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POSITIVE SOLUTIONS FOR PERTURBATIONS OF THE ROBIN EIGENVALUE PROBLEM PLUS AN INDEFINITE POTENTIAL

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Abstract. We study perturbations of the eigenvalue problem for the negative Laplacian plus an indefinite and unbounded potential and Robin boundary condition. First we consider the case of a sublinear perturbation and then of a superlinear perturbation. For the first case we show that for $\lambda < \widehat{\lambda}_1$ ($\widehat{\lambda}_1$ being the principal eigenvalue) there is one positive solution which is unique under additional conditions on the perturbation term. For $\lambda \geq \widehat{\lambda}_1$ there are no positive solutions. In the superlinear case, for $\lambda < \widehat{\lambda}_1$ we have at least two positive solutions and for $\lambda \geq \widehat{\lambda}_1$ there are no positive solutions. For both cases we establish the existence of a minimal positive solution $\bar{u}_\lambda$ and we investigate the properties of the map $\lambda \mapsto \bar{u}_\lambda$.

1. Introduction.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear parametric Robin problem with an indefinite and unbounded potential $\xi(z)$:

$$\left\{ \begin{array}{l} -\Delta u(z) + \xi(z)u(z) = \lambda u(z) + f(z, u(z)) \ \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \ \text{ on } \partial\Omega, \ u \geq 0. \end{array} \right\} \qquad (P_\lambda)$$

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In this problem $\lambda \in \mathbb{R}$ is a parameter and $\xi(\cdot)$ is a potential function which is indefinite (that is, $\xi(\cdot)$ is sign changing) and unbounded from below. We can think of (P_λ) as a perturbation of the standard eigenvalue problem for the differential operator $u \mapsto -\Delta u + \xi(z)u$ with Robin boundary condition. We look for positive solutions and consider two cases: a sublinear perturbation $f(z,\cdot)$ and a superlinear perturbation $f(z,\cdot)$. For both cases we determine the dependence of the set of positive solutions as the parameter $\lambda \in \mathbb{R}$ varies.

We mention that the standard eigenvalue problems for the Robin Laplacian have recently been studied by D'Agui, Marano & Papageorgiou [4] and by Papageorgiou & Rădulescu [11]. Additional existence and multiplicity results for parametric Robin and Neumann problems can be found in the works of Papageorgiou & Rădulescu [12, 13].

Let $\widehat{\lambda}_1 \in \mathbb{R}$ be the principal eigenvalue of the operator $u \mapsto -\Delta u + \xi(z)u$ with Robin boundary condition. In the *sublinear* case (that is, when $f(z,\cdot)$ is sublinear near $+\infty$) we show that for $\lambda \geq \widehat{\lambda}_1$ problem (P_λ) has no positive solution, whereas for $\lambda < \widehat{\lambda}_1$ problem (P_λ) has at least one positive solution. In fact, we show that under an additional monotonicity condition on the quotient $x \mapsto \frac{f(z,x)}{x}$ on $(0,+\infty)$, this positive solution is unique. In the *superlinear* case (that is, when $f(z,\cdot)$ is superlinear near $+\infty$) the situation changes and uniqueness of the solution fails. In fact, we show that the equation exhibits a kind of bifurcation phenomenon. Namely, for $\lambda \geq \widehat{\lambda}_1$ problem (P_λ) has no positive solution, whereas for $\lambda < \widehat{\lambda}_1$ it has at least two positive solutions. For both cases, we show that the problem has a minimal (that is, smallest) positive solution $\bar{u}_\lambda$ and we determine the monotonicity and continuity properties of the map $\lambda \mapsto \bar{u}_\lambda$.

Our approach is variational, based on the critical point theory, together with suitable truncation and perturbation techniques. In the next section, for the convenience of the reader, we recall the main mathematical tools which will be used in the sequel.

2. **Mathematical background.**

Let X be a Banach space and X^* be its topological dual. By $\langle \cdot,\cdot \rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the *Cerami condition* (the *C-condition* for short), if the following is true:

Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1+\|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence.

This compactness-type condition on φ replaces the local compactness of the ambient space X (in most applications X is infinite dimensional and so it is not locally compact). It leads to a deformation theorem, from which one can derive the minimax theory of the critical values of φ. A central result of this theory, is the so-called *mountain pass theorem*, which we state here in a slightly more general version (see, for example, Gasinski & Papageorgiou [6]).

Theorem 2.1. *Assume that $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho$,*

$$\max\{\varphi(u_0),\varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = m_\rho$$

and $c = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1}\varphi(\gamma(t))$, *where* $\Gamma = \{\gamma\in C([0,1],X) : \gamma(0)=u_0, \gamma(1)=u_1\}$. *Then* $c \geq m_\rho$ *and* c *is a critical value of* φ.

The analysis of problem (P_λ) will make use of the Sobolev space $H^1(\Omega)$. This is a Hilbert space with inner product

$$(u,h)_{H^1(\Omega)} = \int_\Omega uh\,dz + \int_\Omega (Du,Dh)_{\mathbb{R}^N}\,dz \text{ for all } u,h\in H^1(\Omega).$$

The norm corresponding to this inner product, is given by

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2} \text{ for all } u\in H^1(\Omega).$$

In addition, we will also use the Banach spaces $C^1(\overline{\Omega})$ and $L^\tau(\partial\Omega)$, $1\le\tau\le\infty$.

The space $C^1(\overline{\Omega})$ is an ordered Banach space with order (positive) cone

$$C_+ = \{u\in C^1(\overline{\Omega}) :\ u(z)\ge 0 \text{ for all } z\in\overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \{u\in C_+ :\ u(z) > 0 \text{ for all } z\in\overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure we can define the Lebesgue spaces $L^\tau(\partial\Omega)$ $(1\le\tau\le\infty)$ in the usual way. The theory of Sobolev spaces says that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega)\to L^2(\partial\Omega)$, known as the *trace map*, such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u\in H^1(\Omega)\cap C(\overline{\Omega}).$$

Therefore we can interpret $\gamma_0(u)$ as representing the *boundary values* of $u\in H^1(\Omega)$. From the general theory of Sobolev spaces, we know that

$$\operatorname{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad\text{and}\quad \ker\gamma_0 = H_0^1(\Omega).$$

In addition, we know that $\gamma_0(\cdot)$ is compact from $H^1(\Omega)$ into $L^\tau(\partial\Omega)$ with $\tau\in\left[1, \frac{2(N-1)}{N-2}\right)$ if $N\ge 3$ and $\tau\in[1,+\infty)$ if $N=1,2$.

In what follows, for the sake of notational simplicity, we shall drop the use of the map γ_0. All restrictions of functions $u\in H^1(\Omega)$ on the boundary $\partial\Omega$, are understood in the sense of traces.

Suppose that $f_0 : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (that is, for all $x\in\mathbb{R}$, $z\mapsto f(z,x)$ is measurable and for a.a. $z\in\Omega$, $x\mapsto f(z,x)$ is continuous) which has subcritical growth in the $x\in\mathbb{R}$ variable. Hence

$$|f_0(z,x)| \le a_0(z)(1+|x|^{r-1}) \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R},$$

with $a_0\in L^\infty(\Omega)_+$, $2\le r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N\ge 3\\ +\infty & \text{if } N=1,2\end{cases}$ (the critical Sobolev exponent). We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0 : H^1(\Omega)\to\mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\gamma(u) - \int_\Omega F_0(z,u(z))\,dz \text{ for all } u\in H^1(\Omega),$$

where $\gamma : H^1(\Omega)\to\mathbb{R}$ is the C^1-functional defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega}\beta(z)u^2\,d\sigma \text{ for all } u\in H^1(\Omega).$$

The next result relates local minimizers of φ_0 in $C^1(\overline{\Omega})$ and in $H^1(\Omega)$, respectively. It is an outgrowth of the regularity theory for such problems (see Wang

[14]) and a more general version of it (with proof) can be found in Papageorgiou & Rădulescu [11]. We mention that the first result of this kind for the space $H_0^1(\Omega)$ (Dirichlet problems) and $\xi \equiv 0$, was proved by Brezis & Nirenberg [3]. Our conditions on the potential $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$ are:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$ if $N \geq 3$ and $s > 1$ if $N = 1, 2$

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geq 0$ for all $z \in \partial\Omega$

Remark. If $\beta \equiv 0$, then we recover the Neumann problem. Hence our present paper includes the Neumann problems as a special case.

Proposition 1. *Assume that hypotheses $H(\xi)$, $H(\beta)$ hold and $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer, that is, there exists $\rho_1 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0 + h)$ for all $h \in C^1(\overline{\Omega})$ with $\|h\|_{C^1(\overline{\Omega})} \leq \rho_1$. Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is a local $H^1(\Omega)$-minimizer of φ_0, that is, there exists $\rho_2 > 0$ such that $\varphi_0(u_0) \leq \varphi_0(u_0 + h)$ for all $h \in H^1(\Omega)$ with $\|h\| \leq \rho_2$.*

Next we recall some basic facts concerning the spectrum of the differential operator $u \mapsto -\Delta u + \xi(z)u$ with Robin boundary condition. So we consider the following linear eigenvalue problem

$$\left\{ \begin{array}{l} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda} u(z) \ \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \ \text{ on } \partial\Omega. \end{array} \right\} \tag{1}$$

By an eigenvalue we mean a $\widehat{\lambda} \in \mathbb{R}$ for which problem (1) has a nontrivial solution $\hat{u} \in H^1(\Omega)$, called an eigenfunction corresponding to $\widehat{\lambda}$. Using the spectral theorem for compact self-adjoint operators on a Hilbert space, we know that the spectrum of (1) consists of a sequence $\{\widehat{\lambda}_k\}_{k\geq 1}$ of distinct eigenvalues such that $\widehat{\lambda}_k \to +\infty$ (see [4, 11]). The first eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ has the following properties:

- $\widehat{\lambda}_1$ is simple with eigenfunctions of constant sign;
- we have
$$\widehat{\lambda}_1 = \inf \left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0 \right]. \tag{2}$$

The infimum in (2) is realized on the corresponding one dimensional eigenspace. Let $\hat{u}_1$ be the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$. If hypotheses $H(\xi)$ and $H(\beta)$ hold, then $\hat{u}_1 \in C^1(\overline{\Omega})$ (see Wang [14]) and by Harnack's inequality (see Gasinski & Papageorgiou [6, p. 731]), we have $\hat{u}_1(z) > 0$ for all $z \in \Omega$. Moreover, if in addition we assume that $\xi^+ \in L^\infty(\Omega)$, then $\hat{u}_1 \in \operatorname{int} C_+$ (via the strong maximum principle, see for example Gasinski & Papageorgiou [6, p. 738]).

As a consequence of these properties, we obtain the following simple lemma.

Lemma 2.2. *If hypotheses $H(\xi)$, $H(\beta)$ hold, $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leq \widehat{\lambda}_1$ for a.a. $z \in \Omega$ and $\vartheta \not\equiv \widehat{\lambda}_1$, then there exists $\hat{c} > 0$ such that*

$$J(u) = \gamma(u) - \int_\Omega \vartheta(z)u^2\, dz \geq \hat{c}\|u\|^2 \ \text{ for all } \ u \in H^1(\Omega).$$

Proof. From the variational characterization of $\widehat{\lambda}_1$ (see (2)), we get that $J \geq 0$. Suppose that the lemma is not true. Exploiting the 2-homogeneity of $J(\cdot)$, we can then find $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$\|u_n\| = 1 \text{ for all } n \in \mathbb{N} \text{ and } J(u_n) \downarrow 0 \text{ as } n \to \infty. \tag{3}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{4}$$

It follows from (3) and (4) that

$$\begin{aligned} & J(u) \leq 0, \\ \Rightarrow\, & \gamma(u) \leq \int_\Omega \vartheta(z) u^2\, dz \leq \widehat{\lambda}_1 \|u\|_2^2 \\ \Rightarrow\, & \gamma(u) = \widehat{\lambda}_1 \|u\|_2^2 \quad (\text{see } (2)), \\ \Rightarrow\, & u = \chi \hat{u}_1 \text{ with } \chi \in \mathbb{R}. \end{aligned} \tag{5}$$

If $\chi = 0$, then $u = 0$ and so

$$\begin{aligned} & \|Du_n\|_2 \to 0 \text{ (see (3))}, \\ \Rightarrow\, & u_n \to 0 \text{ in } H^1(\Omega) \text{ (see (4))}, \end{aligned}$$

which contradicts the fact that $\|u_n\| = 1$ for all $n \in \mathbb{N}$ (see (3)).

