\lambda(z,u)\,dz, \quad u \in W^{1,p}(\Omega).$$

Using (3.63), we can easily show that

$$K_{\tilde{\varphi}_\lambda} \subseteq [u_\lambda, u^*] \cap D_+. \tag{3.64}$$

From (3.63) it is clear that $\tilde{\varphi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\tilde{\varphi}_\lambda(\tilde{u}_\lambda) = \inf\{\tilde{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\},$$

and thus, by (3.64),

$$\tilde{u}_\lambda \in K_{\tilde{\varphi}_\lambda} \subseteq [u_\lambda, u^*] \cap D_+.$$

From (3.59) and (3.63), we see that $\tilde{\varphi}'_\lambda|_{[0,u^*]} = \hat{\varphi}'_\lambda|_{[0,u^*]}$, hence $\tilde{u}_\lambda \in K_{\hat{\varphi}_\lambda}$, and $\tilde{u}_\lambda = u_\lambda$ (see (3.62)). Then, from (3.58), we infer that for $\Sigma_0 = \{z \in \partial\Omega : u_\lambda(z) = u^*(z)\}$, we have that u_λ is a $C^1_*(\Omega)$-minimizer of $\hat{\varphi}_\lambda$, and so u_λ is a $W^{1,p}_*(\Omega)$-minimizer of $\hat{\varphi}_\lambda$ (see Proposition 2.12).

Without any loss of generality, we may assume that $K_{\hat{\varphi}_\lambda}$ is finite. Otherwise, the claim and (3.59) imply that we already have a whole sequence of distinct smooth solutions of (1.1) bigger than u_λ, and so we are done. Then we can find $\rho \in (0, 1)$ small such that

$$\hat{\varphi}_\lambda(u_\lambda) < \inf\{\hat{\varphi}_\lambda(u_\lambda + h) : \|h\| \leqslant \rho,\ h \in W^{1,p}_*(\Omega)\} = \hat{m}^\lambda_\rho. \tag{3.65}$$

In addition, Hypothesis 3.1 (ii) implies that for all $h \in \operatorname{int} C^*_+(\Sigma_0)$, we have

$$\hat{\varphi}_\lambda(u_\lambda + th) \to -\infty \quad \text{as } t \to +\infty. \tag{3.66}$$

From (3.60), (3.65) and (3.66), we see that we can apply Theorem 2.1 (the mountain pass theorem) on the affine space (manifold) $Y = u_\lambda + W^{1,p}_*(\Omega)$ and find $\hat{u}_\lambda \in Y$ such that (see (3.65))

$$\langle \hat{\varphi}'_\lambda(\hat{u}_\lambda), h\rangle = 0 \quad \text{for all } h \in W^{1,p}_*(\Omega),\ \hat{m}^\lambda_\rho \leqslant \hat{\varphi}_\lambda(\hat{u}_\lambda), \tag{3.67}$$

and thus, by choosing $h = (u_\lambda - \hat{u}_\lambda)^+ \in W^{1,p}_*(\Omega)$, $u_\lambda \leqslant \hat{u}_\lambda$.

Also, using the nonlinear Green's identity on the space $W^{1,p}_*(\Omega)$ (see [8, 23]), from (3.67), we infer that $\hat{u}_\lambda \in D_+$ is a solution of (1.1) ($\lambda \in (0, \lambda^*)$). Moreover, from (3.65) we have $\hat{u}_\lambda - u_\lambda \in C_+ \setminus \{0\}$. □

Summarizing the results of this section, we can formulate the following bifurcation-type result.

Theorem 3.13. *Under Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3)*, there exists* $\lambda^* > 0$ *such that the following hold:*

(a) *For all* $\lambda \in (0, \lambda^*)$*, problem* (1.1) *has at least two positive solutions* $u_\lambda, \hat{u}_\lambda \in D_+$*, with* $\hat{u}_\lambda - u_\lambda \in C_+ \setminus \{0\}$.
(b) *For* $\lambda = \lambda^*$*, problem* (1.1) *has at least one positive solution* $u^* \in D_+$.
(c) *For* $\lambda > \lambda^*$*, problem* (1.1) *has no positive solution.*

4 Big, small and minimal positive solutions

In this section we show that as $\lambda \to 0^+$, we can produce positive solutions of problem (1.1) which have $W^{1,p}(\Omega)$-norm, which is arbitrarily big and arbitrarily small. Moreover, we show that for every $\lambda \in (0, \lambda^*)$, problem (1.1) admits a smallest positive solution $u^*_\lambda \in D_+$, and we study the monotonicity and continuity properties of the map $\lambda \mapsto u^*_\lambda$.

