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Positive solutions for superdiffusive mixed problems

Nikolaos S. Papageorgiou[a], Vicenţiu D. Rădulescu[b,c,*], Dušan D. Repovš[d]

[a] National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
[b] Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, University of Craiova, Street A.I. Cuza 13, 200585 Craiova, Romania
[d] Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia

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ABSTRACT

We study a semilinear parametric elliptic equation with superdiffusive reaction and mixed boundary conditions. Using variational methods, together with suitable truncation techniques, we prove a bifurcation-type theorem describing the nonexistence, existence and multiplicity of positive solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $\Sigma_1, \Sigma_2 \subseteq \partial\Omega$ be two $(N-1)$-dimensional C^2-submanifolds of $\partial\Omega$ such that $\partial\Omega = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$, $|\Sigma_1|_{N-1} \in (0, |\partial\Omega|_{N-1})$, and $\overline{\Sigma_1} \cap \overline{\Sigma}_2 = \Gamma$. Here, $|\cdot|_{N-1}$ denotes the $(N-1)$-dimensional Hausdorff (surface) measure and $\Gamma \subset \partial\Omega$ is a $(N-2)$-dimensional C^2-submanifold of $\partial\Omega$.

In this paper, we study the following logistic-type elliptic problem:

$$\left\{\begin{array}{ll} -\Delta u(z) = \lambda u(z)^{q-1} - f(z, u(z)) & \text{in } \Omega, \\ u|_{\Sigma_1} = 0, \ \left.\dfrac{\partial u}{\partial n}\right|_{\Sigma_2} = 0, \ u > 0, \ \lambda > 0. & \end{array}\right\} \qquad (P_\lambda)$$

When $f(z,x) = x^{r-1}$ with $r \in (2, 2^*)$, we get the classical logistic equation, which is important in biological models (see Gurtin & Mac Camy [1]). Depending on the value of $q > 1$, we distinguish three cases:

* Corresponding author at: Department of Mathematics, University of Craiova, Street A.I. Cuza 13, 200585 Craiova, Romania.
E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), vicentiu.radulescu@imar.ro (V.D. Rădulescu), dusan.repovs@guest.arnes.si (D.D. Repovš).

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(i) $1 < q < 2$ (subdiffusive logistic equation); (ii) $2 = q < r$ (equidiffusive logistic equation); (iii) $2 < q < r$ (superdiffusive logistic equation). In this paper, we deal with the third situation (superdiffusive case), which exhibits bifurcation-type phenomena for large values of the parameter $\lambda > 0$ (see also [2]).

Let $E_{\Sigma_1} = \{u \in H^1(\Omega) : u|_{\Sigma_1} = 0\}$. This space is defined as the closure of $C^1_c(\Omega \cup \Sigma_1)$ with respect to the $H^1(\Omega)$-norm. Since $|\Sigma_1|_{N-1} > 0$, we know that for the space E_{Σ_1}, the Poincaré inequality holds (see Gasinski & Papageorgiou [3, Problem 1.139, p. 58]). So, E_{Σ_1} is a Hilbert space equipped with the norm $\|u\| = \|Du\|_2$. Let $\mathcal{A} \in \mathcal{L}(E_{\Sigma_1}, E^*_{\Sigma_1})$ be defined by $\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz$ for all $u, h \in E_{\Sigma_1}$. We denote by N_f the Nemitsky map associated with f, that is, $N_f(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in E_{\Sigma_1}$.

The hypotheses on the perturbation term $f(z, x)$ are the following:

$H(f) : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, x) \geqslant 0$ for all $x > 0$, and

- (i) $f(z, x) \leqslant a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$ and all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $2 < q < r < 2^*$;
- (ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{q-1}} = +\infty$ uniformly for almost all $z \in \Omega$, and the mapping $x \mapsto \frac{f(z,x)}{x}$ is nondecreasing on $(0, +\infty)$ for almost all $z \in \Omega$;
- (iii) $0 \leqslant \liminf_{x\to 0^+} \frac{f(z,x)}{x} \leqslant \limsup_{x\to 0^+} \frac{f(z,x)}{x} \leqslant \hat{\eta}$ uniformly for almost all $z \in \Omega$;
- (iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the function $x \mapsto \hat{\xi}_\rho x - f(z, x)$ is nondecreasing on $[0, \rho]$.