Hence $\chi \neq 0$. Then $u(z) \neq 0$ for all $z \in \Omega$ and by (5) we have

$$\|Du\|_2^2 < \widehat{\lambda}_1 \|u\|_2^2,$$

which contradicts (2). □

We observe that there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \text{ for some } c_0 > 0, \text{ all } u \in H^1(\Omega) \tag{6}$$

(see [4, 13]).

To see (6), we argue by contradiction. So, suppose that the inequality is not true. We can find $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$\gamma(u_n) + n\,\|u_n\|_2^2 < \frac{1}{n}\,\|u_n\| \quad \text{for all } n \in \mathbb{N}\,.$$

Set $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega).$$

Exploiting the sequential weak lower semicontinuity of $\gamma(\,\cdot\,)$ we see that $y = 0$ and $n\|y_n\|_2^2 \to 0$. Therefore $\|y_n\|_2 \to 0$. Finally we can say that

$$\begin{aligned} & 0 \leq \liminf_{n\to\infty} \gamma(y_n) \leq \limsup_{n\to\infty} \gamma(y_n) \leq \lim_{n\to\infty} \left[\frac{1}{n} - n\|y_n\|_2^2\right] = 0, \\ \Rightarrow\;\; & 0 = \lim_{n\to\infty} \gamma(y_n) = \lim_{n\to\infty} \|Dy_n\|_2^2 \text{ (recall } y = 0) \\ \Rightarrow\;\; & y_n \to 0 \text{ in } H^1(\Omega), \text{ a contradiction to the fact that } \|y_n\| = 1. \end{aligned}$$

We say that a Banach space X has the *Kadec-Klee property* if the following is true:

$$u_n \xrightarrow{w} u \text{ in } X \text{ and } \|u_n\| \to \|u\| \Rightarrow u_n \to u \text{ in } X. \tag{7}$$

As a consequence of the parallelogram law, we see that every Hilbert space has the Kadec-Klee property.

Given $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then for every $u \in H^1(\Omega)$ we define

$$u^{\pm}(\cdot) = u(\cdot)^{\pm}.$$

We know that

$$u^{\pm} \in H^1(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^- \text{ for all } u \in H^1(\Omega).$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in H^1(\Omega)$$

(the Nemytskii or superposition map corresponding to the function h).

Given $\varphi \in C^1(X, \mathbb{R})$ (X a Banach space), we set

$$K_\varphi = \{u \in X : \ \varphi'(u) = 0\} \text{ (the critical set of } \varphi).$$

Finally, by $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u, h \in H^1(\Omega).$$

3. **The sublinear case.** In this section we examine problem (P_λ) under the hypothesis that the perturbation term $f(z, \cdot)$ is sublinear near $+\infty$. More precisely, our conditions on the nonlinearity $f(z, x)$ are the following:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, x) > 0$ for all $x > 0$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that $f(z, x) \le a_\rho(z)$ for a.a. $z \in \Omega$, all $0 \le x \le \rho$;

(ii) $\lim\limits_{x \to +\infty} \frac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) there exist $\delta > 0$ and $q \in (1, 2)$ such that $c_1 x^{q-1} \le f(z, x)$ for a.a. $z \in \Omega$, all $0 \le x \le \delta$.

Remarks. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$. Hypothesis H_1(ii) implies that $f(z, \cdot)$ is sublinear near $+\infty$. Hypothesis H_1(iii) says that there is a concave term near the origin.

Examples. The following functions satisfy hypotheses H_1. For the sake of simplicity we drop the z-dependence

$$f_1(x) = x^{q-1} \text{ for all } x \ge 0, \text{ with } 1 < q < 2,$$

$$f_2(x) = \begin{cases} -x^{q-1} \ln x & \text{if } x \in [0, 1] \\ x^{s-1} - x^{p-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < p < s < 2,\ 1 < q < 2.$$

We can improve the properties of the positive solutions of (P_λ), provided we strengthen hypothesis $H(\xi)$.

$H(\xi)'$: $\xi \in L^s(\Omega)$ with $s > N$ if $N \ge 3$, $s > 1$ if $N = 1, 2$ and $\xi^+ \in L^\infty(\Omega)$.

We introduce the following two sets:

$$\mathcal{L} = \{\lambda \in \mathbb{R} : \text{ problem } (P_\lambda) \text{ admits a positive solution}\},$$

$$S(\lambda) = \{\text{the set of positive solutions for problem } (P_\lambda)\}.$$

We start with a simple but useful observation concerning the solution set $S(\lambda)$. For this result the precise behavior of $f(z,\cdot)$ near $+\infty$ and near 0^+ are irrelevant. We only need nonnegativity of $f(z,x)$ and subcritical growth in $x \in \mathbb{R}$. Under these very general conditions, the result is also applicable in the superlinear case (see Section 4). The new hypotheses on the perturbation term $f(z,x)$ are the following:

$\widehat{H}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) \geq 0$ for all $x \geq 0$ and

$$f(z,x) \leq a(z)(1 + x^{r-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0,$$

with $a \in L^\infty(\Omega)_+$, $2 \leq r < 2^*$.

Proposition 2. *If hypotheses $H(\xi)$ (resp. $H(\xi)'$), $H(\beta)$, $\widehat{H}$ hold, then for all $\lambda \in \mathbb{R}$, $S(\lambda) \subseteq C_+ \setminus \{0\}$ (resp. $S(\lambda) \subseteq \operatorname{int} C_+$) (possibly empty).*

Proof. Let $u \in S(\lambda)$. Then we have

$$\left\{ \begin{array}{l} -\Delta u(z) + \xi(z)u(z) = \lambda u(z) + f(z,u(z)) \text{ for a.a. } z \in \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega \end{array} \right\} \tag{8}$$

(see Papageorgiou & Rădulescu [11]).

We introduce the functions

$$k_\lambda(z) = \begin{cases} 0 & \text{if } 0 \leq u(z) \leq 1 \\ \dfrac{f(z,u(z))}{u(z)} + (\lambda - \xi(z)) & \text{if } 1 < u(z) \end{cases}$$

and

$$\vartheta_\lambda(z) = \begin{cases} f(z,u(z)) + (\lambda - \xi(z))u(z) & \text{if } 0 \leq u(z) \leq 1 \\ 0 & \text{if } 1 < u(z). \end{cases}$$

Evidently, $\vartheta_\lambda \in L^s(\Omega)$ (see hypotheses $H(\xi)$ and $\widehat{H}$). Also

$$|k_\lambda(z)| \leq c_2(1 + u(z)^{r-2}) + (\lambda - \xi(z)) \text{ for a.a. } z \in \Omega, \text{ some } c_2 > 0.$$

Note that if $N \geq 3$ (the cases $N = 1, 2$ are straightforward), then

$$\begin{aligned} (r-2)\frac{N}{2} &< \left(\frac{2N}{N-2} - 2\right)\frac{N}{2} \text{ (since } r < 2^*) \\ &= \frac{2N}{N-2} = 2^*. \end{aligned}$$

Since $u \in H^1(\Omega)$, by the Sobolev embedding theorem, we have

$$\begin{aligned} & u^{(r-2)\frac{N}{2}} \in L^1(\Omega), \\ \Rightarrow\ & k_\lambda \in L^{\frac{N}{2}}(\Omega) \text{ (see hypothesis } H(\xi)). \end{aligned}$$

We rewrite (8) as follows

$$\left\{ \begin{array}{l} -\Delta u(z) = k_\lambda(z)u(z) + \vartheta_\lambda(z) \text{ for a.a. } z \in \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

By Lemma 5.1 of Wang [14] we have that $u \in L^\infty(\Omega)$. Using the Calderon-Zygmund estimates (see Lemma 5.2 of Wang [14]), we obtain $u \in W^{2,s}(\Omega)$. Then the Sobolev embedding theorem implies $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s} > 0$. Therefore we have $u \in C_+ \setminus \{0\}$. Suppose that $H(\xi)'$ holds. From (8) we obtain

$$\begin{aligned}
&\Delta u(z) \le (\xi(z) - \lambda)u(z) \text{ for a.a. } z \in \Omega \text{ (see hypotheses } \widehat{H}) \\
&\qquad \le (\|\xi^+\|_\infty + |\lambda|)u(z) \text{ for a.a. } z \in \Omega \text{ (see hypothesis } H(\xi)') \\
\Rightarrow\ & u \in \operatorname{int} C_+ \\
&\qquad \text{(by the strong maximum principle, see [6, p. 738]).}
\end{aligned}$$

Thus, we have proved that when $H(\xi)'$ holds, then $S(\lambda) \subseteq \operatorname{int} C_+$ for all $\lambda \in \mathbb{R}$. ☐

Next, we show that for every $\lambda \ge \widehat{\lambda}_1$ problem (P_λ) has no positive solutions (that is, $S(\lambda) = \emptyset$ for all $\lambda \ge \widehat{\lambda}_1$).

Proposition 3. *If hypotheses $H(\xi)'$, $H(\beta)$, H_1 hold and $\lambda \ge \widehat{\lambda}_1$, then $S(\lambda) = \emptyset$.*

Proof. Let $\lambda \ge \widehat{\lambda}_1$ and suppose that $S(\lambda) \ne \emptyset$. Let $u \in S(\lambda)$. Due to Proposition 4, we know that $u \in \operatorname{int} C_+$. Let $v \in \operatorname{int} C_+$ and consider the function

$$R(v,u)(z) = |Dv(z)|^2 - \left(Du(z), D\left(\frac{v^2}{u}\right)(z)\right)_{\mathbb{R}^N}.$$

From Picone's identity (see, for example, Motreanu, Motreanu & Papageorgiou [9, p. 255]), we derive

$$\begin{aligned}
&0 \le R(v,u)(z) \text{ for a.a. } z \in \Omega, \\
\Rightarrow\ &0 \le \int_\Omega R(v,u)\,dz \\
&= \|Dv\|_2^2 - \int_\Omega \left(Du, D\left(\frac{v^2}{u}\right)\right)_{\mathbb{R}^N} dz \\
&= \|Dv\|_2^2 - \int_\Omega (-\Delta u)\left(\frac{v^2}{u}\right) dz + \int_{\partial\Omega} \beta(z) u \frac{v^2}{u}\,d\sigma \\
&\qquad \text{(using Green's identity, see Gasinski \& Papageorgiou [6, p. 210])} \\
&= \|Dv\|_2^2 - \int_\Omega (\lambda - \xi(z))u\left(\frac{v^2}{u}\right) dz - \int_\Omega f(z,u)\frac{v^2}{u}\,dz + \int_{\partial\Omega} \beta(z) v^2\,d\sigma \\
&< \gamma(v) - \lambda\|v\|_2^2 \quad \left(\text{since } f(z,u)\frac{v^2}{u} > 0 \text{ a.a. } z \in \Omega\right).
\end{aligned}$$

Let $v = \hat{u}_1 \in \operatorname{int} C_+$. Then $0 < \gamma(\hat{u}_1) - \lambda = \widehat{\lambda}_1 - \lambda \le 0$ (recall $\|\hat{u}_1\|_2 = 1$), a contradiction. Therefore for all $\lambda \ge \widehat{\lambda}_1$, we have $S(\lambda) = \emptyset$. ☐

Next, we show that for $\lambda < \widehat{\lambda}_1$ there exist positive solutions.