Theorem 4.1. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, and* $\lambda_n \to 0^+$*, then we can find positive solutions*

$$\hat{u}_n = \hat{u}_{\lambda_n} \in S^{\lambda_n}_+ \subseteq D_+, \quad u_n = u_{\lambda_n} \in S^{\lambda_n}_+ \subseteq D_+ \quad \textit{for all } n \in \mathbb{N}$$

such that $\|\hat{u}_n\| \to +\infty$ *and* $\|u_n\| \to 0$ *as* $n \to \infty$.

Proof. From (3.4) we have

$$F(z,x) \leqslant \frac{\hat{\eta}}{\tau} x^{\tau} + \frac{c_{10}}{r} x^{r} \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{4.1}$$

Then, for all $u \in W^{1,p}(\Omega)$, we have (see Corollary 2.6, Lemmata 2.8 and 2.9, (C3) and (4.1))

$$\begin{aligned} \varphi_{\lambda_n}(u) &\geqslant \frac{c_1}{p(p-1)}\|Du_n\|_p^p + \frac{1}{p}\int_{\Omega} \xi(z)|u|^p\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p\,d\sigma - \frac{\lambda_n\hat{\eta}}{\tau}\|u^+\|_{\tau}^{\tau} - \frac{\lambda_n c_{10}}{r}\|u^+\|_r^r \\ &\geqslant c_{25}\|u\|^p - \lambda_n c_{26}(\|u\|^{\tau} + \|u\|^{r}) \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{4.2}$$

for some $c_{25}, c_{26} > 0$. Let $\|u\| = \lambda_n^{-\alpha}$ with $\alpha > 0$. We set

$$k(\lambda_n) = c_{25}\lambda_n^{-\alpha p} - c_{26}(\lambda_n^{1-\alpha\tau} + \lambda_n^{1-\alpha r}), \quad n \in \mathbb{N}.$$

We choose $\alpha \in (0, \frac{1}{r-p})$ (recall that $r > p$). Then we have $-\alpha p < 1 - \alpha r < 1 - \alpha\tau$ (recall that $\tau < p < r$). So, we see that (recall that $\lambda_n \to 0^+$)

$$k(\lambda_n) \to +\infty \quad \text{as } n \to \infty. \tag{4.3}$$

Then, from (4.2) and (4.3), we infer that there exists $n_1 \in \mathbb{N}$ such that

$$\varphi_{\lambda_n}(u) \geqslant k(\lambda_n) > 0 = \varphi_{\lambda_n}(0) \quad \text{for all } n \geqslant n_1 \text{ and all } \|u\| = \lambda_n^{-\alpha}. \tag{4.4}$$

Hypothesis 3.1 (ii) implies that if $u \in D_+$, then

$$\varphi_{\lambda_n}(tu) \to -\infty \quad \text{as } t \to +\infty, \text{ for all } n \in \mathbb{N}. \tag{4.5}$$

Moreover, as in the claim in the proof of Proposition 3.11, we can check that

$$\varphi_{\lambda_n}(\cdot) \text{ satisfies the C-condition for all } n \in \mathbb{N}. \tag{4.6}$$

Then (4.4), (4.5) and (4.6) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u}_n \in W^{1,p}(\Omega)$ such that (see Hypothesis 3.1 (i))

$$\hat{u}_n \in K_{\varphi_{\lambda_n}} \quad \text{and} \quad k(\lambda_n) \leqslant \varphi_{\lambda}(\hat{u}_n) \leqslant c_{27}(1 + \|\hat{u}_n\|^r) \quad \text{for all } n \geqslant n_1,$$

for some $c_{27} > 0$. Hence (see (4.3)),

$$\hat{u}_n \in S_+^{\lambda_n} \subseteq D_+ \quad \text{for all } n \in \mathbb{N}, \qquad \|\hat{u}_n\| \to \infty.$$