The following functions satisfy hypotheses $H(f)$: (i) $f(x) = x^{r-1}$ for all $x \geqslant 0$ with $2 < q < r < 2^*$; (ii) $f(x) = x^{q-1}\left[\ln(1 + x) + \frac{1}{q}\,\frac{x}{1+x}\right]$ for all $x \geqslant 0$, with $2 < q < 2^*$.

Let $\mathcal{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution}\}$ and let $S(\lambda)$ denote the set of positive solutions of problem (P_λ). Let $\lambda_* = \inf \mathcal{L}$ (if $\mathcal{L} = \emptyset$, then $\inf \emptyset = +\infty$).

By a solution of problem (P_λ), we understand a function $u \in E_{\Sigma_1}$ such that $u \geqslant 0$, $u \neq 0$ and $\langle A(u), h\rangle = \int_\Omega [\lambda u^{q-1} - f(z, u)]h dz$ for all $h \in E_{\Sigma_1}$.

We refer to Bonanno, D'Agui & Papageorgiou [4], Filippucci, Pucci & Rădulescu [5], and Li, Ruf, Guo & Niu [6] for related results. We also refer to the monograph by Pucci & Serrin [7] for more results concerning the abstract setting of this paper.

2. A bifurcation-type theorem

Proposition 1. *If hypotheses $H(f)$ hold, then $S(\lambda) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1/2)$. For all $u \in S(\lambda)$ we have $u(z) > 0$ for all $z \in \Omega$ and $\lambda_* > 0$.*

Proof. From DiBenedetto [8] and Colorado & Peral [9], we know that if $u \in S(\lambda)$ then $u \in C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1/2)$. Moreover, using Harnack's inequality, we deduce that if $u \in S(\lambda)$ then $u(z) > 0$ for all $z \in \Omega$. Let $\hat{\lambda}_1$ be the smallest eigenvalue of $-\Delta$ with mixed boundary conditions. From Colorado & Peral [9, p. 482], we know that $\hat{\lambda}_1 = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in E_{\Sigma_1} \setminus \{0\}\right\} > 0$. By $H(f)$(i), (iii), there exists $\lambda_0 > 0$ such that

$$\lambda_0 x^{q-1} - f(z, x) \leqslant \hat{\lambda}_1 x \text{ for almost all } z \in \Omega, \text{ and all } x \geqslant 0 \tag{1}$$

(recall that $2 < q < r$). Let $\lambda \in (0, \lambda_0)$ and suppose that $\lambda \in \mathcal{L}$. Then there exists $u_\lambda \in S(\lambda)$ and by using Green's identity, we get

$$A(u_\lambda) = \lambda u_\lambda^{q-1} - N_f(u_\lambda) \text{ in } E^*_{\Sigma_1}. \tag{2}$$

We act on (2) with $u_\lambda \in E_{\Sigma_1}$ and obtain $\|Du_\lambda\|_2^2 = \lambda\|u_\lambda\|_q^q - \int_\Omega f(z, u_\lambda)u_\lambda dz < \hat{\lambda}_1\|u_\lambda\|_2^2$ (see (1) and recall that $\lambda < \lambda_0, u_\lambda(z) > 0$ for all $z \in \Omega$), which contradicts the definition of $\hat{\lambda}_1$. Therefore $\lambda \notin \mathcal{L}$ and we have $0 < \lambda_0 \leqslant \lambda_* = \inf \mathcal{L}$. $\square$

Proposition 2. *If hypotheses $H(f)$ hold, then $\mathcal{L} \neq \emptyset$ and "$\lambda \in \mathcal{L}, \eta > \lambda \Rightarrow \eta \in \mathcal{L}$".*