Proposition 4. *If hypotheses $H(\xi)'$, $H(\beta)$, H_1 hold and $\lambda < \widehat{\lambda}_1$, then $S(\lambda) \ne \emptyset$.*

Proof. Let $\mu > 0$ be as in (6) and consider the Carathéodory function $g_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g_\lambda(z,x) = \begin{cases} 0 & \text{if } x \le 0 \\ (\lambda + \mu)x + f(z,x) & \text{if } 0 < x. \end{cases}$$

We set $G_\lambda(z,x)=\int_0^x g_\lambda(z,s)\,ds$ and consider the C^1-functional $\varphi_\lambda : H^1(\Omega)\to\mathbb{R}$ defined by

$$\varphi_\lambda(u)=\frac{1}{2}\gamma(u)+\frac{\mu}{2}\|u\|_2^2-\int_\Omega G_\lambda(z,u)dz \text{ for all } u\in H^1(\Omega).$$

Hypotheses H_1(i),(ii) imply that given $\varepsilon>0$, we can find $c_3=c_3(\varepsilon)>0$ such that

$$F(z,x)\le\frac{\varepsilon}{2}x^2+c_3 \text{ for a.a. } z\in\Omega,\ \text{all } x\ge 0. \tag{9}$$

Using (9), we obtain

$$\begin{aligned}\varphi_\lambda(u)&\ge\frac{1}{2}\gamma(u)-\frac{\lambda+\varepsilon}{2}\|u^+\|_2^2-c_3|\Omega|_N\\&\ge\frac{1}{2}\gamma(u)-\frac{\lambda+\varepsilon}{2}\|u\|_2^2-c_3|\Omega|_N.\end{aligned} \tag{10}$$

Choosing $\varepsilon\in(0,\widehat{\lambda}_1-\lambda)$ (recall $\lambda<\widehat{\lambda}_1$) and using Lemma 2.2, from (10) we obtain

$$\begin{aligned}&\varphi_\lambda(u)\ge\frac{\hat{c}}{2}\|u\|^2-c_3|\Omega|_N,\\&\Rightarrow\varphi_\lambda \text{ is coercive.}\end{aligned}$$

Also, invoking the Sobolev embedding theorem and the compactness of the trace map, we see that φ_λ is sequentially weakly lower semicontinuous. Therefore by the Weierstrass theorem, we can find $u_\lambda\in H^1(\Omega)$ such that

$$\varphi_\lambda(u_\lambda)=\inf\left[\varphi_\lambda(u):\ u\in H^1(\Omega)\right]. \tag{11}$$

Let $t\in(0,1)$ be so small that $t\hat{u}_1(z)\in(0,\delta]$ for all $z\in\overline{\Omega}$ (recall that $\hat{u}_1\in\operatorname{int}C_+$ and note that $\delta>0$ is as in hypothesis H_1(iii)). Then

$$\begin{aligned}\varphi_\lambda(t\hat{u}_1)&\le\frac{t^2}{2}\gamma(\hat{u}_1)-\frac{t^2}{2}\lambda-\frac{t^q}{q}\|\hat{u}_1\|_q^q\\&\qquad\text{(see hypothesis } H_1\text{(iii) and recall } \|\hat{u}_1\|_2=1)\\&=\frac{t^2}{2}(\widehat{\lambda}_1-\lambda)-\frac{t^q}{q}\|\hat{u}_1\|_q^q.\end{aligned} \tag{12}$$

Since $q<2$, choosing $t\in(0,1)$ even smaller if necessary, from (12) we obtain

$$\begin{aligned}&\varphi_\lambda(t\hat{u}_1)<0,\\&\Rightarrow\varphi_\lambda(u_\lambda)<0=\varphi_\lambda(0) \text{ (see (11)), hence } u_\lambda\ne 0.\end{aligned}$$

By (11) we have

$$\varphi_\lambda'(u_\lambda)=0,$$

$$\begin{aligned}\Rightarrow\ \langle A(u_\lambda),h\rangle+\int_\Omega\xi(z)u_\lambda h\,dz+\int_{\partial\Omega}\beta(z)u_\lambda h\,d\sigma+\mu\int_\Omega u_\lambda h\,dz\\=\int_\Omega g_\lambda(z,u_\lambda)h\,dz \text{for all } h\in H^1(\Omega).\end{aligned} \tag{13}$$

In (13) we choose $h=-u_\lambda^-\in H^1(\Omega)$. Then

$$\begin{aligned}&\gamma(u_\lambda^-)+\mu\|u_\lambda^-\|_2^2=0,\\&\Rightarrow c_0\|u_\lambda^-\|^2\le 0 \text{ (see (6))},\\&\Rightarrow u_\lambda\ge 0,\ u_\lambda\ne 0.\end{aligned}$$

Thus, equation (13) becomes

$$\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z) u_\lambda h\, dz + \int_{\partial\Omega} \beta(z) u_\lambda h\, d\sigma = \int_\Omega (\lambda u_\lambda + f(z,u_\lambda)) h\, dz$$
$$\text{for all } h \in H^1(\Omega),$$

$$\Rightarrow -\Delta u_\lambda(z) + \xi(z) u_\lambda(z) = \lambda u_\lambda(z) + f(z, u_\lambda(z)) \text{ for a.a. } z \in \Omega,$$
$$\frac{\partial u_\lambda}{\partial n} + \beta(z) u_\lambda = 0 \text{ on } \partial\Omega$$

(see Papageorgiou & Rădulescu [11]),

$$\Rightarrow u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+ \text{ (see Proposition 2) and so } S(\lambda) \neq \emptyset \text{ for } \lambda < \widehat{\lambda}_1.$$

□

In fact, we can show that for every $\lambda < \widehat{\lambda}_1$ problem (P_λ) has a smallest positive solution. To this end note that given $\tau \in \left(\frac{2N}{N-1}, 2^*\right)$, because of hypotheses H_1(i),(iii), we can find $c_4(\lambda) > 0$ with $\lambda \mapsto c_4(\lambda)$ bounded on bounded subsets of $\mathbb{R}$ such that

$$\lambda x + f(z,x) \geq c_1 x^{q-1} - c_4(\lambda) x^{\tau-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{14}$$

This unilateral growth restriction on the reaction term of problem (P_λ) leads to the following auxiliary Robin problem:

$$\left\{ \begin{array}{l} -\Delta u(z) + \xi(z) u(z) = c_1 u(z)^{q-1} - c_4(\lambda) u(z)^{\tau-1} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z) u = 0 \text{ on } \partial\Omega, \ u \geq 0. \end{array} \right\} \tag{Au_λ}$$

Proposition 5. *If hypotheses $H(\xi)'$, $H(\beta)$ hold and $\lambda \in \mathbb{R}$, then problem (Au_λ) admits a unique positive solution $u_*^\lambda \in int C_+$.*

Proof. First, we establish the existence of a positive solution for problem (Au_λ). So, we introduce the C^1-functional $\psi_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u^-\|_2^2 - \frac{c_1}{q}\|u^+\|_q^q + \frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^\tau \text{ for all } u \in H^1(\Omega).$$

By Lemma 2.2, we have

$$\begin{aligned} \psi_\lambda(u) &= \frac{1}{2}\gamma(u^-) + \frac{\mu}{2}\|u^-\|_2^2 + \frac{1}{2}\gamma(u^+) + \frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^\tau - \frac{c_1}{q}\|u^+\|_q^q \\ &\geq \frac{\hat{c}}{2}\|u^-\|^2 + \frac{1}{2}\gamma(u^+) + \frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^\tau - \frac{c_1}{q}\|u^+\|_q^q. \end{aligned} \tag{15}$$

Since for $N \geq 3$, $s > N$, by the Sobolev embedding theorem, we have $u^2 \in L^{s'}(\Omega)$ and so due to Hölder's inequality,

$$\left| \int_\Omega \xi(z)(u^+)^2\, dz \right| \leq \|\xi\|_s \|u^+\|_{2s'}^2.$$

Note that $s' < N' = \frac{N}{N-1}$ (recall that for $1 \leq \tau < \infty$, $\tau' \in (1, +\infty]$ and $\frac{1}{\tau} + \frac{1}{\tau'} = 1$), hence $2s' < \frac{2N}{N-1} < \tau$. Therefore

$$\left| \int_\Omega \xi(z)(u^+)^2\, dz \right| \leq c_5 \|u^+\|_\tau^2 \text{ for some } c_5 > 0.$$

Thus we have

$$\begin{aligned}
&\frac{1}{2}\gamma(u^+) + \frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^\tau - \frac{c_1}{q}\|u^+\|_q^q \\
\geq&\frac{1}{2}\|Du^+\|_2^2 + \frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^\tau - c_5\|u^+\|_\tau^2 - c_6\|u^+\|_\tau^q \quad \text{for some } c_6 > 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(recall that } q < \tau) \\
\geq&\frac{1}{2}\|Du^+\|_2^2 + \frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^\tau - c_7\left(\|u^+\|_\tau^2 + 1\right) \quad \text{for some } c_7 > 0 \\
=&\frac{1}{2}\|Du^+\|_2^2 + \left[\frac{c_4(\lambda)}{\tau}\|u^+\|_\tau^{\tau-2} - c_7\right]\|u^+\|_\tau^2 - c_7.
\end{aligned} \tag{16}$$

We return to (15), use (16) and recall that $y \mapsto \left[\|y\|_\tau^2 + \|Dy\|_2^2\right]^{1/2}$ is an equivalent norm on the Sobolev space $H^1(\Omega)$ (see, for example, Gasinski & Papageorgiou [6, p. 227]). So, from (16) we infer that $\psi_\lambda(\cdot)$ is coercive.

The cases $N = 1, 2$ are straightforward because

- if $N = 1$, then $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ (compactly);
- if $N = 2$, then $H^1(\Omega) \hookrightarrow L^\tau(\Omega)$ for all $\tau \in [1, +\infty)$ (compactly).

The Sobolev embedding theorem and the compactness of the trace map, imply that ψ_λ is sequentially weakly lower semicontinuous. So, we can find $u_*^\lambda \in H^1(\Omega)$ such that

$$\psi_\lambda(u_*^\lambda) = \inf\left[\psi_\lambda(u) : \ u \in H^1(\Omega)\right]. \tag{17}$$

As before (see the proof of Proposition 4), exploiting the fact that $q < 2 < \tau$, we obtain

$$\begin{aligned}
&\psi_\lambda(u_*^\lambda) < 0 = \psi_\lambda(0), \\
\Rightarrow\ &u_*^\lambda \neq 0.
\end{aligned}$$

By (17) we have

$$\begin{aligned}
&\psi'_\lambda(u_*^\lambda) = 0, \\
\Rightarrow\ &\langle A(u_*^\lambda), h\rangle + \int_\Omega \xi(z) u_*^\lambda h\, dz + \int_{\partial\Omega} \beta(z) u_*^\lambda h\, d\sigma - \mu \int_\Omega (u_*^\lambda)^- h\, dz \\
&= c_1 \int_\Omega ((u_*^\lambda)^+)^{q-1} h\, dz - c_4(\lambda) \int_\Omega ((u_*^\lambda)^+)^{\tau-1} h\, dz \quad \text{for all } h \in H^1(\Omega).
\end{aligned} \tag{18}$$

In (18) we choose $h = -(u_*^\lambda)^- \in H^1(\Omega)$. Then

$$\begin{aligned}
&\gamma((u_*^\lambda)^-) + \mu\|(u_*^\lambda)^-\|_2^2 = 0, \\
\Rightarrow\ &\hat{c}\|(u_*^\lambda)^-\|^2 \leq 0 \quad (\text{see } (6)), \\
\Rightarrow\ &u_*^\lambda \geq 0,\ u_*^\lambda \neq 0.
\end{aligned}$$

Next, we infer from (18) that u_*^λ is a positive solution of (Au_λ). Again, using Lemmata 5.1 and 5.2 of Wang [14], we infer that $u_*^\lambda \in C_+ \setminus \{0\}$. Moreover, we have

$$\begin{aligned}
&\Delta u_*^\lambda(z) \leq c_4(\lambda) u_*^\lambda(z)^{\tau-1} + \xi(z) u_*^\lambda(z) \quad \text{for a.a. } z \in \Omega, \\
\Rightarrow\ &\Delta u_*^\lambda(z) \leq \left[c_4(\lambda)\|u_*^\lambda\|_\infty^{\tau-2} + \|\xi^+\|_\infty\right] u_*^\lambda(z) \quad \text{for a.a. } z \in \Omega \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{see hypothesis } H(\xi)'), \\
\Rightarrow\ &u_*^\lambda \in \operatorname{int} C_+ \text{ (by the strong maximum principle)}.
\end{aligned}$$

Next, we prove the uniqueness of this positive solution. So let $v_*^\lambda \in H^1(\Omega)$ be another positive solution for (Au_λ). As above, we show that $v_*^\lambda \in \operatorname{int} C_+$. Let $t > 0$ be the biggest real number such that $tv_*^\lambda \le u_*^\lambda$ (see Marano & Papageorgiou [8, Proposition 2.1]).

