Gradient-Type Systems on Unbounded Domains of the Heisenberg Group

Giovanni Molica Bisci[1] · Dušan Repovš[2]

Abstract

The purpose of this paper is to study the existence of weak solutions for some classes of one-parameter subelliptic gradient-type systems involving a Sobolev–Hardy potential defined on an unbounded domain Ω_ψ of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ $(n \geq 1)$ whose geometrical profile is determined by two real positive functions ψ_1 and ψ_2 that are bounded on bounded sets. The treated problems have a variational structure, and thanks to this, we are able to prove the existence of an open interval $\Lambda \subset (0, \infty)$ such that, for every parameter $\lambda \in \Lambda$, the system has at least two non-trivial symmetric weak solutions that are uniformly bounded with respect to the Sobolev $HW_0^{1,2}$-norm. Moreover, the existence is stable under certain small subcritical perturbations of the nonlinear term. The main proof, crucially based on the Palais principle of symmetric criticality, is obtained by developing a group-theoretical procedure on the unitary group $\mathbb{U}(n) = U(n) \times \{1\}$ and by exploiting some compactness embedding results into Lebesgue spaces, recently proved for suitable $\mathbb{U}(n)$-invariant subspaces of the Folland–Stein space $HW_0^{1,2}(\Omega_\psi)$. A key ingredient for our variational approach is a very general min–max argument valid for sufficiently smooth functionals defined on reflexive Banach spaces.

Keywords Gradient-type system · Heisenberg group · Variational methods · Principle of symmetric criticality · Symmetric solutions

Mathematics Subject Classification 35R03 · 35H20 · 35J70 · 35A01 · 35A15

✉ Giovanni Molica Bisci
 giovanni.molicabisci@uniurb.it

 Dušan Repovš
 dusan.repovs@guest.arnes.si

1 Dipartimento di Scienze Pure e Applicate (DiSPeA), Università degli Studi di Urbino Carlo Bo - Piazza della Repubblica 13, 61029 Urbino, Italy

2 Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia

1 Introduction

The purpose of the present paper is to study the existence of weak solutions for subelliptic systems defined on unbounded domains of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ ($n \geq 1$). More precisely, let $\psi_1, \psi_2 : [0, \infty) \to \mathbb{R}$ be two functions that are bounded on bounded sets, with $\psi_1(r) < \psi_2(r)$ for every $r \geq 0$. Define

$$\Omega_\psi = \{q \in \mathbb{H}^n : q = (z, t) \text{ with } \psi_1(|z|) < t < \psi_2(|z|)\}, \tag{1.1}$$

where $|z| = \sqrt{\sum_{i=1}^n |z_i|^2}$, and assume that Ω_ψ contains the origin $O = (0, 0) \in \mathbb{H}^n$.

We deal here with the following singular subelliptic problem:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u - \nu V(q) u + u = \lambda K(q) \partial_1 F(u, v) + \mu \partial_1 G(q, u, v) & \text{in } \Omega_\psi \\ -\Delta_{\mathbb{H}^n} v - \nu V(q) v + v = \lambda K(q) \partial_2 F(u, v) + \mu \partial_2 G(q, u, v) & \text{in } \Omega_\psi \\ u = v = 0 \text{ on } \partial\Omega_\psi, \end{cases} \tag{1.2}$$

where $\Delta_{\mathbb{H}^n}$ is the Kohn–Laplace operator defined by

$$\Delta_{\mathbb{H}^n} \varphi = \operatorname{div}_H (D_{\mathbb{H}^n} \varphi)$$

along any $\varphi \in C_0^\infty(\mathbb{H}^n)$, with

$$D_{\mathbb{H}^n} \varphi = (X_1 \varphi, \ldots, X_n \varphi, Y_1 \varphi, \ldots, Y_n \varphi)$$

as in Sect. 2, and $\{X_j, Y_j\}_{j=1}^n$ is the basis of horizontal left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$. The *critical Sobolev exponent* 2^* in the Heisenberg group $\mathbb{H}^n$ is defined as $2^* = 2Q/(Q - 2)$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

On the potentials $V, K : \Omega_\psi \to \mathbb{R}$, we assume that

(h_V) *V is measurable, cylindrically symmetric, i.e. $V(z, t) = V(|z|, t)$, and there exists a constant $C_V > 0$ such that*

$$0 \leq V(z, t) \leq C_V \frac{|z|^2}{r(z, t)^4}, \tag{1.3}$$

for every $q = (z, t) \in \mathbb{H}^n$, where r denotes the Heisenberg norm $r(q) = r(z, t) = (|z|^4 + t^2)^{1/4}, z \in \mathbb{C}^n, t \in \mathbb{R}$;

(h_K) $K \in L^\infty(\Omega_\psi) \cap L^1(\Omega_\psi)$ *is a non-negative cylindrically symmetric function with*

$$\inf_{q \in \Omega_0} K(q) > 0,$$

for some open set $\Omega_0 \subset \Omega_\psi$.

The parameters $\lambda, \mu > 0$ and $\nu \in [0, C_V^{-1} n^2)$. Suppose also that the nonlinearity F satisfies the following hypotheses:

(f_1) *$F : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function with $F(0, 0) = 0$ and there exists $(\eta_0, \zeta_0) \in \mathbb{R}^2$ such that $F(\eta_0, \zeta_0) > 0$;*

(f_2) $\displaystyle \lim_{\eta, \zeta \to 0} \frac{\partial_1 F(\eta, \zeta)}{|\eta|} = \lim_{\eta, \zeta \to 0} \frac{\partial_2 F(\eta, \zeta)}{|\zeta|} = 0;$

(f_3) *there exist $\epsilon > 0$ and $\alpha \in (2, 2^*)$ such that*

$$|\partial_1 F(\eta, \zeta)| \leq \epsilon \left(|\eta| + |\zeta| + |\eta|^{\alpha-1} \right),$$

and

$$|\partial_2 F(\eta, \zeta)| \leq \epsilon \left(|\eta| + |\zeta| + |\zeta|^{\alpha-1} \right),$$

for every $(\eta, \zeta) \in \mathbb{R}^2$;

(f_4) *there exist $p_F, q_F \in (0, 2)$ and suitable real constants $\kappa_j > 0$, such that*

$$F(\eta, \zeta) \leq \kappa_1 |\eta|^{p_F} + \kappa_2 |\zeta|^{q_F} + \kappa_3,$$

for every $(\eta, \zeta) \in \mathbb{R}^2$.

Here and in the sequel, the nonlinearities $\partial_1 F$ and $\partial_2 F$ denote the partial derivatives of F with respect to the first variable and the second variable, respectively.

Furthermore, for the nonlinear term $G : \Omega_\psi \times \mathbb{R}^2 \to \mathbb{R}$, we assume the following conditions:

(g_1) *$G : \Omega_\psi \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $(\eta, \zeta) \mapsto G(q, \eta, \zeta)$ is of class C^1 and $G(q, 0, 0) = 0$ for every $q \in \Omega_\psi$;*

(g_2) $\displaystyle \lim_{\eta, \zeta \to 0} \frac{\partial_1 G(q, \eta, \zeta)}{|\eta|} = \lim_{\eta, \zeta \to 0} \frac{\partial_2 G(q, \eta, \zeta)}{|\zeta|} = 0,$ *uniformly for every $q \in \Omega_\psi$;*

(g_3) *there exist $\kappa > 0$ and $\beta \in (2, 2^*)$ such that*

$$|\partial_1 G(q, \eta, \zeta)| \leq \kappa \left(|\eta| + |\zeta| + |\eta|^{\beta-1} \right),$$

and

$$|\partial_2 G(q, \eta, \zeta)| \leq \kappa \left(|\eta| + |\zeta| + |\zeta|^{\beta-1} \right),$$

for every $q \in \Omega_\psi$ and $(\eta, \zeta) \in \mathbb{R}^2$;

(g_4) *the function $G(\cdot, \eta, \zeta)$ is cylindrically symmetric for every $(\eta, \zeta) \in \mathbb{R}^2$.*

The nonlinearities $\partial_1 G$ and $\partial_2 G$ denote the partial derivatives of G with respect to the second variable and the third variable, respectively.

Noncompact variational problems have attracted much attention since the late seventies. System (1.2) is a reasonably useful generalization of most studied elliptic problems, with singular potentials and subcritical nonlinearities, which naturally arise in different branches of mathematics. More precisely, differential problems involving a subelliptic operator on an unbounded domain Ω of stratified groups have been intensively studied in recent years by several authors, see, among others, the papers [26,35,36,45,46] and references therein.

As observed in the recent paper [7], if the domain Ω is not bounded, then the Folland–Stein space $HW_0^{1,2}(\Omega)$ need not be compactly embeddable into a Lebesgue space. This lack of compactness produces several difficulties in exploiting variational methods. In order to recover compactness for the unbounded case, a standard hypothesis in the above cited results was the *strongly asymptotically contractive* condition on Ω (see [20] and [35] for related topics).