Next let $\zeta \in (0, \frac{1}{p})$ and consider $\|u\| = \lambda_n^{\zeta}$. Then from (4.2) we have

$$\varphi_{\lambda_n}(u) \geqslant c_{25}\lambda_n^{\zeta p} - c_{26}(\lambda_n^{\zeta\tau+1} + \lambda_n^{\zeta r+1}) = \lambda_n[c_{25}\lambda_n^{\zeta p-1} - c_{26}(\lambda_n^{\zeta\tau} + \lambda_n^{\zeta r})].$$

Let $k_0(\lambda_n) = c_{25}\lambda_n^{\zeta p-1} - c_{26}(\lambda_n^{\zeta\tau} + \lambda_n^{\zeta r})$. Since $\zeta p - 1 < 0$ and $\lambda_n \to 0^+$, we infer that

$$k_0(\lambda_n) \to +\infty \quad \text{as } n \to +\infty.$$

So, we can find $n_2 \in \mathbb{N}$ such that

$$\varphi_{\lambda_n}(u) \geqslant \lambda_n k_0(\lambda_n) > 0 = \varphi_{\lambda_n}(0) \quad \text{for all } n \geqslant n_2 \text{ and all } \|u\| = \lambda_n^{\zeta}. \tag{4.7}$$

Let $\overline{B}_n = \{u \in W^{1,p}(\Omega) : \|u\| \leqslant \lambda_n^{\zeta}\}$, $n \in \mathbb{N}$. Since Hypotheses 2.3 (iv) and 3.1 (iv) and the fact that $\tau < q < p$ imply that for every $n \in \mathbb{N}$, every $u \in D_+$ and for $t \in (0,1)$ small, we have (see the proof of Proposition 3.5)

$$\varphi_{\lambda_n}(tu) < 0, \quad \|tu\| \leqslant \lambda_n^{\zeta} \quad \text{for all } n \in \mathbb{N}. \tag{4.8}$$

From (4.7) and (4.8), we see that

$$0 < \inf_{\partial B_n} \varphi_{\lambda_n}, \quad \inf_{\overline{B}_n} \varphi_{\lambda_n} < 0 \quad \text{for all } n \geqslant n_2. \tag{4.9}$$

Let $\bar{c}_n = \inf_{\partial B_n} \varphi_{\lambda_n} - \inf_{\overline{B}_n} \varphi_{\lambda_n} > 0$ for $n \geqslant n_2$ (see (4.9)). Using the Ekeland variational principle (see, for example, [15, pp. 579]), given $\epsilon \in (0, \tau_n)$ $(n \geqslant n_2)$, we can find $u_\epsilon^n \in B_n = \{u \in W^{1,p}(\Omega) : \|u\| < \lambda_n^\zeta\}$ such that

$$\varphi_{\lambda_n}(u_\epsilon^n) \leqslant \inf_{\overline{B}_n} \varphi_{\lambda_n} + \epsilon, \tag{4.10}$$

$$\varphi_{\lambda_n}(u_\epsilon^n) \leqslant \varphi_{\lambda_n}(y) + \epsilon\|y - u_n\| \quad \text{for all } y \in \overline{B_n},\ n \geqslant n_2. \tag{4.11}$$

Given $t \in W^{1,p}(\Omega)$, for $t > 0$ small, we have $u_\epsilon^n + th \in \overline{B}_n$. So, if in (4.11) we choose $y = u_\epsilon^n + th$, then

$$-\epsilon\|h\| \leqslant \langle \varphi'_{\lambda_n}(u_\epsilon^n), h\rangle \quad \text{for all } h \in W^{1,p}(\Omega),$$

and thus

$$\|\varphi'_{\lambda_n}(u_\epsilon^n)\|_* \leqslant \epsilon \quad \text{for all } n \geqslant n_2. \tag{4.12}$$

Let $\epsilon_m \to 0^+$ and set $u_{\epsilon_m}^n = u_m^n$ for all $m \in \mathbb{N}$, $n \geqslant n_2$. From (4.12) we have

$$\varphi'_{\lambda_n}(u_m^n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \text{ as } m \to \infty,\ n \geqslant n_2. \tag{4.13}$$

But from (4.6) we know that $\varphi_{\lambda_n}(\cdot)$ satisfies the C-condition. So, from (4.10) and (4.13), it follows that at least for a subsequence, we have

$$u_m^n \to u_{\lambda_n} = u_n \quad \text{in } W^{1,p}(\Omega) \text{ as } m \to \infty. \tag{4.14}$$