Suppose that $0 < t < 1$. Let $\rho = \|u_*^\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be such that

$$x \mapsto c_1 x^{q-1} - c_4(\lambda)x^{\tau-1} + \hat{\xi}_\rho x$$

is nondecreasing on $[0,\rho]$. We have

$$\begin{aligned}
-\Delta(tv_*^\lambda) + (\xi(z) + \hat{\xi}_\rho)(tv_*^\lambda) =& t\left[-\Delta v_*^\lambda + (\xi(z)+\hat{\xi}_\rho)v_*^\lambda\right] \\
=& t\left[c_1(v_*^\lambda)^{q-1} - c_4(\lambda)(v_*^\lambda)^{\tau-1} + \hat{\xi}_\rho v_*^\lambda\right] \\
<& c_1(tv_*^\lambda)^{q-1} - c_4(\lambda)(tv_*^\lambda)^{\tau-1} + \hat{\xi}_\rho(tv_*^\lambda) \\
& \qquad\qquad (\text{since } t \in (0,1) \text{ and } q < 2 < \tau) \\
\le& c_1(u_*^\lambda)^{q-1} - c_4(u_*^\lambda)^{\tau-1} + \hat{\xi}_\rho u_*^\lambda \\
& \qquad\qquad (\text{since } tv_*^\lambda \le u_*^\lambda) \\
=& -\Delta u_*^\lambda + (\xi(z)+\hat{\xi}_\rho)u_*^\lambda,
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \Delta(u_*^\lambda - tv_*^\lambda) < (\xi(z)+\hat{\xi}_\rho)(u_*^\lambda - tv_*^\lambda) \le (\|\xi^+\|_\infty + \hat{\xi}_\rho)(u_*^\lambda - tv_*^\lambda) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{see hypothesis } H(\xi)'), \\
&\Rightarrow u_*^\lambda - tv_*^\lambda \in \operatorname{int} C_+ \text{ (by the strong maximum principle).}
\end{aligned}$$

However, this contradicts the maximality of $t \in (0,1)$. Hence $t \ge 1$ and so we have $v_*^\lambda \le u_*^\lambda$.

If in the above argument we reverse the roles of u_* and v_*, we obtain

$$\begin{aligned}
&u_*^\lambda \le v_*^\lambda, \\
\Rightarrow\ &u_*^\lambda = v_*^\lambda
\end{aligned}$$

and this proves the uniqueness of the positive solution of problem (Au_λ). ☐

Remark. We can have an alternative proof of the uniqueness based on the Picone identity. The argument goes as follows. Suppose again that $v_*^\lambda \in H^1(\Omega)$ is another positive solution of (Au_λ). We have $v_*^\lambda \in \operatorname{int} C_+$. Therefore

$$\begin{aligned}
&\int_\Omega \left(\frac{c_1}{(u_*^\lambda)^{2-q}} - c_4(\lambda)(u_*^\lambda)^{\tau-2}\right)\left((u_*^\lambda)^2 - (v_*^\lambda)^2\right) dz \\
&= \int_\Omega \left(c_1(u_*^\lambda)^{q-1} - c_4(\lambda)(u_*^\lambda)^{\tau-1}\right)\left(u_*^\lambda - \frac{(v_*^\lambda)^2}{u_*^\lambda}\right) dz \\
&= \int_\Omega \left(-\Delta_p u_*^\lambda + \xi(z)u_*^\lambda\right)\left(u_*^\lambda - \frac{(v_*^\lambda)^2}{u_*^\lambda}\right) dz \\
&= \int_\Omega \left(Du_*^\lambda, D\left(u_*^\lambda - \frac{(v_*^\lambda)^2}{u_*^\lambda}\right)\right)_{\mathbb{R}^N} dz + \int_\Omega \xi(z)u_*^\lambda\left(u_*^\lambda - \frac{(v_*^\lambda)^2}{u_*^\lambda}\right) dz \\
&\quad + \int_{\partial\Omega} \beta(z)u_*^\lambda\left(u_*^\lambda - \frac{(v_*^\lambda)^2}{u_*^\lambda}\right) d\sigma \\
&\quad (\text{using Green's identity, see [6, p. 210]})
\end{aligned}$$

$$= \|Du_*^\lambda\|_2^2 - \|Dv_*^\lambda\|_2^2 + \int_\Omega R(v_*^\lambda, u_*^\lambda)dz + \int_\Omega \xi(z)((u_*^\lambda)^2 - (v_*^\lambda)^2)dz$$
$$+ \int_{\partial\Omega} \beta(z)((u_*^\lambda)^2 - (v_*^\lambda)^2)d\sigma \tag{19}$$

(see the proof of Proposition 3).

Interchanging the roles of u_*^λ and v_*^λ in the above argument, we obtain

$$\int_\Omega \left(\frac{c_1}{(v_*^\lambda)^{2-q}} - c_4(\lambda)(v_*^\lambda)^{\tau-2}\right)\left((v_*^\lambda)^2 - (u_*^\lambda)^2\right)dz$$
$$= \|Dv_*^\lambda\|_2^2 - \|Du_*^\lambda\|_2^2 + \int_\Omega R(u_*^\lambda, v_*^\lambda)dz + \int_\Omega \xi(z)((v_*^\lambda)^2 - (u_*^\lambda)^2)dz$$
$$+ \int_{\partial\Omega} \beta(z)((v_*^\lambda)^2 - (u_*^\lambda)^2)d\sigma. \tag{20}$$

We add (19) and (20) and use Picone's identity. We obtain

$$0 \le \int_\Omega \left[R(v_*^\lambda, u_*^\lambda) + R(u_*^\lambda, v_*^\lambda)\right]dz$$
$$= \int_\Omega \left[c_1\left(\frac{1}{(u_*^\lambda)^{2-q}} - \frac{1}{(v_*^\lambda)^{2-q}}\right) - c_4(\lambda)\left((u_*^\lambda)^{\tau-2} - (v_*^\lambda)^{\tau-2}\right)\right]\left((u_*^\lambda)^2 - (v_*^\lambda)^2\right)dz. \tag{21}$$

Since the function $x \mapsto \frac{c_1}{x^{2-q}} - c_4(\lambda)x^{\tau-2}$ is strictly decreasing on $(0, +\infty)$ (recall $q < 2 < \tau$), it follows from (21) that

$$u_*^\lambda = v_*^\lambda.$$

So, we again get the uniqueness of the positive solution of (Au_λ).

Note also that since $\lambda \mapsto c_4(\lambda)$ is bounded on bounded sets of $\mathbb{R}$, if $B \subseteq \mathbb{R}$ is bounded and $\hat{c}_4 \ge c_4(\bar{\lambda})$ for all $\bar{\lambda} \in B$, then the unique solution $\bar{u} \in \operatorname{int} C_+$ of

$$\left\{\begin{array}{l} -\Delta u(z) + \xi(z)u(z) = c_1 u(z)^{q-1} - \hat{c}_4 u(z)^{\tau-1} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega, \end{array}\right\}$$

satisfies $\bar{u} \le u_*^{\bar{\lambda}}$ for all $\bar{\lambda} \in B$.

Using Proposition 5, we can produce a lower bound for the solution set $S(\lambda)$.

Proposition 6. *If hypotheses $H(\xi)'$, $H(\beta)$, H_1 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then $u_*^\lambda \le u$ for all $u \in S(\lambda)$.*

Proof. Let $u \in S(\lambda)$ and consider the following Carathéodory function

$$\hat{g}_\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ c_1 x^{q-1} - c_4(\lambda)x^{\tau-1} + \mu x & \text{if } 0 \le x \le u(z) \\ c_1 u(z)^{q-1} - c_4(\lambda)u(z)^{\tau-1} + \mu u(z) & \text{if } u(z) < x. \end{cases} \tag{22}$$

Here $\mu > 0$ is as in (6). We set $\widehat{G}_\lambda(z,x) = \int_0^x \hat{g}_\lambda(z,s)ds$ and consider the C^1-functional $\widehat{\psi}_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega \widehat{G}_\lambda(z,u)dz \text{ for all } u \in H^1(\Omega).$$

Invoking (6) and (22), we see that $\widehat{\psi}_\lambda$ is coercive. It is also sequentially weakly lower semicontinuous. Therefore we can find $\tilde{u}_*^\lambda \in H^1(\Omega)$ such that

$$\widehat{\psi}_\lambda(\tilde{u}_*^\lambda) = \inf\left[\psi_\lambda(u) : u \in H^1(\Omega)\right]. \tag{23}$$

As before (see the proof of Proposition 4), exploiting the fact that $q < 2 < \tau$, we can show that

$$\begin{aligned} &\widehat{\psi}(\tilde{u}_*^\lambda) < 0 = \widehat{\psi}_\lambda(0), \\ \Rightarrow\ &\tilde{u}_*^\lambda \neq 0. \end{aligned}$$

By (23) we have

$$\begin{aligned} &\widehat{\psi}'_\lambda(\tilde{u}_*^\lambda) = 0, \\ \Rightarrow\ &\langle A(\tilde{u}_*^\lambda), h\rangle + \int_\Omega \xi(z)\tilde{u}_*^\lambda h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_*^\lambda h\,d\sigma + \mu\int_\Omega \tilde{u}_*^\lambda h\,dz = \int_\Omega \hat{g}_\lambda(z,\tilde{u}_*^\lambda)h\,dz \\ &\hspace{8cm}\text{for all } h \in H^1(\Omega). \end{aligned} \tag{24}$$

In (24) we choose $h = -(\tilde{u}_*^\lambda)^- \in H^1(\Omega)$ and obtain

$$\begin{aligned} &\gamma((\tilde{u}_*^\lambda)^-) + \mu\|(\tilde{u}_*^\lambda)^-\|_2^2 = 0 \text{ (see (22))} \\ \Rightarrow\ &c_1\|(\tilde{u}_*^\lambda)^-\|^2 \leq 0 \text{ (see (6))} \\ \Rightarrow\ &\tilde{u}_*^\lambda \geq 0,\ \tilde{u}_*^\lambda \neq 0. \end{aligned}$$

Next, in (24) we choose $(\tilde{u}_*^\lambda - u)^+ \in H^1(\Omega)$. Then

$$\begin{aligned} &\langle A(\tilde{u}_*^\lambda), (\tilde{u}_*^\lambda - u)^+\rangle + \int_\Omega \xi(z)\tilde{u}_*^\lambda(\tilde{u}_*^\lambda - u)^+dz + \int_{\partial\Omega} \beta(z)\tilde{u}_*^\lambda(\tilde{u}_*^\lambda - u)^+d\sigma \\ &\quad + \int_\Omega \mu\tilde{u}_*^\lambda(\tilde{u}_*^\lambda - u)^+dz \\ &= \int_\Omega \left(c_1u^{q-1} - c_4(\lambda)u^{\tau-1}\right)(\tilde{u}_*^\lambda - u)^+dz + \mu\int_\Omega u(\tilde{u}_*^\lambda - u)^+dz \text{ (see (22))} \\ &\leq \int_\Omega (\lambda u + f(z,u))(\tilde{u}_*^\lambda - u)^+dz + \mu\int_\Omega u(\tilde{u}_*^\lambda - u)^+dz \text{ (see (14))} \\ &= \langle A(u), (\tilde{u}_*^\lambda - u)^+\rangle + \int_\Omega \xi(z)u(\tilde{u}_*^\lambda - u)^+dz + \int_{\partial\Omega} \beta(z)u(\tilde{u}_*^\lambda - u)^+d\sigma \\ &+ \int_\Omega \mu u(\tilde{u}_*^\lambda - u)^+dz, \qquad\qquad (\text{since } u \in S(\lambda)) \end{aligned}$$

$$\begin{aligned} \Rightarrow\ &\gamma((\tilde{u}_*^\lambda - u)^+) + \mu\|(\tilde{u}_*^\lambda - u)^+\|_2^2 \leq 0, \\ \Rightarrow\ &c_0\|(\tilde{u}_*^\lambda - u)^+\|^2 \leq 0 \text{ (see (6))}, \\ \Rightarrow\ &\tilde{u}_*^\lambda \leq u. \end{aligned}$$

Therefore we have proved that

$$\tilde{u}_*^\lambda \in [0,u] = \{v \in H^1(\Omega) : 0 \leq v(z) \leq u(z) \text{ for a.a } z \in \Omega\},\ \tilde{u}_*^\lambda \neq 0. \tag{25}$$

By (22) and (25), we see that equation (24) becomes

$$\langle A(\tilde{u}_*^\lambda), h\rangle + \int_\Omega \xi(z)\tilde{u}_*^\lambda h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_*^\lambda h\,d\sigma = \int_\Omega \left[c_1(\tilde{u}_*^\lambda)^{q-1} - c_4(\lambda)(u_*^\lambda)^{\tau-1}\right] dz$$
$$\text{for all } h \in H^1(\Omega),$$

$\Rightarrow$ $\tilde{u}_*^\lambda$ is a positive solution of (Au_λ),

$\Rightarrow$ $\tilde{u}_*^\lambda = u_*^\lambda \in \operatorname{int} C_+$ (see Proposition 5).