Now, we observe that a strongly asymptotically contractive domain Ω is geometrically thin at infinity. Following [1], in the presence of symmetries, by replacing the contractive assumption on Ω with a technical geometrical hypothesis, we are able to treat here subelliptic gradient-type systems on the Heisenberg group $\mathbb{H}^n$, in which the domain Ω_ψ is possibly large at infinity. We also notice that if the functions ψ_1 and ψ_2 are bounded, the domain Ω_ψ is strongly asymptotically contractive and the entire space $HW_0^{1,2}(\Omega_\psi)$ is compactly embedded in $L^q(\Omega_\psi)$ for every $q \in (2, 2^*)$. We refer to [1,26,36,45] for further details.

In the main result of the present paper (see Theorem 9), we prove the existence of an open interval $\Lambda \subset (0, \infty)$ such that, for every parameter $\lambda \in \Lambda$, the system (1.2) has at least two symmetric weak solutions that are uniformly bounded with respect to the Sobolev $HW_0^{1,2}$-norm. This existence result is stable in the presence of a small perturbation term G for which the structural conditions (g_1)–(g_4) are satisfied.

In order to prove Theorem 9, we find critical points of the energy functional associated with problem (1.2) by means of a mini-max theorem and the well-known Palais principle of symmetric criticality (see, respectively, Theorems 6 and 5). More precisely, our strategy is to use the topological unitary group $\mathbb{U}(n) = U(n) \times \{1\}$. Indeed, this group acts continuously on $HW_0^{1,2}(\Omega_\psi)$ by

$$(\widehat{\tau} \sharp u)(q) = u(\tau^{-1}z, t) \quad \text{for all } q = (z, t) \in \mathbb{H}^n,$$

and the T-invariant closed subspace $HW_{0,T}^{1,2}(\Omega_\psi)$ associated to the subgroup

$$T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\}, \quad n = \sum_{i=1}^{\ell} n_i, \quad \text{with} \quad n_i \geq 1 \text{ and } \ell \geq 1,$$

is compactly embedded in the Lebesgue space $L^q(\Omega_\psi)$, for every $q \in (2, 2^*)$, as proved in [1]; see Lemmas 2 and 3. A similar argument works for strip-like domains

$\Omega = \omega \times \mathbb{R}^{n-m}$, where $\omega \subset \mathbb{R}^m$ is bounded and $n - m \geq 2$, yielding the space of cylindrically symmetric functions on $H_0^1(\Omega)$ via the group $T = id_{\mathbb{R}^m} \times O(n - m)$ (see [21] and [31]).

Successively, thanks to the left invariance of the standard Haar measure μ of the Heisenberg group $\mathbb{H}^n$, with respect to the natural action of the group $* : \mathbb{U}(n) \times HW_0^{1,2}(\Omega_\psi) \to HW_0^{1,2}(\Omega_\psi)$, given by

$$\widehat{\tau} * q = (\tau z, t) \quad \text{for all } \widehat{\tau} = (\tau, 1) \in \mathbb{U}(n), \, q = (z, t) \in \mathbb{H}^n,$$

(see Chapter III § 2 No 4 of Bourbaki [13] and Chapter 7 § 1 No 1 of Bourbaki [14] for related topics) the principle of symmetric criticality of Palais, see Theorem 5 below, can be applied to the associated energy Euler–Lagrange functional

$$I_{\lambda,\mu}(u, v) = \frac{1}{2} \left(\|u\|^2 + \|v\|^2 \right) - \lambda \int_{\Omega_\psi} K(q) F(u(q), v(q)) \mathrm{d}\mu(q)$$

$$- \mu \int_{\Omega_\psi} G(q, u(q), v(q)) \mathrm{d}\mu(q),$$

for every $(u, v) \in HW_{0,T}^{1,2}(\Omega_\psi) \times HW_{0,T}^{1,2}(\Omega_\psi)$, allowing a variational approach to the problem (1.2).

The methods used here may be suitable for other purposes, too. Indeed, we recall that a similar variational approach has been used in a different context, in order to prove multiplicity results for elliptic problems defined in Euclidean strip-like domains and involving the p–Laplacian operator (see [32, Theorem 2.2]). More precisely, in [32] the author studied gradient-type systems of the form

$$\begin{cases} -\Delta_p u = \lambda F_u(x, u, v) & \text{in } \Omega \\ -\Delta_q v = \lambda F_v(x, u, v) & \text{in } \Omega \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where the nonlinearities F_u and F_v denote the partial derivatives of F with respect to the second variable and the third variable, respectively, and $\Omega = \omega \times \mathbb{R}^l$, where ω is a bounded open subset of the Euclidean space $\mathbb{R}^m$ with smooth boundary, $m \geq 1$, $l \geq 2$ and $1 < p, q < m + l$. Recently, nonlocal gradient–type systems have been investigated by exploiting similar variational arguments (see [15, Theorem 3.1]). In both papers [15,32], crucial roles are played by some invariant subgroups of the orthogonal group $O(n)$ and Lions' embedding results (see [34, Théorèmes III.2 and III.3]).

Theorem 9 extends the existence results obtained in [15,32] to the Heisenberg group setting. In addition, in our case, the presence of the Sobolev–Hardy term makes the search of weak solutions much more delicate. Indeed, in order to handle the singular term in (1.2), it is crucial to introduce the subelliptic Hardy–Sobolev inequality

$$\int_{\mathbb{H}^n} \psi^2 \frac{|\varphi|^2}{r^2} \mathrm{d}\mu(q) \leq \left(\frac{2}{Q - 2} \right)^2 \int_{\mathbb{H}^n} |D_{\mathbb{H}^n} \varphi|_{\mathbb{H}^n}^2 \mathrm{d}\mu(q), \tag{1.4}$$

for every $\varphi \in C_0^\infty(\mathbb{H}^n \setminus \{O\})$, where the main geometrical function ψ is defined by

$$\psi(q) = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n} = \frac{|z|}{r(q)} \quad \text{for all } q = (z, t) \in \mathbb{H}^n, \text{ with } q \neq O, \qquad (1.5)$$

and $0 \le \psi \le 1$, $\psi(0, t) \equiv 0$, $\psi(z, 0) \equiv 1$.

Here, r denotes the Heisenberg norm $r(q) = r(z, t) = (|z|^4 + t^2)^{1/4}$, $z = (x, y) \in \mathbb{C}^n$, $t \in \mathbb{R}$. Furthermore, direct calculations show that

$$\Delta_{\mathbb{H}^n} r = \frac{2n + 1}{r} \psi^2 \quad \text{in } \mathbb{H}^n \setminus \{O\}.$$

For details, we refer to [37, Sect. 2.1].

An important incentive to the study of subelliptic systems on the entire Heisenberg group $\mathbb{H}^n$ was recently provided by Pucci et al. ([11,12,42]), see also the papers [22,25,38,50–52].

In particular, the existence of nontrivial solutions for a subelliptic Schrödinger–Hardy system in the Heisenberg group $\mathbb{H}^n$ was investigated in [42], where the author considered the following problem:

$$\begin{cases} -\Delta_{\mathbb{H}^n}^p u + V(q)|u|^{p-2}u - \gamma \psi^p \dfrac{|u|^{p-2}u}{r(q)^p} = \lambda H_u(q, u, v) + \dfrac{\alpha}{p^*}|v|^\beta |u|^{\alpha-2}u \\[4mm] -\Delta_{\mathbb{H}^n}^p v + V(q)|v|^{p-2}v - \gamma \psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = \lambda H_v(q, u, v) + \dfrac{\beta}{p^*}|u|^\alpha |v|^{\beta-2}v, \end{cases} \qquad (1.6)$$

where γ and λ are real parameters, $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, $1 < p < Q$, the exponent $\alpha > 1$ and $\beta > 1$ are such that $\alpha + \beta = p^*$, $p^* = pQ/(Q - p)$, and $\Delta_{\mathbb{H}^n}^p$ is the p–Laplacian operator on $\mathbb{H}^n$, which is defined by

$$\Delta_{\mathbb{H}^n}^p \varphi = \text{div}_H(|D_{\mathbb{H}^n}\varphi|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n}\varphi)$$

along any $\varphi \in C_0^\infty(\mathbb{H}^n)$, that is $\Delta_{\mathbb{H}^n}^p$ is the familiar horizontal p-Laplacian operator. The potential function V satisfy the following condition:

$(\mathcal{V})$ $V \in C(\mathbb{H}^n)$ and $\inf_{q \in \mathbb{H}^n} V(q) = V_0 > 0$.

The nonlinearities H_u and H_v denote the partial derivatives of H with respect to the second variable and the third variable, respectively, and H satisfies

$(\mathcal{H})$ $H \in C^1(\mathbb{H}^n \times \mathbb{R}^2, \mathbb{R}^+)$, $H_z(q, 0, 0) = 0$ for all $q \in \mathbb{H}^n$ and there exist μ and s such that $p < \mu \le s < p^*$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality

$$|H_z(q, w)| \le \mu \varepsilon |w|^{\mu-1} + q C_\varepsilon |w|^{s-1}, \quad w = (u, v), \ |w| = \sqrt{u^2 + v^2},$$

holds for any $(q, w) \in \mathbb{H}^n \times \mathbb{R}^2$, where $H_w = (H_u, H_v)$, and also

$$0 \le \mu H(q, w) \le H_w(q, w) \cdot w \quad \text{for all } (q, w) \in \mathbb{H}^n \times \mathbb{R}^2$$

is valid.