Proof. Fix $\lambda > 0$ and let $\varphi_\lambda : E_{\Sigma_1} \to \mathbb{R}$, $\varphi_\lambda(u) = \frac{1}{2} \parallel Du|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q + \int_\Omega F(z,u)dz$, where $F(z,x) = \int_0^x f(z,s)ds$. Then $\varphi_\lambda \in C^1(E_{\Sigma_1})$ and φ_λ is sequentially weakly lower semicontinuous. Hypotheses $H(f)$(i), (ii) imply that given $\xi > 0$, we can find $c_1 = c_1(\xi) > 0$ such that $F(z,x) \geqslant \frac{\xi}{q}x^q - c_1$ for almost all $z \in \Omega$ and for all $x \geqslant 0$. Thus, for all $u \in E_{\Sigma_1}$ we have $\varphi_\lambda(u) \geqslant \frac{1}{2}\|Du\|_2^2 + \frac{\xi-\lambda}{2}\|u^+\|_q^q - c_1|\Omega|_N$. Choosing $\xi > \lambda$, we deduce that φ_λ is coercive. So, by the Weierstrass–Tonelli theorem, there exists $u_\lambda \in E_{\Sigma_1}$ such that

$$\varphi_\lambda(u_\lambda) = \inf\{\varphi_\lambda(u) : u \in E_{\Sigma_1}\} = m_\lambda. \tag{3}$$

Fix $\bar{u} \in E_{\Sigma_1} \cap C(\overline{\Omega})$ with $u(z) > 0$ for all $z \in \Omega$. For large enough $\lambda > 0$ we have $\varphi_\lambda(\bar{u}) < 0$, hence $\varphi_\lambda(u_\lambda) = m_\lambda < 0 = \varphi_\lambda(0)$ (see (3)). Thus, $u_\lambda \neq 0$. By (3), $\varphi'_\lambda(u_\lambda) = 0$, hence

$$A(u_\lambda) = \lambda(u_\lambda^+)^{q-1} - N_f(u_\lambda) \text{ in } E_{\Sigma_1}^*. \tag{4}$$

We act on (4) with $-u_\lambda^- \in E_{\Sigma_1}$ and obtain $\|Du_\lambda^-\|_2^2 = 0$, hence $u_\lambda \geqslant 0$. So, relation (4) becomes $A(u_\lambda) = \lambda u_\lambda^{q-1} - N_f(u_\lambda)$. By Green's identity, $u_\lambda \in S(\lambda)$, hence $\lambda \in \mathcal{L} \neq \emptyset$.

Next, let $\lambda \in \mathcal{L}$ and $\eta > \lambda$. Choose $\vartheta \in (0,1)$ such that $\lambda = \vartheta^{q-2}\eta$ (recall that $2 < q$). Also, let $u_\lambda \in S(\lambda) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1/2)$. Let $\underline{u} = \vartheta u_\lambda$. Then

$$A(\underline{u}) = \vartheta A(u_\lambda) = \vartheta\left[\lambda u_\lambda^{q-1} - N_f(u_\lambda)\right] \text{ in } E_{\Sigma_1}^*. \tag{5}$$

From hypothesis $H(f)$(ii) and since $u_\lambda(z), \underline{u}(z) > 0$ for all $z \in \Omega$, we have for a.a. $z \in \Omega$

$$\frac{f(z,\underline{u}(z))}{\underline{u}(z)} \leqslant \frac{f(z,u_\lambda(z))}{u_\lambda(z)} \Rightarrow f(z,\underline{u}(z)) \leqslant \vartheta f(z,u_\lambda(z)) \text{ (recall that } \underline{u} = \vartheta u_\lambda). \tag{6}$$

Using (5) in (6) and since $\vartheta \in (0,1)$, we obtain

$$A(\underline{u}) \leqslant \vartheta^{q-1}\eta u_\lambda^{q-1} - N_f(\underline{u}) \leqslant \eta \underline{u}^{q-1} - N_f(\underline{u}) \text{ in } E_{\Sigma_1}^*. \tag{7}$$

We introduce the following Carathéodory truncation of the reaction term in problem (P_η)

$$g_\eta(z,x) = \begin{cases} \eta\underline{u}(z)^{q-1} - f(z,\underline{u}(z)) & \text{if } x \leqslant \underline{u}(z) \\ \eta x^{q-1} - f(z,x) & \text{if } \underline{u}(z) < x. \end{cases} \tag{8}$$

Let $G_\eta(z,x) = \int_0^x g_\eta(z,s)ds$ and define $\hat{\varphi}_\eta : E_{\Sigma_1} \to \mathbb{R}$ by $\hat{\varphi}_\eta(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_\eta(z,u)dz$.