Therefore $u_*^\lambda \leq u$ for all $u \in S(\lambda)$. ∎

This lower bound leads to the existence of a smallest positive solution for problem (P_λ), $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$.

Proposition 7. *If hypotheses $H(\xi)'$, $H(\beta)$, H_1 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in S(\lambda) \subseteq int C_+$.*

Proof. As in Filippakis & Papageorgiou [5], we can show that $S(\lambda)$ is downward directed, that is, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \leq u_1$, $u \leq u_2$. For completeness we sketch a proof. So, given $\varepsilon > 0$ consider the function

$$\vartheta_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s < -\varepsilon \\ s & \text{if } s \in [-\varepsilon, \varepsilon] \\ \varepsilon & \text{if } s > \varepsilon\,. \end{cases}$$

Evidently, $\vartheta_\varepsilon(\cdot)$ is Lipschitz and so $\vartheta_\varepsilon((u_1 - u_2)^-) \in H^1(\Omega)$. Moreover, the chain rule for Sobolev functions implies that

$$D\vartheta_\varepsilon((u_1 - u_2)^-) = \vartheta_\varepsilon'((u_1 - u_2)^-)D(u_1 - u_2)^-.$$

Let $\psi \in C^1(\overline{\Omega}))$. Then we introduce the test functions

$$\eta_1 = \vartheta_\varepsilon((u_1 - u_2)^-)\psi \quad \text{and} \quad \eta_2 = (\varepsilon - \vartheta_\varepsilon((y_1 - y_2)^-)\psi),$$

which belong to $H^1(\Omega) \cap L^\infty(\Omega)$. We have

$$\langle \gamma'(u_1), \eta_1\rangle = \lambda \int_\Omega u_1\eta_1 dz + \int_\Omega f(z, u_1)\eta_1 dz,$$

$$\langle \gamma'(u_2), \eta_2\rangle = \lambda \int_\Omega u_2\eta_2 dz + \int_\Omega f(z, u_2)\eta_2 dz.$$

We add these two equalities and divide by $\varepsilon > 0$. Taking into account that

$$\frac{1}{\varepsilon}\,\vartheta_\varepsilon((u_1 - u_2)^-)(z) \to \chi_{\{u_1 < u_2\}}(z) \quad \text{for a.a. } z \in \Omega \text{ as } \varepsilon \to 0^+$$

and

$$\chi_{\{u_1 \geq u_2\}} = 1 - \chi_{\{u_1 < u_2\}}$$

we obtain

$$\langle \gamma'(u_1), \chi_{\{u_1<u_2\}}\psi\rangle + \langle \gamma'(u_2), \chi_{\{u_1 \geq u_2\}}\psi\rangle =$$
$$\lambda \int_{\{u_1<u_2\}} u_1\psi dz + \lambda \int_{\{u_1\geq u_2\}} u_2\psi dz + \int_{\{u_1<u_2\}} f(z, u_1)\psi dz + \int_{\{u_1\geq u_2\}} f(z, u_2)\psi dz.$$

So, if $\bar{u} = \min\{u_1, u_2\}$, then $\bar{u}$ is an upper solution of (P_λ) and so by standard truncation techniques we can find $u \in S(\lambda)$ such that $0 \leq u \leq \bar{u} \leq \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right.$.

Then Lemma 3.10 of Hu & Papageorgiou [7, p. 178], implies that there exist $u_n \in S(\lambda)$, $n \in \mathbb{N}$, $\{u_n\}_{n\geq 1}$ decreasing such that

$$\inf S(\lambda) = \inf_{n\geq 1} u_n.$$

For every $n \in \mathbb{N}$, we have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h\,dz + \int_{\partial\Omega} \beta(z)u_n h\,d\sigma = \int_\Omega [\lambda u_n + f(z,u_n)]h\,dz \quad \text{for all } h \in H^1(\Omega). \tag{26}$$

Evidently, $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} \bar{u}_\lambda \text{ in } H^1(\Omega) \text{ and } u_n \to \bar{u}_\lambda \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{27}$$

In (26) we pass to the limit as $n \to \infty$ and use (27). Then

$$\langle A(\bar{u}_\lambda), h\rangle + \int_\Omega \xi(z)\bar{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)\bar{u}_\lambda h\,d\sigma = \int_\Omega [\lambda \bar{u}_\lambda + f(z,\bar{u}_\lambda)]h\,dz \quad \text{for all } h \in H^1(\Omega),$$

$$\Rightarrow \bar{u}_\lambda \text{ is a solution of } (P_\lambda).$$

Due to Proposition 6 we know that

$$\begin{aligned} & u_*^\lambda \leq u_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow\ & u_*^\lambda \leq \bar{u}_\lambda \text{ (see (27))}, \\ \Rightarrow\ & \bar{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+ \text{ and } \bar{u}_\lambda = \inf S(\lambda). \end{aligned}$$

□

We examine the monotonicity and continuity properties of the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$.

Proposition 8. *If hypotheses $H(\xi)'$, $H(\beta)$, H_1 hold, then the map $\lambda \mapsto \bar{u}_\lambda$ is nondecreasing and left continuous from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$.*

Proof. First, we establish the monotonicity of the map $\lambda \to \bar{u}_\lambda$. So let $\lambda < \eta < \widehat{\lambda}_1$ and consider $\bar{u}_\eta \in S(\eta) \subseteq \operatorname{int} C_+$ the minimal positive solution of problem (P_η). We introduce the following Carathéodory function

$$e_\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ (\lambda+\mu)x + f(z,x) & \text{if } 0 \leq x \leq u_\eta(z) \\ (\lambda+\mu)u_\eta(z) + f(z,u_\eta(z)) & \text{if } u_\eta(z) < x. \end{cases} \tag{28}$$

As always, $\mu > 0$ is as in (6). We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and consider the C^1-functional $w_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$w_\lambda(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega E_\lambda(z,u)dz \text{ for all } u \in H^1(\Omega).$$

It follows from (6) and (28) that $w_\lambda(\cdot)$ is coercive. It is also sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in H^1(\Omega)$ such that

$$w_\lambda(\tilde{u}_\lambda) = \inf\left[w_\lambda(u) : u \in H^1(\Omega)\right] < 0 = w_\lambda(0) \quad \text{(as before since } q < 2). \tag{29}$$

By (29) we have $\tilde{u}_\lambda \neq 0$ and

$$\begin{aligned} & w'_\lambda(\tilde{u}_\lambda) = 0, \\ \Rightarrow & \langle A(\tilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)\tilde{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)\tilde{u}_\lambda h\,d\sigma + \mu\int_\Omega \tilde{u}_\lambda h\,dz = \int_\Omega e_\lambda(z,\tilde{u}_\lambda)h\,dz \\ & \text{for all } h \in H^1(\Omega). \end{aligned} \tag{30}$$

As in the proof of Proposition 6, choosing in (30), first $h = -\tilde{u}_\lambda^- \in H^1(\Omega)$ and then $(\tilde{u}_\lambda - \bar{u}_\eta)^+ \in H^1(\Omega)$, we can show that

$$\tilde{u}_\lambda \in [0, \bar{u}_\eta] = \left\{v \in H^1(\Omega) : 0 \leq v(z) \leq \bar{u}_\eta(z) \text{ for a.a. } z \in \Omega\right\}, \ \tilde{u}_\lambda \neq 0. \tag{31}$$

Then we can infer from (28), (30), (31) that

$$\begin{aligned} & \tilde{u}_\lambda \in S(\lambda), \\ \Rightarrow & \bar{u}_\lambda \leq \tilde{u}_\lambda \leq \bar{u}_\eta, \\ \Rightarrow & \lambda \mapsto \bar{u}_\lambda \text{ from } \mathcal{L} = (-\infty, \widehat{\lambda}_1) \text{ into } C^1(\overline{\Omega}) \text{ is nondecreasing.} \end{aligned}$$

Next, we establish the left continuity of this map. To this end, let $\lambda_n \to \lambda^-$ with $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$. Evidently, $\{\bar{u}_{\lambda_n}\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded and increasing. Therefore we may assume that

$$\bar{u}_{\lambda_n} \xrightarrow{w} \hat{u}_\lambda \text{ in } H^1(\Omega) \text{ and } \bar{u}_{\lambda_n} \to \hat{u}_\lambda \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{32}$$

We have

$$\begin{aligned} & \langle A(\bar{u}_{\lambda_n}), h\rangle + \int_\Omega \xi(z)\bar{u}_{\lambda_n} h\,dz + \int_{\partial\Omega} \beta(z)\bar{u}_{\lambda_n} h\,d\sigma = \int_\Omega [\lambda_n \bar{u}_{\lambda_n} + f(z,\bar{u}_{\lambda_n})]\,h\,dz \\ & \text{for all } h \in H^1(\Omega), \ \text{all } n \in \mathbb{N}. \end{aligned} \tag{33}$$

In (33) we pass to the limit as $n \to \infty$ and use (32). We obtain

$$\begin{aligned} & \langle A(\hat{u}_\lambda), h\rangle + \int_\Omega \xi(z)\hat{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)\hat{u}_\lambda h\,d\sigma = \int_\Omega [\lambda\hat{u}_\lambda + f(z,\hat{u}_\lambda)]\,h\,dz \\ & \text{for all } h \in H^1(\Omega). \end{aligned} \tag{34}$$

Set $B = \{\lambda_n\}_{n\geq 1}$ and let $\hat{c}_4 \geq c_4(\bar{\lambda})$ for all $\bar{\lambda} \in B$ (recall that $\lambda \mapsto c_4(\lambda)$ is bounded on bounded sets of $\mathbb{R}$). Let $\bar{u} \in \operatorname{int} C_+$ be the unique positive solution of the following semilinear Robin problem

$$\left\{ \begin{array}{l} -\Delta u(z) + \xi(z)u(z) = c_1 u(z)^{q-1} - \hat{c}_4 u(z)^{\tau-1} \ \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \ \text{ on } \partial\Omega. \end{array} \right\}$$

(see Proposition 5).

We know that $\bar{u} \leq \bar{u}_{\lambda_n}$ for all $n \in \mathbb{N}$ (see the remark following Proposition 5). Hence

$$\begin{aligned} & \bar{u} \leq \hat{u}_\lambda, \\ \Rightarrow & \hat{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+. \end{aligned}$$

We claim that $\hat{u}_\lambda = \bar{u}_\lambda$. If this is not true, then we can find $z_0 \in \overline{\Omega}$ such that

$$\bar{u}_\lambda(z_0) < \hat{u}_\lambda(z_0). \tag{35}$$

From Wang [14], we know that we can find $M_1 > 0$ and $\alpha \in (0,1)$ such that

$$\bar{u}_{\lambda_n} \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|\bar{u}_{\lambda_n}\|_{C^1(\overline{\Omega})} \leq M_1 \text{ for all } n \in \mathbb{N}. \tag{36}$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (32), we obtain

$$\begin{aligned} &\bar{u}_{\lambda_n} \to \hat{u}_\lambda \text{ in } C^1(\overline{\Omega}), \\ \Rightarrow\ &\bar{u}_\lambda(z_0) < \bar{u}_{\lambda_n}(z_0) \text{ for all}\, n \geq n_0 \end{aligned} \tag{37}$$

a contradiction to the monotonicity of $\lambda \mapsto \bar{u}_\lambda$ (recall that $\lambda_n < \lambda$ for all $n \in \mathbb{N}$). So $\hat{u}_\lambda = \bar{u}_\lambda$ and this proves the left continuity of $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ (see (37)). □

We can improve the monotonicity of the map $\lambda \mapsto \bar{u}_\lambda$ provided that we strengthen the conditions on $f(z,\cdot)$.

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses H_2(i), (ii), (iii) are the same as hypotheses H_1(i), (ii), (iii) and

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z,x) + \hat{\xi}_\rho x$$

is nondecreasing on $[0,\rho]$.

Remark. The new condition on $f(z,\cdot)$ is satisfied, if for a.a. $z \in \Omega$, $f(z,\cdot)$ is differentiable and $z \mapsto f'_x(z,\cdot)$ is locally $L^\infty(\Omega)$-bounded (just use the mean value theorem).