In this very interesting paper [42], an existence result is obtained by an application of the mountain pass theorem and the Ekeland variational principle. We emphasize that the assumptions adopted here are milder and in any case much different from the usual hypotheses in problem (1.6). Moreover, to the best of our knowledge, Theorem 9 is the first multiplicity result for subelliptic gradient-type systems on unbounded domains of $\mathbb{H}^n$. In contrast with [42], the Hilbertian setting, i.e. $p = 2$, is peculiar for our approach in order to recover the compactness properties stated in Lemma 3 (see also Remark 4). In a forthcoming work, we plan to come back to problem (1.6) and prove some multiplicity results by exploiting suitable group-theoretical arguments and variational methods.

Now, let us recall that the Folland–Stein horizontal Sobolev space $HW_0^{1,2}(\Omega_\psi)$ is the completion of $C_0^\infty(\Omega_\psi)$ with respect to the Hilbertian norm

$$\|u\|_{HW_0^{1,2}(\Omega_\psi)} = \left(\int_{\Omega_\psi} |D_{\mathbb{H}^n} u(q)|_{\mathbb{H}^n}^2 \mathrm{d}\mu(q) + \int_{\Omega_\psi} |u(q)|^2 \mathrm{d}\mu(q) \right)^{1/2},$$

$$\langle u, \varphi \rangle = \int_{\Omega_\psi} \left(D_{\mathbb{H}^n} u(q), D_{\mathbb{H}^n} \varphi(q) \right)_{\mathbb{H}^n} \mathrm{d}\mu(q) + \int_{\Omega_\psi} u(q)\varphi(q) \mathrm{d}\mu(q).$$

As far as we know, the abstract framework and Theorem 9 in the subelliptic setting are new also in the non-singular case. A special and meaningful case of our main result reads as follows:

Theorem 1 *Let $\Omega_\psi \subset \mathbb{H}^n$ as in (1.1), and let $K : \Omega_\psi \to \mathbb{R}$ be a potential satisfying (h_K). Furthermore, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying (f_1)–(f_4).*

Then there exist a number $\sigma > 0$ and a nonempty open set $\Lambda \subset (0, \infty)$ such that, for every $\lambda \in \Lambda$, the following system

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + u = \lambda K(q)\partial_1 F(u, v) & \text{in } \Omega_\psi \\ -\Delta_{\mathbb{H}^n} v + v = \lambda K(q)\partial_2 F(u, v) & \text{in } \Omega_\psi \\ u = v = 0 \text{ on } \partial\Omega_\psi, \end{cases} \quad (1.7)$$

has at least two solutions $(u_{\lambda,\mu}^{(j)}, v_{\lambda,\mu}^{(j)}) \in HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi) \times HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi)$, with $j \in \{1, 2\}$, lying in the ball

$$\{(u, v) \in HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi) \times HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi) : \|(u, v)\| \leq \sigma\},$$

where

$$\|(u, v)\| = \|u\|_{HW_0^{1,2}(\Omega_\psi)} + \|v\|_{HW_0^{1,2}(\Omega_\psi)},$$

and

$$HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi) = \{u \in HW_0^{1,2}(\Omega_\psi) : u(z, t) = u(|z|, t) \text{ for all } q = (z, t) \in \Omega_\psi\},$$

is the linear subspace of cylindrically symmetric functions of $HW_0^{1,2}(\Omega_\psi)$.

The manuscript is organized as follows. In Section 2, we give some notations, and we recall some properties of the functional space we work in. In order to apply critical point methods to problem (1.2), we need to work in a subspace of the functional space $X = HW_0^{1,2}(\Omega_\psi) \times HW_0^{1,2}(\Omega_\psi)$. In particular, we provide some tools which will be useful along the paper (see Lemmas 7 and 8). Finally, in Sect. 3, we study system (1.2) and prove our multiplicity result (see Theorem 9). An example of an application is given in Example 10.

For general references on the subject and on methods used in the paper, we refer to the monographs [10,33], as well as papers [6,18,40,49] and the references therein. See also [6–9,28] for related topics.

2 Abstract Framework

In this section, we briefly recall some basic facts on the Heisenberg group and the functional Folland–Stein space $HW_0^{1,2}(\Omega_\psi)$; see [23,24]. The simplest example of Carnot group of step two is provided by the Heisenberg group $\mathbb{H}^n$ of topological dimension $m = 2n + 1$ and homogeneous dimension $Q = 2n + 2$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non–Abelian group law

$$q \circ q' = \left(z + z', t + t' + 2 \sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right)$$

for all $q, q' \in \mathbb{H}^n$, with

$$q = (z, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t), \quad q' = (z', t') = (x_1', \ldots, x_n', y_1', \ldots, y_n', t').$$

The vector fields for $j = 1, \ldots, n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad Z = \frac{\partial}{\partial t}, \tag{2.1}$$

constitute a basis $\mathcal{B}^*$ for the real graded Lie algebra $\mathfrak{H} = \bigoplus_{k=1}^{2} \mathfrak{H}_k$ of left invariant vector fields on $\mathbb{H}^n$. More precisely, the first graded component $\mathfrak{H}_1$ is generated by $\mathcal{B}_1^* = \{X_j, Y_j : j = 1, \ldots, 2n\}$ and the second graded component $\mathfrak{H}_2$ is generated by $\mathcal{B}_2^* = \{Z\}$. The basis $\mathcal{B}^*$ satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators are zero.

The natural inner product in the span of $\{X_j, Y_j\}_{j=1}^{n}$

$$(W, Z)_{\mathbb{H}^n} = \sum_{j=1}^{n} \left(w^j z^j + \widetilde{w}^j \widetilde{z}^j \right)$$

for $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ and $Z = \{z^j X_j + \widetilde{z}^j Y_j\}_{j=1}^n$ produces the Hilbertian norm

$$|D_{\mathbb{H}^n} u|_{\mathbb{H}^n} = \sqrt{\left(D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} u\right)_{\mathbb{H}^n}}$$

for the horizontal vector field $D_{\mathbb{H}^n} u$. Moreover, if also $v \in C^1(\mathbb{H}^n)$ then the Cauchy–Schwarz inequality

$$\left|\left(D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} v\right)_{\mathbb{H}^n}\right|_{\mathbb{H}^n} \leq |D_{\mathbb{H}^n} u|_{\mathbb{H}^n} |D_{\mathbb{H}^n} v|_{\mathbb{H}^n}$$

continues to be valid.

For any horizontal vector field $W = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ of class $C^1(\mathbb{H}^n; \mathbb{R}^{2n})$, *the horizontal divergence* is defined by

$$\mathrm{div}_H W = \sum_{j=1}^n \left[X_j(w^j) + Y_j(\widetilde{w}^j) \right].$$

If furthermore $g \in C^1(\mathbb{R})$, then the *Leibnitz formula* holds, namely

$$\mathrm{div}_H(gW) = g\,\mathrm{div}_H(W) + \left(D_{\mathbb{H}^n} g, W\right)_{\mathbb{H}^n}.$$

If $u \in C^2(\mathbb{H}^n)$, then the *horizontal Laplacian in* $\mathbb{H}^n$ of u, called the *Kohn–Spencer Laplacian*, is defined as follows:

$$\begin{aligned}
\Delta_{\mathbb{H}^n} u &= \sum_{j=1}^n \left(X_j^2 + Y_j^2 \right) u \\
&= \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4 y_j \frac{\partial^2}{\partial x_j \partial t} - 4 x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2},
\end{aligned}$$

and $\Delta_{\mathbb{H}^n}$ is *hypoelliptic* according to the celebrated Theorem 1.1 due to *Hörmander* [29].

Going back to (1.2), we need to introduce a suitable solution space. Let Ω be a nontrivial open subset of $\mathbb{H}^n$. The Folland–Stein horizontal Sobolev space $HW_0^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$, with respect to the Hilbertian norm

$$\begin{aligned}
\|u\|_{HW_0^{1,2}(\Omega)} &= \left(\int_\Omega |D_{\mathbb{H}^n} u(q)|_{\mathbb{H}^n}^2 d\mu(q) + \int_\Omega |u(q)|^2 d\mu(q) \right)^{1/2}, \\
\langle u, \varphi \rangle &= \int_\Omega \langle D_{\mathbb{H}^n} u(q), D_{\mathbb{H}^n} \varphi(q) \rangle_{\mathbb{H}^n} d\mu(q) + \int_\Omega u(q)\varphi(q) d\mu(q).
\end{aligned} \tag{2.2}$$

Of course, if $\Omega = \mathbb{H}^n$, then $HW^{1,2}(\mathbb{H}^n) = HW_0^{1,2}(\mathbb{H}^n)$, where $HW^{1,2}(\mathbb{H}^n)$ denotes the horizontal Sobolev space of the functions $u \in L^2(\mathbb{H}^n)$ such that $D_{\mathbb{H}^n} u$ exists in

the sense of distributions and $|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}$ is in $L^2(\mathbb{H}^n)$, endowed with the Hilbertian norm (2.2).