Hypotheses $H(f)$(i), (ii) imply that given $\xi > 0$, we can find $c_2 = c_2(\xi) > 0$ such that

$$\eta x^{q-1} - f(z,x) \leqslant (\eta - \xi)x^{q-1} + c_2 \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{9}$$

Then for all $u \in E_{\Sigma_1}$, we have

$$\hat{\varphi}_\eta(u) \geqslant \frac{1}{2}\|Du\|_2^2 + \frac{\xi - \eta}{q}\|u^+\|_q^q - c_3 \text{ for some } c_3 > 0 \text{ (see (8), (9))}. \tag{10}$$

Choosing $\xi > \eta$, we see from (10) that $\hat{\varphi}_\eta$ is coercive. This function is also sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, there exists $u_\eta \in E_{\Sigma_1}$ such that $\hat{\varphi}_\eta(u_\eta) = \inf[\hat{\varphi}_\eta(u) : u \in E_{\Sigma_1}]$, hence $\hat{\varphi}'_\eta(u_\eta) = 0$. We deduce that

$$A(u_\eta) = N_{g_\eta}(u_\eta) \text{ in } E_{\Sigma_1}^*. \tag{11}$$

We act on (11) with $(\underline{u}-u_\eta)^+ \in E_{\Sigma_1}$. By (8) and (7) we have

$$\left\langle A(u_\eta), (\underline{u}-u_\eta)^+\right\rangle = \int_\Omega [\eta \underline{u}^{q-1} - f(z,\underline{u})](\underline{u}-u_\eta)^+ dz \ \geqslant \left\langle A(\underline{u}), (\underline{u}-u_\eta)^+\right\rangle$$

$$\Rightarrow \left\langle A(\underline{u}-u_\eta), (\underline{u}-u_\eta)^+\right\rangle \leqslant 0 \Rightarrow \|D(\underline{u}-u_\eta)^+\|_2^2 \leqslant 0 \Rightarrow \underline{u} \leqslant u_\eta. \tag{12}$$

Using (8) and (12) we see that relation (11) becomes $A(u_\lambda) = \eta u_\eta^{q-1} - N_f(u_\eta)$ in $E^*_{\Sigma_1}$. Thus, by Proposition 1, we have $u_\eta \in S(\eta) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$. Therefore $\eta \in \mathcal{L}$. We also observe that Proposition 2 implies $(\lambda_*, +\infty) \subseteq \mathcal{L}$. □

Proposition 3. *If hypotheses $H(f)$ hold and $\lambda > \lambda_*$, then problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1/2)$ with $0 < u_0(z), \hat{u}(z)$ for all $z \in \Omega$.*

Proof. Let $\mu \in (\lambda_*, \lambda)$. By Proposition 2 we know that $\mu \in \mathcal{L}$. Hence we can find $u_\mu \in S(\mu) \subseteq E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1/2), u_\mu(z) > 0$ for all $z \in \Omega$. We have $A(u_\mu) = \mu u_\mu^{q-1} - N_f(u_\mu)$ in $E^*_{\Sigma_1}$. Next, we define the following Carathéodory function

$$\hat{h}_\lambda(z,x) = \begin{cases} \lambda u_\mu(z)^{q-1} - f(z,u_\mu(z)) & \text{if } x \leqslant u_\mu(z) \\ \lambda x^{q-1} - f(z,x) & \text{if } u_\mu(z) < x. \end{cases} \tag{13}$$

Let $\hat{H}_\lambda(z,x) = \int_0^x \hat{h}_\lambda(z,s)ds$ and let $\hat{\psi}_\lambda : E_{\Sigma_1} \to \mathbb{R}$, $\hat{\psi}_\lambda(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{H}_\lambda(z,u)dz$. Then $\hat{\psi}_\lambda$ is coercive and sequentially weakly lower semicontinuous. Thus, we can find $u_0 \in E_{\Sigma_1}$ such that $\hat{\psi}_\lambda(u_0) = \inf\{\hat{\psi}_\lambda(u) : u \in E_{\Sigma_1}\}$, hence $\hat{\psi}'_\lambda(u_0) = 0$. Thus, $A(u_0) = N_{\hat{h}_\lambda}(u_0)$. Using (13) and reasoning as in the proof of Proposition 2 we deduce that $u_\mu \leqslant u_0$. By Colorado & Peral [9, Theorem 6.6], we have $u_0 \in E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1/2)$ and $u_0 > 0$ in $z\Omega$ (by Harnack's inequality).