Examples. The following functions satisfy hypotheses H_2. As before, for the sake of simplicity, we drop the z-dependence

$$f_1(x) = x^{q-1} \text{ for all } x \geq 0 \text{ with } 1 < q < 2,$$

$$f_2(x) = \begin{cases} x^{q-1} & \text{if } x \in [0,1] \\ 2x^{\tau-1} - x^{s-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q, \tau, s < 2,\ s < \tau,$$

$$f_3(x) = \begin{cases} x^{q-1} - x^{s-1} & \text{if } x \in [0,1] \\ x^{\tau-1} \ln x & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q, s, \tau < 2,\ q < s,$$

$$f_4(x) = \begin{cases} \ln(x^{q-1} + 1) & \text{if } x \in [0,1] \\ x^{\tau-1} - x^{s-1} + c & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q, \tau, s < 2,\ s < \tau,\ c = \ln 2.$$

Proposition 9. *If hypotheses $H(\xi)'$, $H(\beta)$, H_2 hold, then the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is strictly increasing (that is, if $\lambda < \eta < \widehat{\lambda}_1$, then $\bar{u}_\eta - \bar{u}_\lambda \in int\, C_+$).*

Proof. Let $\lambda < \eta < \widehat{\lambda}_1$ and let $\bar{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$, $\bar{u}_\eta \in S(\eta) \subseteq \operatorname{int} C_+$ be the corresponding minimal positive solutions of problems (P_λ) and (P_η), respectively. By Proposition 8 we know that

$$\bar{u}_\lambda \leq \bar{u}_\eta. \tag{38}$$

Let $\rho = \|\bar{u}_\eta\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv). We have

$$\begin{aligned}
&-\Delta\bar{u}_\lambda(z) + (\xi(z) + \hat{\xi}_\rho)\bar{u}_\lambda(z)\\
&= \lambda\bar{u}_\lambda(z) + f(z, \bar{u}_\lambda(z)) + \hat{\xi}_\rho\bar{u}_\lambda(z)\\
&< \eta\bar{u}_\eta(z) + f(z, \bar{u}_\eta(z)) + \hat{\xi}_\rho\bar{u}_\eta(z)\\
&\qquad \text{(see (38), hypothesis } H_2\text{(iv) and recall } \lambda < \eta)\\
&= -\Delta\bar{u}_\eta(z) + (\xi(z) + \hat{\xi}_\rho)\bar{u}_\eta(z) \text{ for a.a. } z \in \Omega,\\
\Rightarrow\, &\Delta(\bar{u}_\eta - \bar{u}_\lambda)(z) \leq (\|\xi^+\|_\infty + \hat{\xi}_\rho)(\bar{u}_\eta - \bar{u}_\lambda)(z) \text{ for a.a. } z \in \Omega\\
&\qquad \text{(see hypothesis } H(\xi)'),\\
\Rightarrow\, &\bar{u}_\eta - \bar{u}_\lambda \in \operatorname{int} C_+\\
&\qquad \text{(by the strong maximum principle, see [6, p. 738]).}
\end{aligned}$$

This proves the strict monotonicity of $\lambda \mapsto \bar{u}_\lambda$. □

In fact, under a monotonicity restriction on the quotient $\frac{f(z,x)}{x}$, we can conclude that for all $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$ problem (P_λ) admits a unique positive solution.

Hence the new conditions on the perturbation $f(z, x)$ are the following:

H_3: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses H_3(i), (ii), (iii) are the same as hypotheses H_1(i), (ii), (iii) and (iv) for a.a. $z \in \Omega$, $x \mapsto \frac{f(z,x)}{x}$ is strictly decreasing on $(0, +\infty)$.

Examples. The following functions satisfy the new conditions. Again, for the sake of simplicity we drop the z-dependence:

$$f_1(x) = x^{q-1} \text{ for all } x \geq 0 \text{ with } 1 < q < 2,$$

$$f_2(x) = \begin{cases} x^{q-1} & \text{if } x \in [0, 1] \\ x^{\tau-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q,\, \tau < q.$$

Proposition 10. *If hypotheses $H(\xi)'$, $H(\beta)$, H_3 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then $S(\lambda)$ is a singleton, that is, $S(\lambda) = \{\bar{u}_\lambda\}$ and $\lambda \mapsto \bar{u}_\lambda$ is nondecreasing and continuous from $(-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$.*

Proof. The nonemptiness of $S(\lambda)$ follows from Proposition 4. Suppose that $\bar{u}_\lambda, \bar{v}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$. We have

$$-\Delta\bar{u}_\lambda(z) + \xi(z)\bar{u}_\lambda(z) = \lambda\bar{u}_\lambda(z) + f(z, \bar{u}_\lambda(z)) \text{ for a.a. } z \in \Omega, \tag{39}$$

$$-\Delta\bar{v}_\lambda(z) + \xi(z)\bar{v}_\lambda(z) = \lambda\bar{v}_\lambda(z) + f(z, \bar{v}_\lambda(z)) \text{ for a.a. } z \in \Omega. \tag{40}$$

We multiply (39) with $\bar{v}_\lambda(z)$ and (40) with $\bar{u}_\lambda(z)$, then integrate both equations over Ω and use Green's identity. We obtain

$$\begin{aligned}
&\int_\Omega (D\bar{u}_\lambda, D\bar{v}_\lambda)_{\mathbb{R}^N} dz + \int_\Omega \xi(z)\bar{u}_\lambda\bar{v}_\lambda dz + \int_{\partial\Omega} \beta(z)\bar{u}_\lambda\bar{v}_\lambda d\sigma\\
=&\lambda \int_\Omega \bar{u}_\lambda\bar{v}_\lambda dz + \int_\Omega f(z, \bar{u}_\lambda)\bar{v}_\lambda dz
\end{aligned} \tag{41}$$

$$\begin{aligned}&\int_\Omega (D\bar{v}_\lambda, D\bar{u}_\lambda)_{\mathbb{R}^N} dz + \int_\Omega \xi(z)\bar{v}_\lambda\bar{u}_\lambda dz + \int_{\partial\Omega} \beta(z)\bar{v}_\lambda\bar{u}_\lambda d\sigma \\ =&\lambda \int_\Omega \bar{v}_\lambda\bar{u}_\lambda dz + \int_\Omega f(z,\bar{v}_\lambda)\bar{u}_\lambda dz. \end{aligned} \tag{42}$$

We subtract (42) from (41) and obtain

$$\begin{aligned}&\int_\Omega [f(z,\bar{u}_\lambda)\bar{v}_\lambda - f(z,\bar{v}_\lambda)\bar{u}_\lambda]\, dz = 0, \\ \Rightarrow &\int_\Omega \left[\frac{f(z,\bar{u}_\lambda)}{\bar{u}_\lambda} - \frac{f(z,\bar{v}_\lambda)}{\bar{v}_\lambda}\right] \bar{u}_\lambda\bar{v}_\lambda dz = 0 \text{ (recall } \bar{u}_\lambda, \bar{v}_\lambda \in \operatorname{int} C_+) \\ \Rightarrow &\, \bar{u}_\lambda = \bar{v}_\lambda \text{ (see hypothesis } H_3\text{(iv))}.\end{aligned}$$

This proves the uniqueness of the positive solution of problem (P_λ). The uniqueness of this positive solution, together with Proposition 8, imply that the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is nondecreasing and continuous. ☐

As before, by strengthening the conditions on the perturbation $f(z,\cdot)$ we can improve the monotonicity of the map $\lambda \to \bar{u}_\lambda$.

The new condition on $f(z,x)$ are the following:

H_4: $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses H_4(i), (ii), (iii), (iv) are the same as the corresponding hypotheses H_3(i), (ii), (iii), (iv) and

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \mapsto f(z,x) + \hat{\xi}_\rho x$ is nondecreasing on $[0,\rho]$.

Remark. The examples after hypotheses H_3, also satisfy hypotheses H_4. Then Propositions 9 and 10 imply the following result.

Proposition 11. *If hypotheses $H(\xi)'$, $H(\beta)$, H_4 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then the set $S(\lambda)$ is a singleton $\{\bar{u}_\lambda\}$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L}$ into $C^1(\overline{\Omega})$ is strictly increasing (that is, if $\lambda < \eta < \widehat{\lambda}_1$, then $\bar{u}_\eta - \bar{u}_\lambda \in int C_+$).*

Summarizing the situation for problem (P_λ) when the perturbation term $f(z,\cdot)$ is sublinear, we can state the following theorem.

Theorem 3.1. *(a) If hypotheses $H(\xi)'$, $H(\beta)$, H_1 hold, then for every $\lambda \geq \widehat{\lambda}_1$ we have $S(\lambda) = \emptyset$, while for every $\lambda < \widehat{\lambda}_1$, $S(\lambda) \neq \emptyset$, $S(\lambda) \subseteq int C_+$, problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in int C_+$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is nondecreasing (that is, $\lambda < \eta < \lambda_1$ implies $\bar{u}_\lambda \leq \bar{u}_\eta$) and left continuous.*

(b) If hypotheses $H(\xi)'$, $H(\beta)$, H_2 hold, then the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is strictly increasing (that is, $\lambda < \eta < \widehat{\lambda}_1$ implies $\bar{u}_\eta - \bar{u}_\lambda \in int C_+$).

(c) If hypotheses $H(\xi)'$, $H(\beta)$, H_3 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then $S(\lambda)$ is a singleton $\{\bar{u}_\lambda\}$ ($\bar{u}_\lambda \in int C_+$) and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is nondecreasing and continuous.

(d) If hypotheses $H(\xi)'$, $H(\beta)$, H_4 hold, then the map $\lambda \mapsto \bar{u}_\lambda$ is strictly increasing from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$.

4. **The superlinear case.** In this section, we investigate the case of a superlinear perturbation $f(z,\cdot)$. Now we cannot have uniqueness of the positive solution and in fact we show that the problem exhibits a kind of bifurcation phenomenon, that is, for $\lambda < \widehat{\lambda}_1$, problem (P_λ) has at least two positive solutions, while for $\lambda \geq \widehat{\lambda}_1$, $S(\lambda) = \emptyset$. We stress that for the superlinearity of $f(z,\cdot)$ we do not use the Ambrosetti-Rabinowitz condition (AR-condition for short).

The hypotheses on the perturbation $f(z,x)$ are the following:

H_5: $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) \geq 0$ for all $x \geq 0$, $f(z,x) > 0$ for a.a. $z \in \Omega_0 \subseteq \Omega$ with $|\Omega_0|_N > 0$, all $x > 0$ and

(i) $f(z,x) \leq a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)_+$ and $2 < r < 2^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega$$

and there exists $q \in \left(\max\left\{1,(r-2)\frac{N}{2}\right\},2^*\right)$ such that

$$0 < \tilde{\xi} \leq \liminf_{x\to+\infty} \frac{f(z,x)x - 2F(z,x)}{x^q} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) $\displaystyle\lim_{x\to 0^+} \frac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$.

Remarks. As in the *sublinear* case, since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, we may assume without any loss of generality that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypothesis H_5(ii) implies that for a.a. $z \in \Omega$, $f(z,\cdot)$ is superlinear near $+\infty$. However, we do not assume the usual for superlinear problems AR-condition. Recall that the AR-condition (unilateral version since it is imposed only on the positive semiaxis), says that there exist $\tau > 2$ and $M_2 > 0$ such that

$$0 < \tau F(z,x) \leq f(z,x)x \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_2 \tag{43}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot,M_2) \tag{44}$$

(see Ambrosetti & Rabinowitz [2] and Mugnai [10]). Integrating (43) and using (44), we obtain

$$c_8 x^\tau \leq F(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_2, \text{ some } c_8 > 0. \tag{45}$$

From (43) and (45) it follows that for a.a. $z \in \Omega$, $f(z,\cdot)$ has at least $(\tau-1)$-polynomial growth near $+\infty$. Our hypothesis is implied by the AR-condition. We may assume that $\tau > \max\left\{1,(r-2)\frac{N}{2}\right\}$. We have

$$\begin{aligned}\frac{f(z,x)x - 2F(z,x)}{x^\tau} &= \frac{f(z,x)x - \tau F(z,x)}{x^\tau} + (\tau-2)\frac{F(z,x)}{x^\tau}\\ &\geq c_8(\tau-2) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_2\\ &\qquad\qquad \text{(see (43) and (45)).}\end{aligned}$$

Thus, hypothesis H_5(ii) is satisfied (recall that $\tau > 2$). This more general superlinearity condition, incorporates in our framework superlinear nonlinearities with *slower* growth near $+\infty$, which fail to satisfy the AR-condition (unilateral version).

Examples. The following functions satisfy hypotheses H_5. As in the previous examples, for the sake of simplicity we drop the z-dependence.

$$f_1(x) = x^{q-1} \text{ for all } x \geq 0, \text{ with } 2 < q < 2^*$$

$$f_2(x) = x \ln(1+x) \text{ for all } x \geq 0.$$

Note that f_1 satisfies the unilateral AR-condition, but f_2 does not.