Since $\nu \in [0, C_V^{-1} n^2)$, condition (h_V) in addition to the Hardy–Sobolev inequality (1.4) and relation (1.5), gives that the norm $\| \cdot \|_{HW_0^{1,2}(\Omega_\psi)}$ which is equivalent to the norm given by

$$\|u\| = \left(\|u\|_{HW_0^{1,2}(\Omega_\psi)}^2 - \nu \int_{\Omega_\psi} V(q)|u(q)|^2 \mathrm{d}\mu(q) \right)^{1/2}, \tag{2.3}$$

for every $u \in HW_0^{1,2}(\Omega_\psi)$.

More precisely, one has

$$\frac{\sqrt{n^2 - \nu C_V}}{n} \|u\|_{HW_0^{1,2}(\Omega_\psi)} \leq \|u\| \leq \|u\|_{HW_0^{1,2}(\Omega_\psi)},$$

for every $u \in HW_0^{1,2}(\Omega_\psi)$.

Take $q_1, q_2 \in \mathbb{H}^n$ and let $H\Gamma_{q_1, q_2}(\mathbb{H}^n)$ be the set of piecewise smooth curves γ, such that $\gamma : [0, 1] \to \mathbb{H}^n$, $\dot{\gamma}(t) \in \mathfrak{H}_1$ a.e. $t \in [0, 1]$, $(\gamma(0), \gamma(1)) = (q_1, q_2)$ and

$$\int_0^1 |\dot{\gamma}(t)|_{\mathbb{H}^n} dt < \infty.$$

Since $H\Gamma_{q_1, q_2}(\mathbb{H}^n) \neq \emptyset$ by the celebrated Chow–Rashevskiĭ theorem [17], it is possible to define the *Carnot–Carathéodory distance* on $\mathbb{H}^n$, as follows:

$$d_{CC}(q_1, q_2) = \inf_{\gamma \in H\Gamma_{q_1, q_2}(\mathbb{H}^n)} \int_0^1 |\dot{\gamma}(t)|_{\mathbb{H}^n} \mathrm{d}t,$$

see [43] for details.

In order to use a variational approach for studying problem (1.2), we need to work in a special functional space. Indeed, one of the difficulties in treating our problem is related to the lack of a compact embedding of $HW_0^{1,2}(\Omega_\psi)$ into suitable Lebesgue spaces. In this respect the standard subelliptic Sobolev spaces are not enough in order to study the problem. We overcome this difficulty by working in a new functional space, whose definition will be given below.

Let $(T, \cdot)$ be a closed topological group with neutral element $\jmath$. The group T is said to *act continuously* on $\mathbb{H}^n$, if there exists a map $\star : T \times \mathbb{H}^n \to \mathbb{H}^n$ such that the following conditions hold:

(T_1) $\jmath \star q = q$ *for every* $q \in \mathbb{H}^n$;
(T_2) $\tau_1 \star (\tau_2 \star q) = (\tau_1 \cdot \tau_2) \star q$ *for every* $\tau_1, \tau_2 \in T$ *and* $q \in \mathbb{H}^n$.

In addition, the action $\star$ is called *left distributed* if

(T_3) $\tau \star (p \circ q) = (\tau \star p) \circ (\tau \star q)$ *for every* $\tau \in T$ *and* $p, q \in \mathbb{H}^n$.

A set $\Omega \subset \mathbb{H}^n$ is said to be *T-invariant*, with respect to the action $\star$, if $T \star \Omega = \Omega$.

Let us consider $T = (T, \cdot)$ be a closed infinite topological group acting continuously and left-distributively on $\mathbb{H}^n$ by the map $\star : T \times \mathbb{H}^n \to \mathbb{H}^n$. Assume that T acts isometrically on the horizontal Folland–Stein space $HW_0^{1,2}(\mathbb{H}^n)$, where the action $\sharp : T \times HW_0^{1,2}(\mathbb{H}^n) \to HW_0^{1,2}(\mathbb{H}^n)$ is defined for every $(\widehat{\tau}, u) \in T \times HW_0^{1,2}(\mathbb{H}^n)$ by

$$(\widehat{\tau} \sharp u)(q) = u(\widehat{\tau}^{-1} \star q) \quad \text{for all } q \in \mathbb{H}^n.$$

In what follows, μ is the natural Haar measure on $\mathbb{H}^n$ while "$\liminf$" is the Kuratowski lower limit of sets.

Let Ω be a nonempty open T-invariant subset of $\mathbb{H}^n$, with boundary $\partial\Omega$, and assume that

($\mathcal{H}$) *for every* $(q_k)_k \subset \mathbb{H}^n$ *such that*

$$\lim_{k \to \infty} d_{CC}(e, q_k) = \infty \quad \text{and} \quad \mu\left(\liminf_{k \to \infty}(q_k \circ \Omega)\right) > 0,$$

where $q_k \circ \Omega = \{q_k \circ q : q \in \Omega\}$, *there exist a subsequence* $(q_{k_j})_j$ *of* $(q_k)_k$ *and a sequence of subgroups* $(T_{q_{k_j}})_j$ *of* T, *with cardinality* $\mathrm{card}(T_{q_{k_j}}) = \infty$, *having the property that for all* $\widehat{\tau}_1, \widehat{\tau}_2 \in T_{q_{k_j}}$, *with* $\widehat{\tau}_1 \neq \widehat{\tau}_2$, *the following holds:*

$$\lim_{j \to \infty} \inf_{q \in \mathbb{H}^n} d_{CC}((\widehat{\tau}_1 \star q_{k_j}) \circ q, (\widehat{\tau}_2 \star q_{k_j}) \circ q) = \infty.$$

A domain Ω of $\mathbb{H}^n$, for which condition ($\mathcal{H}$) holds, is simply called an $\mathcal{H}$ *domain*. Let us denote $\mathbb{U}(n) = U(n) \times \{1\}$, where

$$U(n) = U(n, \mathbb{C}) = \{\tau \in GL(n; \mathbb{C}) : \langle \tau z, \tau z' \rangle_{\mathbb{C}^n} = \langle z, z' \rangle_{\mathbb{C}^n} \text{ for all } z, z' \in \mathbb{C}^n\},$$

that is, $U(n)$ is the usual unitary group. Here $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denotes the standard Hermitian product on $\mathbb{C}^n$, in other words $\langle z, z' \rangle_{\mathbb{C}^n} = \sum_{k=1}^n z_k \cdot \overline{z'_k}$.

Hence, $\mathbb{U}(n)$ is the unitary group endowed with the natural multiplication law $\cdot : \mathbb{U}(n) \times \mathbb{U}(n) \to \mathbb{U}(n)$, which acts continuously and left–distributively on $\mathbb{H}^n$ by the map $* : \mathbb{U}(n) \times \mathbb{H}^n \to \mathbb{H}^n$, defined by

$$\widehat{\tau} * q = (\tau z, t) \quad \text{for all } \widehat{\tau} = (\tau, 1) \in \mathbb{U}(n), \ q = (z, t) \in \mathbb{H}^n,$$

thanks to [1, Lemma 3.1]. If we take $T = \mathbb{U}(n)$, then Ω_ψ is $\mathbb{U}(n)$-invariant and an $\mathcal{H}$ domain, as shown in the proof of of [1, theorem 1.1]. Moreover,

$$HW_{0,\mathbb{U}(n)}^{1,2}(\Omega_\psi) = \{u \in HW_0^{1,2}(\Omega_\psi) : u(z, t) = u(|z|, t) \text{ for all } q = (z, t) \in \Omega_\psi\},$$

that is $HW_{0,\mathbb{U}(n)}^{1,2}(\Omega_\psi) = HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi)$ is the space of cylindrically symmetric functions of $HW_0^{1,2}(\Omega_\psi)$.

Finally, $\mathbb{U}(n)$ acts isometrically on the horizontal Folland–Stein space $HW_0^{1,2}(\mathbb{H}^n)$, where the action $\sharp : \mathbb{U}(n) \times HW_0^{1,2}(\mathbb{H}^n) \to HW_0^{1,2}(\mathbb{H}^n)$ is defined for every $(\widehat{\tau}, u)$ in $\mathbb{U}(n) \times HW_0^{1,2}(\mathbb{H}^n)$ by

$$(\widehat{\tau} \sharp u)(q) = u(\widehat{\tau}^{-1} * q) = u(\tau^{-1}z, t) \quad \text{for all } q = (z, t) \in \mathbb{H}^n, \tag{2.4}$$

by force of [1, Lemma 3.2].

Now, let $T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\}$, where $n = \sum_{i=1}^{\ell} n_i$, with $n_i \geq 1$ and $\ell \geq 1$ and consider the closed subspace

$$HW_{0,T}^{1,2}(\Omega_\psi) = \{u \in HW_0^{1,2}(\Omega_\psi) \, : \, \widehat{\tau} \sharp u = u \text{ for all } \widehat{\tau} \in T\}.$$

By keeping the same notation, we naturally extend the function $u \in HW_{0,T}^{1,2}(\Omega_\psi)$ to the entire group $\mathbb{H}^n$ by zero on $\mathbb{H}^n \setminus \Omega_\psi$.