Let $\rho_0 = \|u_0\|_\infty$ and let $\hat{\xi}_{\rho_0} > 0$ be as postulated in hypothesis $H(f)$(iv). We have

$$\left\{\begin{aligned} &-\Delta u_0(z) + \hat{\xi}_{\rho_0} u_0(z) = \lambda u_0(z)^{q-1} - f(z,u_0(z)) + \hat{\xi}_{\rho_0} u_0(z) \text{ in } \Omega, \\ &u_0|_{\Sigma_1} = 0, \left.\frac{\partial u_0}{\partial n}\right|_{\Sigma_2} = 0 \end{aligned}\right\} \tag{14}$$

and

$$\left\{\begin{aligned} &-\Delta u_\mu(z) + \hat{\xi}_{\rho_0} u_\mu(z) = \mu u_\mu(z)^{q-1} - f(z,u_\mu(z)) + \hat{\xi}_{\rho_0} u_\mu(z) \text{ in } \Omega, \\ &\hat{u}_\mu|_{\Sigma_1} = 0, \left.\frac{\partial u_\mu}{\partial n}\right|_{\Sigma_2} = 0. \end{aligned}\right\} \tag{15}$$

Let $\hat{y} = u_0 - u_\mu \geqslant 0$. Since $\lambda > \mu, u_0 \geqslant u_\mu$, from (14), (15), and $H(f)$(iv) we have

$$\begin{aligned} &-\Delta \hat{y}(z) + \hat{\xi}_{\rho_0} \hat{y}(z) = \lambda u_0(z)^{q-1} - \mu u_\mu(z)^{q-1} + [\hat{\xi}_{\rho_0} u_0(z) - f(z,u_0(z))] - \\ &-[\hat{\xi}_{\rho_0} u_\mu(z) - f(z,u_\mu(z))] \geqslant 0 \text{ in } \Omega. \end{aligned}$$

Let $v_1 \in E_{\Sigma_1}$ be the unique function satisfying $-\Delta v(z) + \hat{\xi}_{\rho_0} v(z) = 1$ Ω, $v|_{\Sigma_1} = 0$, and $\frac{\partial v}{\partial n}\big|_{\Sigma_2} = 0$. Then $v_1 \in C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1/2)$ (see [8,9]) and $v_1 > 0$ in Ω. By Lemma 2.1 of Barletta, Livrea & Papageorgiou [10] (see also Lemma 5.3 of Colorado & Peral [9]), we can find $\vartheta > 0$ such that

$$\vartheta v_1(z) \leqslant u_\mu(z) \text{ and } \vartheta v_1(z) \leqslant \hat{y}(z) \Rightarrow \vartheta v_1(z) \leqslant u_\mu(z) \leqslant u_0(z) - \vartheta v_1(z) \text{ for all } z \in \overline{\Omega}. \tag{16}$$

Let $\hat{C}_1 = \left\{ y \in E_{\Sigma_1} \cap C(\overline{\Omega}) : \left\| \frac{y}{v_1} \right\|_\infty < \infty \right\}$ and $[u_\mu) = \{u \in E_{\Sigma_1} : u_\mu(z) \leqslant u(z), \text{ a.a. } z \in \Omega\}$. We claim that if $\bar{B}_1(0) := \{y \in \hat{C}_1 : \left\| \frac{y}{v_1} \right\|_\infty \leqslant 1\}$, then $u_0 - \vartheta \bar{B}_1(0) \subseteq [u_\mu) \cap \hat{C}_1$. To see this, let $y \in \bar{B}_1(0)$. Then

$$-v_1(z) \leqslant y(z) \leqslant v_1(z) \text{ for all } z \in \overline{\Omega}. \tag{17}$$