Using Proposition 2 and reasoning exactly as in the proof of Proposition 3, we obtain the following result.

Proposition 12. *If hypotheses $H(\xi)'$, $H(\beta)$, H_5 hold and $\lambda \geq \lambda_1$, then $S(\lambda) = \emptyset$.*

So according to this proposition, we have $\mathcal{L} \subseteq (-\infty, \widehat{\lambda}_1)$. In fact we will show that equality holds.

Proposition 13. *If hypotheses $H(\xi)'$, $H(\beta)$, H_5 hold, then $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$.*

Proof. We fix $\lambda \in (-\infty, \widehat{\lambda}_1)$ and consider the Carathéodory function

$$k_\lambda(z,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \lambda x + f(z,x) & \text{if } 0 < x. \end{cases} \tag{46}$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)ds$ and consider the C^1-functional $\varphi_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u^-\|_2^2 - \int_\Omega K_\lambda(z,u)dz \text{ for all } u \in H^1(\Omega).$$

Here, $\mu > 0$ is as in (6). Hypotheses H_5(i), (iii) imply that given $\varepsilon > 0$, we can find $c_9 = c_9(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{\varepsilon}{2}x^2 + c_9 x^r \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{47}$$

Choosing $\varepsilon \in (0, \widehat{\lambda}_1 - \lambda)$ (recall $\lambda < \widehat{\lambda}_1$), for all $u \in H^1(\Omega)$ we obtain

$$\begin{aligned} \varphi_\lambda(u) &\geq \frac{1}{2}\gamma(u^-) + \frac{\mu}{2}\|u^-\|_2^2 + \frac{1}{2}\gamma(u^+) - \frac{\lambda+\varepsilon}{2}\|u^+\|_2^2 - c_9\|u^+\|_r^r \\ &\qquad \text{(see (46), (47))} \\ &\geq \frac{c_0}{2}\|u^-\|^2 + \frac{\hat{c}}{2}\|u^+\|^2 - c_{10}\|u\|^r \text{ for some } c_{10} > 0 \\ &\qquad \text{(see (6) and Lemma 2.2)} \\ &\geq c_{11}\|u\|^2 - c_{10}\|u\|^r \text{ for some } c_{11} > 0. \end{aligned} \tag{48}$$

Since $r > 2$, from (48) we infer that $u = 0$ is a strict local minimizer of φ_λ. It is easy to see that $K_{\varphi_\lambda} \subseteq C_+$ (see (46)). Thus, we may assume that $u = 0$ is an isolated critical point of φ_λ or otherwise we already have a whole sequence of distinct positive solutions of (P_λ) which converge to zero in $H^1(\Omega)$. Therefore we can find $\rho \in (0,1)$ small such that

$$\varphi_\lambda(0) = 0 < \inf\left[\varphi_\lambda(u) : \|u\| = \rho\right] = m_\rho^\lambda \tag{49}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 29).

Hypothesis H_5(ii) implies that given any $u \in \operatorname{int} C_+$, we have

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{50}$$

Claim. *φ_λ satisfies the C-condition.*

Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$|\varphi_\lambda(u_n)| \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \in \mathbb{N}, \tag{51}$$

$$(1 + \|u_n\|)\varphi_\lambda'(u_n) \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{52}$$

By (52) we have

$$\begin{aligned}\left|\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n h\,dz + \int_{\partial\Omega} \beta(z) u_n h\,d\sigma + \mu \int_\Omega u_n^- h\,dz - \int_\Omega k_\lambda(z, u_n) h\,dz\right| \\ \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in H^1(\Omega) \text{ with } \varepsilon_n \to 0^+.\end{aligned} \tag{53}$$

In (53) we choose $h = -u_n^- \in H^1(\Omega)$. Using (46), we obtain

$$\begin{aligned}&\left|\gamma(u_n^-) + \mu\|u_n^-\|_2^2\right| \leq \varepsilon_n \text{ for all } n \in \mathbb{N},\\ &\Rightarrow c_0\|u_n^-\|^2 \leq \varepsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (6))}\\ &\Rightarrow u_n^- \to 0 \text{ in } H^1(\Omega).\end{aligned} \tag{54}$$

It follows from (51) and (54) that

$$\gamma(u_n^+) - \int_\Omega \left[\lambda(u_n^+)^2 + 2F(z, u_n^+)\right] dz \leq M_4 \quad \text{for some } M_4 > 0, \text{ all } n \in \mathbb{N}. \tag{55}$$

On the other hand, if in (53) we choose $h = u_n^+ \in H^1(\Omega)$, then

$$-\gamma(u_n^+) + \int_\Omega \left[\lambda(u_n^+)^2 + f(z, u_n^+)u_n^+\right] dz \leq \varepsilon_n \text{ for all } n \in \mathbb{N} \quad \text{(see (46))}. \tag{56}$$

Adding (55) and (56), we obtain

$$\int_\Omega \left[f(z, u_n^+)u_n^+ - 2F(z, u_n^+)\right] dz \leq M_5 \text{ for some } M_5 > 0, \text{ all } n \in \mathbb{N}. \tag{57}$$

Hypotheses H_5(i), (ii) imply that we can find $\tilde{\xi}_0 \in (0, \xi_0)$ and $c_{12} > 0$ such that

$$\tilde{\xi}_0 x^q - c_{12} \leq f(z, x)x - 2F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{58}$$

We use (58) in (57) and infer that

$$\{u_n^+\}_{n\geq 1} \subseteq L^q(\Omega) \text{ is bounded.} \tag{59}$$

First, suppose that $N \neq 2$. By hypothesis H_5(ii), we see that we may assume without loss of generality that $q < r < 2^*$. So, we can find $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{2^*}. \tag{60}$$

Invoking the interpolation inequality (see, for example, Gasinski & Papageorgiou [6, p. 905]), we have

$$\begin{aligned}
\|u_n^+\|_r &\le \|u_n^+\|_q^{1-t}\|u_n^+\|_{2^*}^t \\
&\le M_6\|u_n^+\|^t \text{ for some } M_6>0,\ \text{all } n\in\mathbb{N} \text{ (see (59))},\\
\Rightarrow \|u_n^+\|_r^r &\le M_7\|u_n^+\|^{tr} \text{ for some } M_7=M_6^r>0,\ \text{all } n\in\mathbb{N}.
\end{aligned} \tag{61}$$

In (53) we choose $h=u_n^+\in H^1(\Omega)$. Then

$$\begin{aligned}
\gamma(u_n^+) &\le \varepsilon_n+\int_\Omega\left[\lambda(u_n^+)^2+f(z,u_n^+)u_n^+\right]dz \text{ for all } n\in\mathbb{N} \text{ (see (46))},\\
\Rightarrow \gamma(u_n^+) &\le c_{13}(1+\|u_n^+\|_r^r) \text{ for some } c_{13}>0,\ \text{all } n\in\mathbb{N}\\
&\qquad \text{(see hypothesis } H_5\text{(i) and recall that } 2<r)\\
&\le c_{14}(1+\|u_n^+\|^{tr}) \text{ for some } c_{14}>0, \text{all } n\in\mathbb{N}\\
&\qquad \text{(see (61))}.
\end{aligned} \tag{62}$$

By hypothesis H_5(i), we see that we can always assume that r is close to 2^*, hence $q\ge 2$ (see hypothesis H_5(ii)). Then (59) implies that $\{u_n^+\}_{n\ge1}\subseteq L^2(\Omega)$ is bounded and so by (62) we have

$$\begin{aligned}
\gamma(u_n^+)+\mu\|u_n^+\|_2^2 &\le c_{15}(1+\|u_n^+\|^{tr}) \text{ for some } c_{15}>0,\ \text{all } n\in\mathbb{N},\\
\Rightarrow c_0\|u_n^+\|^2 &\le c_{15}(1+\|u_n^+\|^{tr}) \text{ for all } n\in\mathbb{N}\\
&\qquad\qquad \text{(see (6))}.
\end{aligned} \tag{63}$$

Due to (60) and hypothesis H_5(ii), we see that

$$\begin{aligned}
&t\,r<2,\\
&\Rightarrow \{u_n^+\}_{n\ge1}\subseteq H^1(\Omega) \text{ is bounded (see (63))}.
\end{aligned} \tag{64}$$

If $N=2$, then $2^*=+\infty$ and the Sobolev embedding theorem says that $H^1(\Omega)\hookrightarrow L^\eta(\Omega)$ for all $\eta\in[1,+\infty)$. Let $\eta>r>q$ and $t\in(0,1)$ such that

$$\begin{aligned}
\frac{1}{r}&=\frac{1-t}{q}+\frac{t}{\eta},\\
\Rightarrow t\,r&=\frac{\eta(r-q)}{\eta-q}.
\end{aligned}$$

Note that

$$\frac{\eta(r-q)}{\eta-q}\to r-q \text{ as } \eta\to+\infty=2^* \text{ and } r-q<2 \text{ (see hypothesis } H_5\text{(ii))}.$$

Therefore the previous argument works if instead of 2^* we use $\eta>1$ big such that $t\,r<2$. We again obtain (64). It follows from (54) and (64) that

$$\{u_n\}_{n\ge1}\subseteq H^1(\Omega) \text{ is bounded}.$$

Thus, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n\to u \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega). \tag{65}$$

In (53) we choose $h=u_n-u\in H^1(\Omega)$, pass to the limit as $n\to\infty$ and use (65). Then

$$\begin{aligned}
&\lim_{n\to\infty}\langle A(u_n),u_n-u\rangle=0,\\
&\Rightarrow \|Du_n\|_2\to\|Du\|_2,
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow Du_n \to Du \text{ in } L^2(\Omega,\mathbb{R}^N) \\
&\quad \text{(by the Kadec-Klee property, see (65))} \\
&\Rightarrow u_n \to u \text{ in } H^1(\Omega) \text{ (see (65))}.
\end{aligned}
$$

Therefore φ_λ satisfies the C-condition and this proves the Claim.

Then (49), (50) and the Claim, permit the use of Theorem 2.1 (the mountain pass theorem) and so we can find $u_\lambda \in H^1(\Omega)$ such that

$$
\begin{aligned}
&u_\lambda \in K_{\varphi_\lambda} \text{ and } m_\rho^\lambda \leq \varphi_\lambda(u_\lambda), \\
\Rightarrow\ & u_\lambda \neq 0 \text{ (see (49)) and so } u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+.
\end{aligned}
$$

□

Next, we show that for every $\lambda \in (-\infty, \widehat{\lambda}_1)$, problem (P_λ) has a smallest positive solution.

First, let us recall the following lemma from Hu and Papageorgiou [7, p. 178].

Lemma A. *If $(\Omega, \sum, \mu)$ is a finite measure space and $\mathcal{D}$ is a family of $\mathbb{R}_+$-valued measurable functions which is downward directed, then there exists a unique (modulo equality μ-a.e.) function $h : \Omega \to \mathbb{R}_+$ such that*
(a) $h(\omega) \leq u(\omega)$ μ-a.e. in Ω for all $u \in \mathcal{D}$;
(b) if $g : \Omega \to \mathbb{R}$ is a measurable function such that

$$g(\omega) \leq u(\omega) \quad \mu\text{-a.e. in } \Omega \text{ for all } u \in \mathcal{D}$$

then $g(\omega) \leq h(\omega)$ μ-a.e., that is, $h = \inf \mathcal{D}$. Moreover, there is a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \mathcal{D}$ such that $\inf_{n\geq 1} u_n = h = \inf \mathcal{D}$.

Proposition 14. *If hypotheses $H(\xi)'$, $H(\beta)$, H_5 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$.*

Proof. We again have that $S(\lambda)$ is downward directed (that is, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \leq u_1$, $u \leq u_2$; see Filippakis & Papageorgiou [5]). Therefore Lemma A above implies that we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq S(\lambda)$ such that

$$\inf S(\lambda) = \inf_{n\geq 1} u_n.$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n h\, dz + \int_{\partial\Omega} \beta(z) u_n h\, d\sigma = \int_\Omega [\lambda u_n + f(z,u_n)] h dz \tag{66}$$
$$\text{for all } h \in H^1(\Omega), \text{ all } n \in \mathbb{N}.$$

Choosing $h = u_n \in H^1(\Omega)$ in (66), we obtain

$$\gamma(u_n) = \lambda \|u_n\|_2^2 + \int_\Omega f(z,u_n) u_n dz \text{ for all } n \in \mathbb{N}. \tag{67}$$

Since $u_n \leq u_1 \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$ (recall that $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is decreasing) and by (67), (6) and hypotheses $H(\beta)'$, $H(\beta)$, H_5(i), we can conclude that

$$\{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded}.$$

Hence, we may assume that

$$u_n \xrightarrow{w} \bar{u}_\lambda \text{ in } H^1(\Omega) \text{ and } u_n \to \bar{u}_\lambda \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{68}$$

Returning to (66), passing to the limit as $n \to \infty$ and using (68), we obtain

$$\langle A(\bar{u}_\lambda), h\rangle + \int_\Omega \xi(z)\bar{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)\bar{u}_\lambda h\,d\sigma = \int_\Omega [\lambda \bar{u}_\lambda + f(z,\bar{u}_\lambda)]h\,dz \quad \text{for all } h \in H^1(\Omega),$$

$\Rightarrow \bar{u}_\lambda \geq 0$ is a solution of (P_λ).