2.1 Sobolev Embedding Results

Following Folland and Stein [24], we can easily deduce the following embedding property that will be crucial in this paper.

Lemma 2 *Let $q \in [2, 2^*]$ and $T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\}$, where $n = \sum_{i=1}^{\ell} n_i$, with $n_i \geq 1$ and $\ell \geq 1$. Then the embeddings*

$$HW_{0,T}^{1,2}(\Omega_\psi) \hookrightarrow HW_0^{1,2}(\Omega_\psi) \hookrightarrow L^q(\Omega_\psi)$$

are continuous. Hence, there exists a constant k_q such that

$$\|u\|_q = \|u\|_{L^q(\Omega_\psi)} \leq k_q \|u\|_{HW_0^{1,2}(\Omega_\psi)} \quad \text{for all } u \in HW_0^{1,2}(\Omega_\psi),$$

where k_q depends on q and n.

Moreover, since the norms $\| \cdot \|_{HW_0^{1,2}(\Omega_\psi)}$ and $\| \cdot \|$ are equivalent, there exists a constant

$$c_q = k_q \frac{n}{\sqrt{n^2 - \nu C_V}},$$

such that

$$\|u\|_q \leq c_q \|u\|, \tag{2.5}$$

for every $u \in HW_0^{1,2}(\Omega_\psi)$.

On the other hand, by [27,30,48], we know that if $\mathcal{O}$ is a bounded open set of $\mathbb{H}^n$ then the embedding

$$HW_0^{1,2}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O}) \tag{2.6}$$

is compact for all q, with $1 \leq q < 2^*$.

Moreover, by [1, Theorems 1.1 and 3.1], the main compactness statement reads as follows:

Lemma 3 *Let $T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\}$, where $n = \sum_{i=1}^{\ell} n_i$, with $n_i \geq 1$ and $\ell \geq 1$. Then the embedding*

$$HW_{0,T}^{1,2}(\Omega_\psi) \hookrightarrow L^q(\Omega_\psi)$$

is compact for any $q \in (2, 2^*)$. We also have that

$$HW_{0,T}^{1,2}(\Omega_\psi) = Fix_T(HW_0^{1,2}(\Omega_\psi)),$$

where

$$Fix_T(HW_0^{1,2}(\Omega_\psi)) = \{u \in HW_0^{1,2}(\Omega_\psi) : \widehat{\tau} \natural u = u \text{ for all } \widehat{\tau} \in T\},$$

and $\natural : T \times HW_0^{1,2}(\mathbb{H}^n) \to HW_0^{1,2}(\mathbb{H}^n)$ is the action defined in (2.4)*.*

Remark 4 The above lemma is a consequence of a more general result stated in [1, Theorem 3.1]. More precisely, let $\mathbb{G} = (\mathbb{G}, \circ)$ be a Carnot group of step r and homogeneous dimension $Q > 2$, with the neutral element denoted by e. Let $T = (T, \cdot)$ be a closed infinite topological group acting continuously and left distributively on $\mathbb{G}$ by the map $\circledast : T \times \mathbb{G} \to \mathbb{G}$. Assume furthermore that T acts isometrically on $HW_0^{1,2}(\mathbb{G})$, where the action $\natural : T \times HW_0^{1,2}(\mathbb{G}) \to HW_0^{1,2}(\mathbb{G})$ is defined by

$$(\widehat{\tau} \natural u)(q) = u(\widehat{\tau}^{-1} \circledast q) \quad \text{for all } q \in \mathbb{G}.$$

Let $\mathbb{G}_0$ be an $\mathcal{H}$ domain (see [7]). Then, the following embedding

$$HW_{0,T}^{1,2}(\mathbb{G}_0) \hookrightarrow L^q(\mathbb{G}_0)$$

is compact for every $q \in (2, 2^*)$. This result was inspired by Tintarev and Fieseler [47].

2.2 Weak Formulation and *T*-Invariance

The natural solution space for (1.2) is

$$X = HW_0^{1,2}(\Omega_\psi) \times HW_0^{1,2}(\Omega_\psi),$$

with associated norm

$$\|(u, v)\| = \|u\| + \|v\|.$$

Let us consider the action $\pi_\sharp : T \times X \to X$ given by

$$\pi_\sharp(\widehat{\tau}, (u, v)) = (\widehat{\tau}\sharp u, \widehat{\tau}\sharp v),$$

for every $\widehat{\tau} \in T$ and $(u, v) \in X$.

The above definition immediately yields

$$Fix_T(X) = Fix_T(HW_0^{1,2}(\Omega_\psi)) \times Fix_T(HW_0^{1,2}(\Omega_\psi)),$$

where

$$Fix_T(X) = \{(u, v) \in X : \pi_\sharp(\widehat{\tau}, (u, v)) = (u, v) \text{ for all } \widehat{\tau} \in T\}.$$

A function $(u, v) \in X$ is said to be a (*weak*) *solution* of problem (1.2) if

$$
\begin{aligned}
\langle u, \varphi \rangle_{X,\nu} + \langle v, \psi \rangle_{X,\nu} = {} & \lambda \int_{\Omega_\psi} K(q)\partial_1 F(u(q), v(q))\varphi(q)\mathrm{d}\mu(q) \\
& + \lambda \int_{\Omega_\psi} K(q)\partial_2 F(u(q), v(q))\varphi(q)\mathrm{d}\mu(q) \\
& + \mu \int_{\Omega_\psi} \partial_1 G(q, u(q), v(q))\psi(q)\mathrm{d}\mu(q) \\
& + \mu \int_{\Omega_\psi} \partial_2 G(q, u(q), v(q))\psi(q)\mathrm{d}\mu(q), \quad (2.7)
\end{aligned}
$$

for any $(\varphi, \psi) \in X$, where we set

$$\langle u, \varphi \rangle_{X,\nu} = \langle u, \varphi \rangle - \nu \int_{\Omega_\psi} V(q)u(q)\varphi(q)\mathrm{d}\mu(q),$$

and

$$\langle v, \psi \rangle_{X,\nu} = \langle v, \psi \rangle - \nu \int_{\Omega_\psi} V(q)v(q)\psi(q)\mathrm{d}\mu(q).$$

Problem (1.2) has a variational nature and the Euler–Lagrange functional $I_{\lambda,\mu} : X \to \mathbb{R}$ associated to (1.2) is given by

$$
\begin{aligned}
I_{\lambda,\mu}(u, v) = {} & \frac{1}{2}\left(\|u\|^2 + \|v\|^2\right) - \lambda \int_{\Omega_\psi} K(q)F(u(q), v(q))\mathrm{d}\mu(q) \\
& - \mu \int_{\Omega_\psi} G(q, u(q), v(q))\mathrm{d}\mu(q),
\end{aligned}
$$

for every $(u, v) \in X$.

Clearly, the functional $I_{\lambda,\mu}$ is well-defined in X and, thanks to (f_1)–(f_3) as well as (g_1)–(g_3), it is of class $C^1(X)$. Moreover, for every $(u, v) \in X$

$$
\begin{aligned}
\langle I'_{\lambda,\mu}(u, v), (\varphi, \psi) \rangle = {} & \langle u, \varphi \rangle - \nu \int_{\Omega_\psi} V(q) u(q) \varphi(q) \mathrm{d}\mu(q) \\
& + \langle v, \psi \rangle - \nu \int_{\Omega_\psi} V(q) v(q) \psi(q) \mathrm{d}\mu(q) \\
& - \lambda \int_{\Omega_\psi} K(q) \partial_1 F(u(q), v(q)) \varphi(q) \mathrm{d}\mu(q) \\
& - \lambda \int_{\Omega_\psi} K(q) \partial_2 F(u(q), v(q)) \varphi(q) \mathrm{d}\mu(q) \\
& - \mu \int_{\Omega_\psi} \partial_1 G(q, u(q), v(q)) \psi(q) \mathrm{d}\mu(q) \\
& - \mu \int_{\Omega_\psi} \partial_2 G(q, u(q), v(q)) \psi(q) \mathrm{d}\mu(q), \quad (2.8)
\end{aligned}
$$

for all $(\varphi, \psi) \in X$. Hence, the critical points of $I_{\lambda,\mu}$ in X are exactly the (weak) solutions of (1.2).

Let

$$
Y_T = HW^{1,2}_{0,T}(\Omega_\psi) \times HW^{1,2}_{0,T}(\Omega_\psi) \subset X
$$

be endowed with the induced norm $\|\cdot\|$, where $T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\} \subseteq \mathbb{U}(n)$, with $n = \sum_{i=1}^{\ell} n_i$, $n_i \geq 1$ and $\ell \geq 1$.

A pair $(u, v) \in Y_T$ is said to be a (*weak*) *solution of problem* (1.2) *only in the* Y_T *sense* if equality (2.7) holds for every $(\varphi, \psi) \in Y_T$.

Then $(u, v) \in Y_T$ is a solution of (1.2) in the entire space $HW^{1,2}_0(\Omega_\psi)$ if the *principle of symmetric criticality* of Palais given in [41] holds.