Fix $z \in \overline{\Omega}$. If $y(z) > 0$, then $0 \leqslant u_\mu(z) \leqslant u_\mu(z) + \vartheta y(z) \leqslant u_\mu(z) + \vartheta v_1(z) \leqslant u_0(z)$ (see (16), (17)), hence $u_\mu(z) \leqslant u_0(z) - \vartheta y(z)$. If $y(z) < 0$, then $0 \leqslant u_\mu(z) - \vartheta v_1(z) \leqslant u_\mu(z) + \vartheta y(z) \leqslant u_\mu(z) + \vartheta v_1(z) \leqslant u_0(z)$ (see (16), (17)), hence $u_\mu(z) \leqslant u_0(z) - \vartheta y(z)$. We conclude that $u_\mu \in u_0 - \vartheta \bar{B}_1(0)$, which proves the claim. It follows that

$$u_0 \in \operatorname{int}_{\hat{C}_1} [u_\mu) \cap C(\overline{\Omega}). \tag{18}$$

By (13) it is clear that

$$\hat{\psi}_\lambda(u) = \varphi_\lambda(u) + c_4 \text{ for some } c_4 \in \mathbb{R} \text{ and for all } u \in [u_\mu)\,. \tag{19}$$

It follows from (18) and (19) that u_0 is a local $\hat{C}_1$-minimizer of φ_λ.

Claim. *u_0 is a local E_{Σ_1}-minimizer of φ_λ.*

Suppose that this assertion is not true. Then for every $\rho > 0$, we have $\inf\{\varphi_\lambda(u_0 + y) : y \in E_{\Sigma_1},\ \|y\| \leqslant \rho\} < \varphi_\lambda(u_0)$. By the Weierstrass–Tonelli theorem, there exists $y_\rho \in E_{\Sigma_1} \setminus \{0\},\ \|y_\rho\| \leqslant \rho$ such that $\varphi_\lambda(u_0 + y_\rho) = \inf\{\varphi_\lambda(u_0 + y) : y \in E_{\Sigma_1}, \|y\| \leqslant \rho\} < \varphi_\lambda(y_0)$. By the Lagrange multiplier rule, there exists $\vartheta \leqslant 0$ such that $(1 - \vartheta)\langle A(u_\rho), h\rangle = \lambda \int_\Omega (u_\rho^+)^{q-1} h dz - \int_\Omega f(z, u_\rho) h dz$ for all $h \in E_{\Sigma_1}$, with $u_\rho = u_0 + y_\rho \in E_{\Sigma_1}$. It follows that $\Delta u_\rho(z) = \frac{1}{1-\vartheta}[\lambda u_\rho^+(z)^{q-1} - f(z, u_\rho(z))]$ in Ω, hence

$$-\Delta u_\rho(z) + \hat{\xi}_{\rho_0} u_\rho(z) = \frac{1}{1-\vartheta}[\lambda u_\rho^+(z)^{q-1} + f(z, u_\rho(z))] + \hat{\xi}_{\rho_0} u_\rho(z) \text{ in } \Omega, \tag{20}$$

with $\hat{\xi}_{\rho_0} > 0$ as before resulting from hypothesis $H(f)$(iv) (recall that $\rho_0 = \|u_0\|_\infty$). Also,

$$-\Delta u_0(z) + \hat{\xi}_{\rho_0} u_0(z) = \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \hat{\xi}_{\rho_0} u_0(z) \text{ in } \Omega. \tag{21}$$

From (20) and (21) we obtain

$$-\Delta y_\rho(z) + \hat{\xi}_{\rho_0} y_\rho(z) = g_\lambda^\rho(z) \text{ in } \Omega \tag{22}$$

with $g_\lambda^\rho(z) = \frac{1}{1-\vartheta}[\lambda u_\rho^+(z)^{q-1} - f(z, u_\rho(z))] - \lambda u_0(z)^{q-1} + f(z, u_0(z)) + \hat{\xi}_{\rho_0} y_\rho(z)$. By (22) and Colorado & Peral [9], there exist $c_5 > 0$ and $\alpha \in (0, 1/2)$ such that

$$y_\rho \in C^{0,\alpha}(\overline{\Omega}) \text{ and } \|y_\rho\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant c_5 \text{ for all } \rho \in (0,1]\,. \tag{23}$$