From (4.10) and (4.14), we infer that

$$\varphi_{\lambda_n}(u_n) = \inf_{\overline{B}_n} \varphi_{\lambda_n} \quad \text{for all } n \geqslant n_2,$$

hence $u_n \in B_n$, and so $u_n \in K_{\varphi_{\lambda_n}}$ for all $n \geqslant n_2$ (see (4.9)). Therefore, we have $u_n \in S_+^\lambda \subseteq D_+$ and $\|u_n\| < \lambda_n^\zeta$ for all $n \geqslant n_2$. Thus, $\|u_n\| \to 0$ as $n \to \infty$ (recall that $\lambda_n \to 0^+$). □

For every $\lambda \in (0, \lambda^*)$, we show that problem (1.1) admits a minimal positive solution u_λ^* and determine the monotonicity and continuity properties of the map $\lambda \mapsto u_\lambda^*$.

Theorem 4.2. *If Hypotheses 2.3 and 3.1, and conditions* (C1)–(C3) *hold, and* $\lambda \in (0, \lambda^*)$, *then problem* (1.1) *has a smallest positive solution* $u_\lambda^* \in S_+^\lambda \subseteq D_+$ *and the map* $\lambda \mapsto u_\lambda^*$ *from* $(0, \lambda^*)$ *into* $C^1(\overline{\Omega})$ *is*

- *"strictly monotone", in the sense that*

$$\vartheta < \lambda \implies u_\lambda^* - u_\vartheta^* \in \operatorname{int} C_+^*(\Sigma_0), \quad \textit{with } \Sigma_0 = \{z \in \partial\Omega : u_\lambda^*(z) = u_\vartheta^*(z)\};$$

- *"left continuous", that is, if* $\lambda_n \to \lambda^- < \lambda^*$, *then* $u_{\lambda_n} \to u_\lambda$ *in* $C^1(\overline{\Omega})$.

Proof. From [20, Lemma 3.10], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq S_+^\lambda$ such that (see the proof of Proposition 3.5)

$$\inf S_+^\lambda = \inf_{n\geqslant 1} u_n, \quad u_n \leqslant \tilde{u}_\lambda \quad \text{for all } n \in \mathbb{N}.$$

Evidently, $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \quad \text{in } W^{1,p}(\Omega), \qquad u_n \to u_\lambda^* \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{4.15}$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\,d\sigma = \lambda \int_\Omega f(z, u_n)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{4.16}$$

In (4.16) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$. Passing to the limit as $n \to \infty$ and using (4.15), we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_\lambda^*\rangle = 0,$$

and so (see Proposition 2.7)

$$u_n \to u_\lambda^* \quad \text{in } W^{1,p}(\Omega). \tag{4.17}$$

If in (4.16) we pass to the limit as $n \to \infty$ and use (4.17), then

$$\langle A(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p-1} h\,dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1} h\,d\sigma = \lambda \int_\Omega f(z, u_\lambda^*) h\,dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

which implies that u_λ^* is a nonnegative solution of (1.1) (see [28]).

Hypotheses 3.1 (i) and (iv) imply that we can find $c_{28} > 0$ such that

$$f(z, x) \geqslant \hat{\eta}_0 x^{\tau-1} - c_{28} x^{r-1} \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{4.18}$$

We consider the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda(\hat{\eta}_0 u(z)^{\tau-1} - c_{28} u(z)^{r-1}) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

with $u > 0$, $\lambda > 0$. As in the proof of Proposition 3.4 (there we had the auxiliary problem (3.19)), problem (4.19) has a unique positive solution $\overline{u}_\lambda^* \in D_+$ for all $\lambda > 0$ and (see (4.18))

$$\overline{u}_\lambda^* \leqslant u \quad \text{for all } u \in S_+^\lambda.$$

So, we have

$$\overline{u}_\lambda^* \leqslant u_n \quad \text{for all } n \in \mathbb{N},$$

which implies $\overline{u}_\lambda^* \leqslant u_\lambda^*$, and so $u_\lambda^* \in S_\lambda^*$ and $u_\lambda^* = \inf S_+^\lambda$.

From Corollary 3.9, we infer the strict monotonicity of the map $\lambda \mapsto u_\lambda^*$.