To prove this, let us recall the well-known principle of symmetric criticality of Palais stated in the general form, proved in [19] for reflexive strictly convex Banach spaces. For details and comments, we refer to [16, Section 5].

More precisely, let $E = (E, \|\cdot\|_E)$ be a reflexive strictly convex Banach space. Suppose that $\mathcal{G}$ is a subgroup of isometries $g : E \to E$, that is g is linear and $\|gu\|_E = \|u\|_E$ for all $u \in E$.

Consider the $\mathcal{G}$–invariant closed subspace of E

$$
\Sigma_{\mathcal{G}} = \{u \in E : gu = u \text{ for all } g \in \mathcal{G}\}.
$$

By [19, Proposition 3.1], we have the following result:

Theorem 5 *Let E, $\mathcal{G}$ and Σ be as before and let I be a C^1 functional defined on E such that*

$$
I(gu) = I(u), \quad \forall u \in E
$$

for every $g \in \mathcal{G}$.

Then $u \in \Sigma_\mathcal{G}$ is a critical point of I if and only if u is a critical point of $\mathcal{J} = I|_{\Sigma_\mathcal{G}}$.

We recall that Ω_ψ is a nonempty open subset of $\mathbb{H}^n$, which is T-invariant. Furthermore, we recall that from the invariance point of view, the unitary groups play the same role in the Heisenberg setting as the orthogonal groups in the Euclidean framework.

Thus, we we apply the principle of symmetric criticality to the Sobolev space Y_T under the action $\pi_\sharp : T \times X \to X$ defined in (2.4). Let us again denote

$$\Phi(u, v) = \frac{1}{2} \left(\|u\|^2 + \|v\|^2 \right), \quad \forall (u, v) \in X.$$

Clearly,

$$\begin{aligned}
\Phi(\pi_\sharp(\widehat{\tau}, (u, v))) &= \Phi(\widehat{\tau}\sharp u, \widehat{\tau}\sharp v) \\
&= \frac{1}{2} \left(\|\widehat{\tau}\sharp u\|^2 + \|\widehat{\tau}\sharp v\|^2 \right) \\
&= \frac{1}{2} \left(\|u\|^2 + \|v\|^2 \right) \\
&= \Phi(u, v), \quad \text{for every } (u, v) \in X, \ \widehat{\tau} \in T,
\end{aligned} \tag{2.9}$$

since T acts isometrically on $HW_0^{1,2}(\Omega_\psi)$ as proved in [1, Lemma 3.2]. Thus, the functional Φ is T-invariant.

Moreover, the functional $\Upsilon_{\lambda,\mu} : X \to \mathbb{R}$ given by

$$\begin{aligned}
\Upsilon_{\lambda,\mu}(u, v) &= \lambda \int_{\Omega_\psi} K(q) F(u(q), v(q)) \mathrm{d}\mu(q) \\
&\quad + \mu \int_{\Omega_\psi} G(q, u(q), v(q)) \mathrm{d}\mu(q), \quad \text{for every } (u, v) \in X, \ \widehat{\tau} \in T,
\end{aligned}$$

is T-invariant by assumptions (h_K) and (g_4).

Indeed, let us fix $\widehat{\tau} \in T$ and $(u, v) \in X$. Then putting $\tau^{-1} * q = p$, we get by (T_1)–(T_3)

$$\begin{aligned}
\Upsilon_{\lambda,\mu}(\pi_\sharp(\widehat{\tau}, (u, v))) &= \Upsilon_{\lambda,\mu}(\widehat{\tau}\sharp u, \widehat{\tau}\sharp v) \\
&= \lambda \int_{\Omega_\psi} K(q) F(\widehat{\tau}\sharp u(q), \widehat{\tau}\sharp v(q)) \mathrm{d}\mu(q) \\
&\quad + \mu \int_{\Omega_\psi} G(q, \widehat{\tau}\sharp u(q), \widehat{\tau}\sharp v(q)) \mathrm{d}\mu(q) \\
&= \lambda \int_{\Omega_\psi} K(q) F(u(\widehat{\tau}^{-1} * q), v(\widehat{\tau}^{-1} * q)) \mathrm{d}\mu(q) \\
&\quad + \mu \int_{\Omega_\psi} G(q, u(\widehat{\tau}^{-1} * q), v(\widehat{\tau}^{-1} * q)) \mathrm{d}\mu(q)
\end{aligned}$$

$$= \lambda \int_{\widehat{\tau}*\Omega_\psi} K(\widehat{\tau}*p)F(u(p), v(p))\mathrm{d}\mu(\widehat{\tau}*p)$$

$$+ \mu \int_{\widehat{\tau}*\Omega_\psi} G(\widehat{\tau}*p, u(p), v(p))\mathrm{d}\mu(\widehat{\tau}*p)$$

$$= \lambda \int_{\Omega_\psi} K(p)F(u(p), v(p))\mathrm{d}\mu(p)$$

$$+ \mu \int_{\Omega_\psi} G(p, u(p), v(p))\mathrm{d}\mu(p)$$

$$= \Upsilon_{\lambda,\mu}(u, v),$$

since $T*\Omega_\psi = \Omega_\psi$ and K, G are T-invariant by assumption. Moreover, the left $*$ invariance of the measure μ (keeping in mind that the Jacobian of the change of variables has determinant 1) implies

$$d\mu(\widehat{\tau}*p) = \mathrm{d}\mu(p) \quad \text{for all } p \in \Omega_\psi,$$

which is exactly formula (10) from [14], where 1 is the multiplier of μ. See also [2, Chapter 4].

Thus, $I_{\lambda,\mu}$ is T-invariant in X with respect to the action $\pi_\sharp : T \times X \to X$.

Hence, the principle of symmetric criticality of Palais ensures that $(u, v) \in Y_T$ is a solution of problem (1.2) if and only if (u, v) is a critical point of the functional $\mathcal{J}_{\lambda,\mu} : Y_T \to \mathbb{R}$, where $\mathcal{J}_{\lambda,\mu} = I_{\lambda,\mu}|_{Y_T}$.

We will employ first an abstract theorem by Ricceri [44, Theorem 4] merging together minimax and critical point theory to derive the existence of two local minima for the energy $\mathcal{J}_{\lambda,\mu}$. For the convenience of the reader and to make our exposition self–contained, we state this abstract tool below, is the version rephrased in terms of the weak topology.

Theorem 6 *Let E be a reflexive Banach space, $D \subseteq \mathbb{R}$ an interval, and $\Psi : E \times D \to \mathbb{R}$ a function satisfying the following:*

(Ψ_1) $\Psi(x, \cdot)$ *is concave in D for every $x \in E$;*
(Ψ_2) $\Psi(\cdot, \lambda)$ *is continuous, coercive and sequentially weakly lower semicontinuous in E, for every $\lambda \in D$;*
(Ψ_3) $\sup_{\lambda \in D} \inf_{x \in E} \Psi(x, \lambda) < \inf_{x \in E} \sup_{\lambda \in D} \Psi(x, \lambda).$

Then for every $\zeta > \sup_{\lambda \in D} \inf_{x \in E} \Psi(x, \lambda)$, there exists an a nonempty open set $\Lambda \subseteq D$ with the following property:

for every $\lambda \in \Lambda$ and every sequentially weakly lower semicontinuous functional $\Theta : E \to \mathbb{R}$, there exists $\delta > 0$ such that, for every $\mu \in (0, \delta)$, the functional $\mathcal{E}_{\lambda,\mu} : E \to \mathbb{R}$ given by

$$\mathcal{E}_{\lambda,\mu}(x) = \Psi(x, \lambda) + \mu\Theta(x)$$

has at least two local minima lying in the set

$$E_\zeta^\lambda = \{x \in E : \Psi(x, \lambda) < \zeta\}.$$

2.3 Growth Condition and Regularity Properties

Here, we use the structural assumptions on F to get some bounds from above for the nonlinear term and its derivatives. This part is quite standard and does not take into account the subelliptic features of the problem: the reader familiar with these nonlinear analysis estimates may go directly to Lemma 8.