Exploiting the compact embedding of $C^{0,\alpha}(\overline{\Omega})$ into $C(\overline{\Omega})$, we have $y_\rho \to 0$ in $C(\overline{\Omega})$ as $\rho \to 0^+$. Thus, by the definition of g_λ^ρ, there exists $\tau_\rho^* > 0$ such that

$$\|g_\lambda^\rho\|_\infty \leqslant \tau_\rho^* \text{ for all } \rho \in (0,1] \text{ and } \tau_\rho^* \to 0^+ \text{ as } \rho \to 0^+. \tag{24}$$

Let $\hat{y}_\rho = \frac{1}{\tau_\rho^*} y_\rho$. Then by (24) $-\Delta(\hat{y}_\rho - v_1)(z) + \hat{\xi}_{\rho_0}(\hat{y}_\rho - v_1)(z) = \frac{1}{\tau_\rho^*} g_\lambda^\rho(z) - 1 \leqslant 0$. We deduce that $\|D(\hat{y}_\rho - v_1)^+\|_2^2 + \hat{\xi}_{\rho_0}\|(\hat{y}_\rho - v_1)^+\|_2^2 \leqslant 0$, hence $y_\rho \leqslant \tau_\rho^* v_1$.

Also, we have $-\Delta(-\hat{y}_\rho - v_1)(z) + \hat{\xi}_{\rho_0}(-\hat{y}_\rho - v_1)(z) = -\frac{1}{\tau_\rho^*} g_\lambda^\rho(z) - 1 \leqslant 0$ in Ω and so as above we obtain that $-\tau_\rho^* v_1 \leqslant y_\rho$. Therefore we have proved that $-\tau_\rho^* v_1 \leqslant y_\rho \leqslant \tau_\rho^* v_1$. These relations show that $y_\rho \in \hat{C}_1$ and $\left\|\frac{y_\rho}{v_1}\right\|_\infty \leqslant \tau_\rho^*$ for all $\rho \in (0,1]$, hence $y_\rho \to 0$ in $\hat{C}_1$ as $\rho \to 0^+$. Therefore for small $\rho \in (0,1]$ we have $\varphi_\lambda(u_0 + y_\rho) < \varphi_\lambda(u_0)$, which contradicts the fact that u_0 is a local $\hat{C}_1$-minimizer of φ_λ. This proves the claim.

Since $f \geqslant 0$, for all $u \in E_{\Sigma_1}$ we have $\varphi_\lambda(u) \geqslant \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{q}\|u^+\|_q^q \geqslant \frac{1}{2}\|Du\|_2^2 - c_6\|Du\|_2^q$ for some $c_6 > 0$. Since $q > 2$, we deduce that $u = 0$ is a local minimizer of φ_λ. We assume that the set of critical points of φ_λ

is finite (otherwise we already have an infinity of positive solutions for (P_λ) for $\lambda > \lambda_*$ and so we are done) and that $\varphi_\lambda(0) \leqslant \varphi_\lambda(u_0)$ (the reasoning is similar if the opposite inequality holds). The claim implies that we can find small enough $\rho \in (0, \|u_0\|)$ such that $0 = \varphi_\lambda(0) \leqslant \varphi_\lambda(u) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda^\rho$. Thus, we can apply the mountain pass theorem. So, there exists $\hat{u} \in E_{\Sigma_1}$ such that $\varphi'_\lambda(\hat{u}) = 0$ and $m_\lambda^\rho \leqslant \varphi_\lambda(\hat{u})$, hence $\hat{u} \notin \{0, u_0\}$, $\hat{u} \in S_\lambda \subseteq E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$, and $\hat{u} > 0$ in Ω. □

Proposition 4. *If hypotheses $H(f)$ hold, then $\lambda_* \in \mathcal{L}$, that is, $\mathcal{L} = [\lambda^*, +\infty)$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq (\lambda_*, +\infty)$ be such that $\lambda_n \downarrow \lambda_*$. We find $u_n \in S(\lambda_n)$ such that

$$A(u_n) = \lambda u_n^{q-1} - N_f(u_n) \text{ in } E_{\Sigma_1}^* \text{ for all } n \in \mathbb{N}. \tag{25}$$

Hypotheses $H(f)$(i), (ii) imply that given any $\xi > 0$, we find $c_7 = c_7(\xi) > 0$ such that

$$f(z, x) \geqslant \xi x^{q-1} - c_7 \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{26}$$