We will show that $\bar{u}_\lambda \neq 0$. Arguing by contradiction, suppose that $\bar{u}_\lambda = 0$. Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$. So we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } u_n \to y \text{ in } L^r(\Omega) \text{ and } L^2(\partial\Omega). \tag{69}$$

By (66) we have

$$\langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n h\,dz + \int_{\partial\Omega} \beta(z) y_n h\,d\sigma = \int_\Omega \left[\lambda y_n + \frac{N_f(u_n)}{\|u_n\|}\right] h\,dz \quad \text{for all } h \in H^1(\Omega), \text{ all } n \in \mathbb{N}. \tag{70}$$

Let $\rho = \|u_1\|_\infty$. Hypotheses H_5(i), (iii) imply that

$$\begin{aligned} & f(z,x) \leq c_{16}\, x \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\rho], \text{ some } c_{16} > 0, \\ \Rightarrow\ & \frac{f(z,u_n(z))}{\|u_n\|} \leq c_{18}\, y_n(z) \text{ for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \\ \Rightarrow\ & \left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geq 1} \subseteq L^2(\Omega) \text{ is bounded (see (69)).} \end{aligned} \tag{71}$$

By passing to a subsequence if necessary and using hypothesis H_5(iii) we infer that

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} 0 \text{ in } L^2(\Omega) \tag{72}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 14). In (70), we first choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (69) and (71). Then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

$$\Rightarrow y_n \to y \text{ in } H^1(\Omega) \quad \text{(by the Kadec-Klee property)} \tag{73}$$

$$\Rightarrow \|y\| = 1. \tag{74}$$

Next in (70) we choose $h = y_n \in H^1(\Omega)$ and pass to the limit as $n \to \infty$. Using (72) and (73) we obtain

$$\gamma(y) = \lambda\|y\|_2^2 < \widehat{\lambda}_1\|y\|_2^2 \text{ (see (74))},$$

a contradiction to (2). Hence, $\bar{u}_\lambda \neq 0$ and therefore

$$\bar{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+,\ \bar{u}_\lambda = \inf S(\lambda).$$

$\square$

Reasoning as in the proof of Proposition 8, we can establish the monotonicity and continuity properties of the map $\lambda \mapsto \bar{u}_\lambda$.

Proposition 15. *If hypotheses $H(\xi)'$, $H(\beta)$, H_5 hold, then the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is nondecreasing and continuous.*

As before (see Proposition 9 and hypotheses H_2), by strengthening the conditions on $f(z,\cdot)$, we can improve the monotonicity of $\lambda \mapsto \bar{u}_\lambda$.

The new conditions on $f(z,x)$ are the following:

H_6: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses H_6(i), (ii), (iii) are the same as the corresponding hypotheses H_5(i), (ii), (iii) and

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z,x) + \hat{\xi}_\rho x$$

is nondecreasing on $[0,\rho]$.

Reasoning as in the proof of Proposition 7, we obtain.

Proposition 16. *If hypotheses $H(\xi)'$, $H(\beta)$, H_6 hold, then the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is strictly increasing (that is, if $\lambda < \eta < \widehat{\lambda}_1$, then $\bar{u}_\eta - \bar{u}_\lambda \in int C_+$).*

Next, we show that for all admissible $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, problem (P_λ) has at least two positive solutions, which are ordered.

Proposition 17. *If hypotheses $H(\xi)'$, $H(\beta)$, H_6 hold and $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$, then problem (P_λ) admits at least two positive solutions*

$$u_\lambda, \hat{u}_\lambda \in int C_+,\ \hat{u}_\lambda - u_\lambda \in int C_+.$$

Proof. By Proposition 13 we already have one positive solution $u_\lambda \in \operatorname{int} C_+$. We may assume that u_λ is the minimal positive solution (that is, $u_\lambda = \bar{u}_\lambda$, see Proposition 14). We consider the following Carathéodory function

$$\tilde{g}_\lambda(z,x) = \begin{cases} (\lambda+\mu)u_\lambda(z) + f(z,u_\lambda(z)) & \text{if } x \le u_\lambda(z) \\ (\lambda+\mu)x + f(z,x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{75}$$

We set $\widetilde{G}_\lambda(z,x) = \int_0^x \tilde{g}_\lambda(z,s)ds$ and consider the C^1-functional $\widetilde{\psi}_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\psi}_\lambda(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega \widetilde{G}_\lambda(z,u)dz \text{ for all } u \in H^1(\Omega).$$

From (75), we see that on $[u_\lambda) = \{u \in H^1(\Omega) :\ u_\lambda(z) \le u(z) \text{ for a.a. } z \in \Omega\}$ we have

$$\begin{aligned} &\widetilde{\psi}_\lambda = \varphi_\lambda + \tilde{\xi}_\lambda \text{ for some } \widetilde{\xi}_\lambda \in \mathbb{R}, \\ \Rightarrow\ &\widetilde{\psi}_\lambda \text{ satisfies the C-condition} \end{aligned} \tag{76}$$

(see the Claim in the proof of Proposition 13).

Claim. *We may assume that $u_\lambda \in int C_+$ is a local minimizer of the functional $\widetilde{\psi}_\lambda$.*

Let $\lambda < \eta < \widehat{\lambda}_1$ and let $u_\eta \in S(\eta)$. From Proposition 16, we obtain $u_\eta - u_\lambda \in \operatorname{int} C_+$. We introduce the following truncation of $\tilde{g}_\lambda(z,\cdot)$:

$$g^*_\lambda(z,x) = \begin{cases} \widetilde{g}_\lambda(z,x) & \text{if } x \le u_\eta(z) \\ \widetilde{g}_\lambda(z,u_\eta(z)) & \text{if } u_\eta(z) < x. \end{cases} \tag{77}$$

Evidently, this is a Carathéodory function. We set $G_\lambda^*(z,x) = \int_0^x g_\lambda^*(z,s)ds$ and consider the C^1-functional $\psi_\lambda^* : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^*(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega G_\lambda^*(z,u)dz \text{ for all } u \in H^1(\Omega).$$

As before (see the proof of Proposition 8), we can show that

$$K_{\psi_\lambda^*} \subseteq [u_\lambda, u_\eta] = \left\{u \in H^1(\Omega) : u_\lambda(z) \le u(z) \le u_\eta(z) \text{ for a.a. } z \in \Omega\right\}. \tag{78}$$

Moreover, by (6) and (77), we see that ψ_λ^* is coercive. It is also sequentially weakly lower semicontinuous. So we can find $u_\lambda^* \in H^1(\Omega)$ such that

$$\begin{aligned} &\psi_\lambda^*(u_\lambda^*) = \inf\left[\psi_\lambda^*(u) : u \in H^1(\Omega)\right], \\ \Rightarrow\ & u_\lambda^* \in K_{\psi_\lambda^*} \subseteq [u_\lambda, u_\eta] \text{ (see (78))}, \\ \Rightarrow\ & u_\lambda^* \in S(\lambda) \subseteq \operatorname{int} C_+ \text{ (see (77))}. \end{aligned}$$

If $u_\lambda^* \ne u_\lambda$, then this is the second positive solution of problem (P_λ) and, as we will see in the last part of the proof, we have $u_\lambda^* - u_\lambda \in \operatorname{int} C_+$, so we are done. So, suppose $u_\lambda^* = u_\lambda$. We have

$$\psi_\lambda^*|_{[0,u_\eta]} = \widetilde{\psi}_\lambda|_{[0,u_\eta]} \text{ (see (75) and (77))}.$$

Since $u_\lambda = u_\lambda^*$ and $u_\eta - u_\lambda \in \operatorname{int} C_+$ (see Proposition 16), it follows that

$$\begin{aligned} & u_\lambda \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widetilde{\psi}_\lambda, \\ \Rightarrow\ & u_\lambda \text{ is a local } H^1(\Omega)\text{-minimizer of } \widetilde{\psi}_\lambda \text{ (see Proposition 1)}. \end{aligned}$$

This proves the Claim.

We assume that K_{ψ_λ} is finite or otherwise we already have a sequence of distinct positive solutions, all strictly bigger than u_λ (note that $K_{\widetilde{\psi}_\lambda} \subseteq [u_\lambda) = \left\{u \in H^1(\Omega) : u_\lambda(z) \le u(z) \text{ for a.a. } z \in \Omega\right\}$) and so we are done. Using the Claim, we can find $\rho \in (0,1)$ small such that

$$\widetilde{\psi}_\lambda(u_\lambda) < \inf\left[\widetilde{\psi}_\lambda(u) : \|u - u_\lambda\| = \rho\right] = \widetilde{m}_\rho^\lambda. \tag{79}$$

Hypothesis H_6(ii) implies that

$$\widetilde{\psi}_\lambda(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty. \tag{80}$$

Then (76), (79), (80) permit the use of Theorem 2.1 (the mountain pass theorem). Therefore we can find $\hat{u}_\lambda \in H^1(\Omega)$ such that

$$\begin{aligned} & \hat{u}_\lambda \in K_{\widetilde{\psi}_\lambda} \subseteq [u_\lambda) \text{ and } \widetilde{m}_\rho^\lambda \le \widetilde{\psi}_\lambda(\hat{u}_\lambda), \\ \Rightarrow\ & \hat{u}_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+ \text{ (see (75))}, u_\lambda \le \hat{u}_\lambda,\ u_\lambda \ne \hat{u}_\lambda. \end{aligned}$$

Moreover, if $\rho = \|\hat{u}_\lambda\|_\infty$ and $\hat{\xi}_\rho > 0$ is as postulated by hypothesis H_6(iv), then

$$\begin{aligned} & -\Delta u_\lambda(z) + (\xi(z) + \hat{\xi}_\rho)u_\lambda(z) \\ & = (\lambda + \hat{\xi}_\rho)u_\lambda(z) + f(z, u_\lambda(z)) \\ & \le (\lambda + \hat{\xi}_\rho)\hat{u}_\lambda(z) + f(z, \hat{u}_\lambda(z)) \\ & \qquad\qquad \text{(recall } u_\lambda \le \hat{u}_\lambda \text{ and see hypothesis } H_6\text{(iv))} \\ & = -\Delta \hat{u}_\lambda(z) + (\xi(z) + \hat{\xi}_\rho)\hat{u}_\lambda(z) \text{ for a.a. } z \in \Omega, \end{aligned}$$

$$\Rightarrow \Delta(\hat{u}_\lambda - u_\lambda)(z) \leq (\|\xi^+\|_\infty + \hat{\xi}_\rho)(\hat{u}_\lambda - u_\lambda)(z) \text{ for a.a. } z \in \Omega$$
$$\text{(see hypothesis } H(\xi)'\text{)}$$
$$\Rightarrow \hat{u}_\lambda - u_\lambda \in \operatorname{int} C_+ \text{ (by the strong maximum principle).}$$

□

Summarizing the situation for problem (P_λ) when the perturbation $f(z,\cdot)$ is superlinear, we can state the following theorem.

Theorem 4.1. *If hypotheses $H(\xi)'$, $H(\beta)$, H_6 hold, then*

(a) for all $\lambda \geq \widehat{\lambda}_1$ problem (P_λ) has no positive solution (that is, $S(\lambda) = \emptyset$);

(b) for every $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$ problem (P_λ) has at least two positive solutions u_λ, $\hat{u}_\lambda \in int C_+$, $\hat{u}_\lambda - u_\lambda \in int C_+$;

(c) for every $\lambda \in \mathcal{L} = (-\infty, \widehat{\lambda}_1)$ problem (P_λ) has a smallest positive solution $\bar{u}_\lambda$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \widehat{\lambda}_1)$ is strictly increasing and left continuous.

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