Lemma 7 *Assume that conditions (f_1)–(f_3) hold. Then for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that*

(i) $\max\{|\partial_1 F(\eta, \zeta)|, |\partial_2 F(\eta, \zeta)|\} \le \varepsilon (|\eta| + |\zeta|) + c_\varepsilon \left(|\eta|^{\alpha-1} + |\zeta|^{(\alpha-1)}\right),$
(ii) for every $(\eta, \zeta) \in \mathbb{R}^2$

$$|F(\eta, \zeta)| \le \varepsilon \left(|\eta|^2 + 2|\eta||\zeta| + |\zeta|^2\right)$$
$$+ c_\varepsilon \left(|\eta|^\alpha + |\zeta|^\alpha + |\zeta|^{(\alpha-1)}|\eta| + |\zeta||\eta|^{(\alpha-1)}\right).$$

Proof Let $\varepsilon > 0$. First, we will prove that there exists $c_{\varepsilon,1} > 0$ such that

$$|\partial_1 F(\eta, \zeta)| \le \varepsilon (|\eta| + |\zeta|) + c_{\varepsilon,1} \left(|\eta|^{\alpha-1} + |\zeta|^{(\alpha-1)}\right), \tag{2.10}$$

for every $(\eta, \zeta) \in \mathbb{R}^2$. Indeed, by (f_2), it follows in particular, that

$$\lim_{\eta, \zeta \to 0} \frac{\partial_1 F(\eta, \zeta)}{|\eta| + |\zeta|} = 0.$$

Thus, there exists $\delta_\varepsilon > 0$ such that if $|\eta| + |\zeta| < \delta_\varepsilon$, then $|\partial_1 F(\eta, \zeta)| \le \varepsilon(|\eta| + |\zeta|)$. On the other hand, if $|\eta| + |\zeta| \ge \delta_\varepsilon$, condition (f_3) yields

$$|\partial_1 F(\eta, \zeta)| \le \epsilon \left(|\eta| + |\zeta| + |\eta|^{\alpha-1}\right)$$
$$= \epsilon \left((|\eta| + |\zeta|)^{\alpha-1}(|\eta| + |\zeta|)^{2-\alpha} + |\eta|^{\alpha-1}\right) \tag{2.11}$$
$$\le \epsilon \left((|\eta| + |\zeta|)^{\alpha-1}\delta_\varepsilon^{2-\alpha} + |\eta|^{\alpha-1}\right).$$

Moreover, since $\alpha \in (2, 2^*)$ and bearing in mind that

$$(|\eta| + |\zeta|)^{\alpha-1} \le 2^{\alpha-2}(|\eta|^{\alpha-1} + |\zeta|^{\alpha-1}), \quad \forall (\eta, \zeta) \in \mathbb{R}^2$$

inequality (2.11) gives

$$|\partial_1 F(\eta, \zeta)| \le c_{\varepsilon,1} \left(|\eta|^{\alpha-1} + |\zeta|^{\alpha-1}\right), \tag{2.12}$$

for every $(\eta, \zeta) \in \mathbb{R}^2$ with $|\eta| + |\zeta| \geq \delta_\varepsilon$. Hence, relation (2.10) immediately follows by (2.11) and (2.12). Arguing as above, it follows that there exists $c_{\varepsilon,2} > 0$ such that

$$|\partial_2 F(\eta, \zeta)| \leq \varepsilon (|\eta| + |\zeta|) + c_{\varepsilon,2} \left(|\eta|^{\alpha-1} + |\zeta|^{(\alpha-1)}\right), \qquad (2.13)$$

for every $(\eta, \zeta) \in \mathbb{R}^2$. In conclusion, relation (i) holds by (2.10) and (2.13) if taking $c_\varepsilon = \max\{c_{\varepsilon,1}, c_{\varepsilon,2}\}$.

In order to prove part (ii), we make use of the Mean Value Theorem in two variables. More precisely, by (f_1), it follows that

$$\begin{aligned}
|F(\eta, \zeta)| &= |F(\eta, \zeta) - F(0, 0)| \\
&= |\nabla F(c\eta, c\zeta) \cdot (\eta, \zeta)| \\
&\leq |\partial_1 F(c\eta, c\zeta)||\eta| + |\partial_2 F(c\eta, c\zeta)||\zeta|,
\end{aligned} \qquad (2.14)$$

for every $(\eta, \zeta) \in \mathbb{R}^2$ and some $c \in (0, 1)$. Now, by using part (ii), it follows that

$$\max\{|\partial_1 F(c\eta, c\zeta)|, |\partial_2 F(c\eta, c\zeta)|\} \leq \varepsilon (|\eta| + |\zeta|) + c_\varepsilon \left(|\eta|^{\alpha-1} + |\zeta|^{(\alpha-1)}\right), \qquad (2.15)$$

for every $(\eta, \zeta) \in \mathbb{R}^2$. A direct computation shows that (ii) follows by (2.14) and (2.15). □

The next result is a consequence of Lemma 7 and can be viewed as a counterpart of some contributions obtained in several different contexts (see, among others, the paper [32]) to the case of subelliptic gradient–type systems defined on the domain Ω_ψ. We emphasize that a key ingredient of the proof is given by Lemmas 2 and 3 . They express peculiar and intrinsic aspects of the problem under consideration.

Lemma 8 *Let $T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\}$, where $n = \sum_{i=1}^{\ell} n_i$, with $n_i \geq 1$ and $\ell \geq 1$. Furthermore, let $\Omega_\psi \subset \mathbb{H}^n$ as in (1.1), $K : \Omega_\psi \to \mathbb{R}$ be a potential such that (h_K) holds, and $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying (f_1)–(f_3).*
Then the functional $\mathfrak{F} : Y_T \to \mathbb{R}$ given by

$$\mathfrak{F}(u, v) = \int_{\Omega_\psi} K(q) F(u(q), v(q)) d\mu(q), \quad \forall (u, v) \in Y_T$$

is sequentially weakly continuous on Y_T.

Proof In order to prove that $\mathfrak{F}$ is a sequentially weakly continuous functional, arguing by contradiction, we assume that there exists a sequence $\{(u_j, v_j)\}_{j \in \mathbb{N}} \subset Y_T$ which weakly converges to an element $(\widetilde{u}, \widetilde{v}) \in Y_T$, and such that

$$|\mathfrak{F}(u_j, v_j) - \mathfrak{F}(\widetilde{u}, \widetilde{v})| > \varepsilon_0, \qquad (2.16)$$

for every $j \in \mathbb{N}$ and some $\varepsilon_0 > 0$. Now, fixing $(u, v) \in Y_T$, one has

$$
\begin{aligned}
\mathfrak{F}'(u, v)(\varphi, \psi) &= \int_{\Omega_\psi} K(q) \partial_1 F(u(q), v(q)) \varphi(q) \mathrm{d}\mu(q) \\
&\quad + \int_{\Omega_\psi} K(q) \partial_2 F(u(q), v(q)) \phi(q) \mathrm{d}\mu(q), \quad \forall (\varphi, \psi) \in Y_T.
\end{aligned}
\tag{2.17}
$$

Invoking (2.16), the Mean Value Theorem ensures that

$$
0 < \varepsilon_0 \leq |\mathfrak{F}'(w_j, y_j)(u_j - \widetilde{u}, v_j - \widetilde{v})|,
\tag{2.18}
$$

where

$$
w_j = u_j + \theta_j(\widetilde{u} - u_j),
$$

and

$$
y_j = v_j + \theta_j(\widetilde{v} - v_j),
$$

for some $\theta_j \in (0, 1)$, for every $j \in \mathbb{N}$.

By using Lemma 7 (part (i)), the Hölder inequality yields

$$
\begin{aligned}
|\mathfrak{F}'(w_j, y_j)(u_j - \widetilde{u}, v_j - \widetilde{v})| &\leq \int_{\Omega_\psi} K(q) |\partial_1 F(w_j(q), y_j(q))| |u_j(q) - \widetilde{u}(q)| \mathrm{d}\mu(q) \\
&\quad + \int_{\Omega_\psi} K(q) |\partial_2 F(w_j(q), y_j(q))| |v_j(q) - \widetilde{v}(q)| \mathrm{d}\mu(q) \\
&\leq \varepsilon \bigg(\int_{\Omega_\psi} K(q)(|w_j(q)| + |y_j(q)|) |u_j(q) - \widetilde{u}(q)| \mathrm{d}\mu(q) \\
&\quad + \int_{\Omega_\psi} K(q)(|w_j(q)| + |y_j(q)|) |v_j(q) - \widetilde{v}(q)| \mathrm{d}\mu(q) \bigg) \\
&\quad + c_\varepsilon \bigg(\int_{\Omega_\psi} K(q)(|w_j(q)|^{\alpha-1} + |y_j(q)|^{\alpha-1}) |u_j(q) - \widetilde{u}(q)| \mathrm{d}\mu(q) \\
&\quad + \int_{\Omega_\psi} K(q)(|w_j(q)|^{\alpha-1} + |y_j(q)|^{\alpha-1}) |v_j(q) - \widetilde{v}(q)| \mathrm{d}\mu(q) \bigg) \\
&\leq \varepsilon \|K\|_\infty \bigg((\|w_j\|_2 + \|y_j\|_2)(\|u_j - \widetilde{u}\|_2 + \|v_j - \widetilde{v}\|_2) \bigg) \\
&\quad + c_\varepsilon \|K\|_\infty \bigg((\|w_j\|_\alpha^{\alpha-1} + \|y_j\|_\alpha^{\alpha-1})(\|u_j - \widetilde{u}\|_\alpha + \|v_j - \widetilde{v}\|_\alpha) \bigg).
\end{aligned}
\tag{2.19}
$$

Now, it is easy to note that the sequences $\{w_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ are bounded in $HW_{0,T}^{1,2}(\Omega_\psi)$. Moreover, due to the compactness Lemma 3, $u_j \to \widetilde{u}$ and $v_j \to \widetilde{v}$

in $L^{\alpha}(\Omega_{\psi})$. Consequently, the last expression in (2.19) tends to zero and this fact contradicts (2.16).