We act on (25) with $u_n \in E_{\Sigma_1}$ and then use (26). We obtain $\|Du\|_2^2 \leqslant (\lambda_n - \xi)\|u_n\|_q^q + c_7|\Omega|_N$. Choosing $\xi > \lambda_1 \geqslant \lambda_n$ for all $n \in \mathbb{N}$, we have $\|Du_n\|_2^2 \leqslant c_7|\Omega|_N$ for all $n \in \mathbb{N}$, hence $\{u_n\}_{n\geqslant 1} \subseteq E_{\Sigma_1}$ is bounded. By passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } E_{\Sigma_1} \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{27}$$

In (25) we pass to the limit as $n \to \infty$ and use (27). Then $A(u_*) = \lambda_* u_*^{q-1} - N_f(u_*)$. Thus, $u_* \in E_{\Sigma_1}$ and $u_* \geqslant 0$ is a solution of (P_{λ_*}). We also notice that $\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0$, hence $\|Du_n\|_2 \to \|Du_*\|_2$. Using the Kadec–Klee property we deduce that $u_n \to u_*$ in E_{Σ_1}.

Claim. $u_* \neq 0$.

Arguing by contradiction, suppose that $u_* = 0$. Then $\|u_n\| \to 0$. Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$. From (25) we have

$$A(y_n) = \lambda_n u_n^{q-2} y_n - \frac{N_f(u_n)}{\|u_n\|} \text{ for all } n \in \mathbb{N}. \tag{28}$$

From hypotheses $H(f)$(i), (iii), we see that we can find $\eta > \hat{\eta}$ and $c_8 > 0$ such that

$$f(z, x) \leqslant \eta x + c_8 x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0 \Rightarrow \{N_f(u_n)\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.} \tag{29}$$

By [9], there exist $\alpha \in (0, 1/2)$ and $c_9 > 0$ such that $u_n \in C^{0,\alpha}(\overline{\Omega})$, $\|u_n\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant c_9$ for all $n \in \mathbb{N}$. Since $C^{0,\alpha}(\overline{\Omega})$ is compactly embedded compactly in $C(\overline{\Omega})$, we deduce that

$$u_n \to 0 \text{ in } C(\overline{\Omega}). \tag{30}$$

Recall that $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } E_{\Sigma_1} \text{ and } y_n \to y \text{ in } L^2(\Omega), y \geqslant 0. \tag{31}$$

It follows from (29), (30) and (31) that $\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded. Thus, by hypothesis $H(f)$(iii), we have at least for a subsequence (see [11]),

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} \eta_0 y \text{ in } L^2(\Omega) \text{ with } 0 \leqslant \eta_0(z) \leqslant \hat{\eta} \text{ for almost all } z \in \Omega. \tag{32}$$

We act on (28) with $y_n - y \in E_{\Sigma_1}$ and pass to the limit as $n \to \infty$. Using (30), (31) and (32) we obtain $\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0$. By the Kadec–Klee property we have $y_n \to y$, hence $\|y\| = 1$, $y \geqslant 0$. In (28) we pass to the limit as $n \to \infty$ and use (30), (32). Then $A(y) = -\eta_0 y$. Thus, by (32) we have $\|Dy\|_2^2 = -\int_\Omega \eta_0 y^2 dz \leqslant 0$, hence $y = 0$, a contradiction. This shows that the claim is true. Hence $u_* \in S(\lambda_*) \subseteq E_{\Sigma_1} \cap C(\overline{\Omega})$ and so $\lambda_* \in \mathcal{L}$. □

Summarizing, we can state the following bifurcation-type theorem.

Theorem 5. *If hypotheses $H(f)$ hold, then there exists $\lambda_* > 0$ such that*

(a) *for all $\lambda > \lambda_*$, problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in E_{\Sigma_1} \cap C(\overline{\Omega})$;*
(b) *for $\lambda = \lambda_*$, problem (P_λ) has at least one positive solution $u_* \in E_{\Sigma_1} \cap C(\overline{\Omega})$;*
(c) *for $\lambda \in (0, \lambda_*)$, problem (P_λ) has no positive solutions.*

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