Finally, suppose that $\{\lambda_n, \lambda\}_{n\geqslant 1} \subseteq (0, \lambda^*)$ and $\lambda_n \to \lambda^-$. Then (see the proof of Proposition 3.5)

$$u_{\lambda_n}^* \leqslant \tilde{u}_{\lambda^*} \quad \text{for all } n \in \mathbb{N},$$

and so $\{u_{\lambda_n}^*\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. From [25], we know that there exist $\alpha \in (0,1)$ and $M_5 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant M_5 \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we have

$$u_{\lambda_n}^* \to \tilde{u}_\lambda^* \quad \text{in } C^1(\overline{\Omega}) \tag{4.20}$$

(here we have the original sequence since it is increasing).

Suppose that $\tilde{u}_\lambda^* \neq u_\lambda^*$. Then we can find $z_0 \in \Omega$ such that $u_\lambda^*(z_0) < \tilde{u}_\lambda^*(z_0)$, and therefore, by (4.20),

$$u_\lambda^*(z_0) < u_{\lambda_n}^*(z_0) \quad \text{for all } n \geqslant n_0.$$

This contradicts the monotonicity of $\lambda \mapsto u_\lambda^*$. Therefore, $\tilde{u}_\lambda^* = u_\lambda$ and the map $\lambda \mapsto u_\lambda^*$ is left continuous. □

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References

[1] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints, *Mem. Amer. Math. Soc.* **196** (2008), no. 915, 1–70.
[2] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Nodal solutions for $(p, 2)$-equations, *Trans. Amer. Math. Soc.* **367** (2015), no. 10, 7343–7372.
[3] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Positive solutions for parametric nonlinear nonhomogeneous Robin problems, *Funkcial. Ekvac.*, to appear.
[4] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), no. 2, 519–543.
[5] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
[6] D. Arcoya and D. Ruiz, The Ambrosetti–Prodi problem for the p-Laplacian operator, *Comm. Partial Differential Equations* **31** (2006), no. 4–6, 849–865.
[7] V. Benci, P. D'Avenia, D. Fortunato and L. Pisani, Solitons in several space dimensions: Derrick's problem and infinitely many solutions, *Arch. Ration. Mech. Anal.* **154** (2000), no. 4, 297–324.
[8] E. Casas and L. A. Fernández, A Green's formula for quasilinear elliptic operators, *J. Math. Anal. Appl.* **142** (1989), no. 1, 62–73.
[9] L. Cherfils and Y. Ilyasov, On the stationary solutions of generalized reaction-diffusion equations with $p\&q$ Laplacian, *Commun. Pure Appl. Anal.* **4** (2005), 9–22.
[10] S. Cingolani and M. Degiovanni, Nontrivial solutions for p-Laplace equations with right-hand side having p-linear growth at infinity, *Comm. Partial Differential Equations* **30** (2005), no. 7–9, 1191–1203.
[11] J. I. Díaz and J. E. Saá, Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), no. 12, 521–524.
[12] M. E. Filippakis and N. S. Papageorgiou, Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian, *J. Differential Equations* **245** (2008), no. 7, 1883–1922.
[13] M. Fuchs and V. Osmolovski, Variational integrals on Orlicz–Sobolev spaces, *Z. Anal. Anwend.* **17** (1998), no. 2, 393–415.
[14] J. P. García Azorero, I. Peral Alonso and J. J. Manfredi, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, *Commun. Contemp. Math.* **2** (2000), no. 3, 385–404.
[15] L. Gasiński and N. S. Papageorgiou, *Nonlinear Analysis*, Ser. Math. Anal. Appl. 9, Chapman & Hall/CRC, Boca Raton, 2006.
[16] L. Gasiński and N. S. Papageorgiou, Existence and multiplicity of solutions for Neumann p-Laplacian-type equations, *Adv. Nonlinear Stud.* **8** (2008), no. 4, 843–870.
[17] L. Gasiński and N. S. Papageorgiou, Bifurcation-type results for nonlinear parametric elliptic equations, *Proc. Roy. Soc. Edinburgh Sect. A* **142** (2012), no. 3, 595–623.
[18] M. Guedda and L. Véron, Quasilinear elliptic equations involving critical Sobolev exponents, *Nonlinear Anal.* **13** (1989), no. 8, 879–902.
[19] Z. Guo and Z. Zhang, $W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations, *J. Math. Anal. Appl.* **286** (2003), no. 1, 32–50.
[20] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I: Theory*, Math. Appl. 419, Kluwer Academic Publishers, Dordrecht, 1997.
[21] S. Hu and N. S. Papageorgiou, Multiplicity of solutions for parametric p-Laplacian equations with nonlinearity concave near the origin, *Tohoku Math. J. (2)* **62** (2010), no. 1, 137–162.
[22] S. Hu and N. S. Papageorgiou, Nonlinear Neumann equations driven by a nonhomogeneous differential operator, *Commun. Pure Appl. Anal.* **10** (2011), no. 4, 1055–1078.
[23] N. Kenmochi, Pseudomonotone operators and nonlinear elliptic boundary value problems, *J. Math. Soc. Japan* **27** (1975), 121–149.
[24] G. Li and C. Yang, The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti–Rabinowitz condition, *Nonlinear Anal.* **72** (2010), no. 12, 4602–4613.
[25] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Comm. Partial Differential Equations* **16** (1991), no. 2–3, 311–361.
[26] D. Motreanu, V. V. Motreanu and N. Papageorgiou, *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York, 2014.
[27] D. Mugnai and N. S. Papageorgiou, Wang's multiplicity result for superlinear (p, q)-equations without the Ambrosetti–Rabinowitz condition, *Trans. Amer. Math. Soc.* **366** (2014), no. 9, 4919–4937.
[28] N. S. Papageorgiou and V. D. Rădulescu, Multiple solutions with precise sign for nonlinear parametric Robin problems, *J. Differential Equations* **256** (2014), no. 7, 2449–2479.
[29] N. S. Papageorgiou and V. D. Rădulescu, Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance, *Appl. Math. Optim.* **69** (2014), no. 3, 393–430.

[30] N. S. Papageorgiou and V. D. Rădulescu, Bifurcation near infinity for the Robin p-Laplacian, *Manuscripta Math.* **148** (2015), no. 3–4, 415–433.

[31] N. S. Papageorgiou and V. D. Rădulescu, Nonlinear parametric Robin problems with combined nonlinearities, *Adv. Nonlinear Stud.* **15** (2015), no. 3, 715–748.

[32] N. S. Papageorgiou and V. D. Rădulescu, Resonant $(p, 2)$-equations with asymmetric reaction, *Anal. Appl. (Singap.)* **13** (2015), no. 5, 481–506.

[33] N. S. Papageorgiou and V. D. Rădulescu, Solutions with sign information for nonlinear nonhomogeneous elliptic equations, *Topol. Methods Nonlinear Anal.* **45** (2015), no. 2, 575–600.

[34] N. S. Papageorgiou and V. D. Rădulescu, Infinitely many nodal solutions for nonlinear nonhomogeneous Robin problems, *Adv. Nonlinear Stud.* **16** (2016), no. 2, 287–299.

[35] N. S. Papageorgiou and V. D. Rădulescu, Noncoercive resonant $(p, 2)$-equations, *Appl. Math. Optim.* (2016), DOI 10.1007/s00245-016-9363-3.

[36] N. S. Papageorgiou and V. D. Rădulescu, Nonlinear nonhomogeneous Robin problems with superlinear reaction term, *Adv. Nonlinear Stud.* **16** (2016), no. 4, 737–764.

[37] N. S. Papageorgiou and V. D. Rădulescu, Multiplicity theorems for nonlinear nonhomogeneous Robin problems, *Rev. Mat. Iberoam.* **33** (2017), no. 1, 251–289.

[38] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, On a class of parametric $(p, 2)$-equations, *Appl. Math. Optim.* **75** (2017), no. 2, 193–228.

[39] N. S. Papageorgiou and G. Smyrlis, A bifurcation-type theorem for singular nonlinear elliptic equations, *Methods Appl. Anal.* **22** (2015), no. 2, 147–170.

[40] N. S. Papageorgiou and P. Winkert, Resonant $(p, 2)$-equations with concave terms, *Appl. Anal.* **94** (2015), no. 2, 342–360.

[41] P. Pucci and J. Serrin, *The Maximum Principle*, Progr. Nonlinear Differential Equations Appl. 73, Birkhäuser, Basel, 2007.

[42] M. Sun, M. Zhang and J. Su, Critical groups at zero and multiple solutions for a quasilinear elliptic equation, *J. Math. Anal. Appl.* **428** (2015), no. 1, 696–712.