In conclusion, the functional $\mathfrak{F}$ is sequentially weakly continuous and this completes the proof. □

3 Main Multiplicity Result

With the previous notations, the main result of the present paper reads as follows:

Theorem 9 *Let $T = U(n_1) \times \cdots \times U(n_\ell) \times \{1\}$, where $n = \sum_{i=1}^{\ell} n_i$, with $n_i \geq 1$ and $\ell \geq 1$. Let $\Omega_{\psi} \subset \mathbb{H}^n$ as in (1.1) with $O = (0, 0) \in \Omega_{\psi}$, $v \in [0, C_V^{-1} n^2)$ fixed, and let $V, K : \Omega_{\psi} \to \mathbb{R}$ be potentials satisfying (h_V) and (h_K). Furthermore, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying (f_1)–(f_4).*

Then there exist a number $\sigma > 0$ and a nonempty open set $\Lambda \subset (0, \infty)$ such that, for every $\lambda \in \Lambda$ and every continuous function $G : \Omega_{\psi} \times \mathbb{R}^2 \to \mathbb{R}$ satisfying (g_1)–(g_4), there exists $\mu_0 > 0$ such that, for each $\mu \in (0, \mu_0)$, the gradient–type system (1.2) has at least two weak solutions $(u_{\lambda,\mu}^{(j)}, v_{\lambda,\mu}^{(j)}) \in Y_T$, with $j \in \{1, 2\}$, lying in the ball

$$\{(u, v) \in Y_T : \|(u, v)\| \leq \sigma\},$$

where

$$\|(u, v)\| = \left(\|u\|_{HW_0^{1,2}(\Omega_{\psi})}^2 - v \int_{\Omega_{\psi}} V(q)|u(q)|^2 \mathrm{d}\mu(q) \right)^{1/2}$$
$$+ \left(\|v\|_{HW_0^{1,2}(\Omega_{\psi})}^2 - v \int_{\Omega_{\psi}} V(q)|v(q)|^2 \mathrm{d}\mu(q) \right)^{1/2}.$$

Proof We will show that the assumptions of Theorem 6 are fulfilled by choosing $E = Y_T$ and $D = [0, \infty)$. Moreover, let us denote

$$\Phi(u, v) = \frac{1}{2} \left(\|u\|^2 + \|v\|^2 \right),$$

and

$$\mathfrak{F}(u, v) = \int_{\Omega_{\psi}} K(q)F(u(q), v(q))\mathrm{d}\mu(q),$$

for every $(u, v) \in Y_T$, and define $\Psi : Y_T \times D \to \mathbb{R}$ as follows:

$$\Psi((u, v), \lambda) = \Phi(u, v) - \lambda \mathfrak{F}(u, v) + \lambda \rho_0$$

for every $(u, v) \in Y_T$, and $\lambda \in D$. Here ρ_0 is a positive and sufficiently small real parameter (see (3.8) below).

We observe that the function $\Psi((u, v), \cdot)$ is concave in D, for every $(u, v) \in Y_T$. Moreover, the functional $\Psi(\cdot, \lambda)$ is continuous and sequential weak lower semicontinuity on Y_T, for every $\lambda \in D$.

Moreover, fixing $\lambda \in D$, on account of (f_4), fixing $\mu_F, \nu_F \in [2, 2^*]$, by Hölder's inequality and Lemma 2, one has

$$\Psi((u, v), \lambda) \geq \Phi(u, v) - \lambda \left(\int_{\Omega_\psi} \left(\kappa_1 K(q) |u(q)|^{p_F} + \kappa_2 K(q) |v(q)|^{q_F} + \kappa_3 K(q) \right) d\mu(q) \right)$$

$$\geq \Phi(u, v) - \lambda \left(\kappa_1 \|K\|_{\mu_F/(\mu_F - p_F)} \|u\|_{\mu_F}^{p_F} + \kappa_2 \|K\|_{\nu_F/(\nu_F - q_F)} \|v\|_{\nu_F}^{q_F} + \kappa_3 \|K\|_1 \right)$$

$$\geq \Phi(u, v) - c\lambda \left(\|u\|^{p_F} + \|v\|^{q_F} + 1 \right)$$

for some $c > 0$. Thus,

$$\lim_{\|(u,v)\| \to \infty} \Psi((u, v), \lambda) = \infty,$$

since $\max\{p_F, q_F\} < 2$.

Hence (Ψ_1) and (Ψ_2) of Theorem 6 are verified. Next, we deal with (Ψ_3). First, let us consider the real function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(\xi) = \sup_{(u,v) \in \Phi^{-1}((-\infty, \xi])} \mathfrak{F}(u, v),$$

for every $\xi \in \mathbb{R}$.

By Lemma 7, we have that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\max\{|\partial_1 F(\eta, \zeta)|, |\partial_2 F(\eta, \zeta)|\} \leq \varepsilon (|\eta| + |\zeta|) + c_\varepsilon \left(|\eta|^{\alpha-1} + |\zeta|^{(\alpha-1)} \right),$$

and

$$|F(\eta, \zeta)| \leq \varepsilon \left(|\eta|^2 + 2|\eta||\zeta| + |\zeta|^2 \right)$$
$$+ c_\varepsilon \left(|\eta|^\alpha + |\zeta|^\alpha + |\zeta|^{(\alpha-1)}|\eta| + |\zeta||\eta|^{(\alpha-1)} \right) \tag{3.1}$$

for any $(\eta, \zeta) \in \mathbb{R}^2$.

Integrating (3.1) and using the Young inequality, we easily get

$$\int_{\Omega_\psi} K(q)|F(u(q), v(q))| d\mu(q) \leq 2\varepsilon \|K\|_\infty \int_{\Omega_\psi} \left(|u(q)|^2 + |v(q)|^2 \right) d\mu(q)$$

$$+ 2c_\varepsilon \|K\|_\infty \int_{\Omega_\psi} \left(|u(q)|^\alpha + |v(q)|^\alpha \right) d\mu(q),$$

for every $(u, v) \in Y_T$.

Furthermore, invoking the embeddings result stated in Lemma 2, since $\nu \in [0, C_V^{-1} n^2)$, we deduce that

$$\int_{\Omega_\psi} K(q)|F(u(q), v(q))|\mathrm{d}\mu(q) \leq \varepsilon c_2 \|K\|_\infty \left(\|u\|^2 + \|v\|^2\right)$$
$$+ c_\varepsilon c_\alpha^\alpha \|K\|_\infty \left(\|u\|^\alpha + \|v\|^\alpha\right).$$

Now, taking into account that the real function defined by $\xi \mapsto (a^\xi + b^\xi)^{1/\xi}, \xi > 0$, $a, b \geq 0$, is nonincreasing, it follows that

$$\|u\|^\alpha + \|v\|^\alpha \leq \left(\|u\|^2 + \|v\|^2\right)^{\alpha/2}$$

and therefore

$$\mathfrak{F}(u, v) \leq 4\varepsilon c_2 \|K\|_\infty \left(\frac{\|u\|^2 + \|v\|^2}{2}\right)$$
$$+ 2^{\alpha/2+1} c_\varepsilon c_\alpha^\alpha \|K\|_\infty \left(\frac{\|u\|^2 + \|v\|^2}{2}\right)^{\alpha/2}.$$

The above inequality yields

$$f(\xi) = \sup_{(u,v)\in\Phi^{-1}((-\infty,\xi])} \mathfrak{F}(u, v)$$
$$= \sup_{(u,v)\in\Phi^{-1}((-\infty,\xi])} \int_{\Omega_\psi} K(q)F(u(q), v(q))\mathrm{d}\mu(q)$$
$$\leq \|K\|_\infty \sup_{(u,v)\in\Phi^{-1}((-\infty,\xi])}$$
$$\times \left(4\varepsilon c_2 \left(\frac{\|u\|^2 + \|v\|^2}{2}\right) + 2^{\alpha/2+1} c_\varepsilon c_\alpha^\alpha \left(\frac{\|u\|^2 + \|v\|^2}{2}\right)^{\alpha/2}\right)$$
$$\leq \|K\|_\infty \left(4c_2\xi + 2^{\alpha/2+1} c_\varepsilon c_\alpha^\alpha \xi^{\alpha/2}\right)$$

for every $\xi > 0$.

Since the nonlinearity f is nonnegative, it follows that

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi} = 0. \tag{3.2}$$

Now, we claim that that there exists $(u_0, v_0) \in Y_T$ such that

$$\int_{\Omega_\psi} K(\sigma)F(u_0(q), v_0(q))\mathrm{d}\mu(q) > 0. \tag{3.3}$$

Indeed, following Balogh and Kristály in [1], we construct a special test function belonging to $HW_{0,T}^{1,2}(\Omega_\psi)$ that will be useful for our purposes. Let

$$\widehat{\Omega}_0 = \bigcup_{\widehat{\tau} \in \mathbb{U}(n)} \{\widehat{\tau} * \Omega_0\},$$

where Ω_0 is the open set of $\mathbb{H}^n$ given in (h_K). Since K is cylindrically symmetric, one has

$$\inf_{q \in \Omega_0} K(q) = \inf_{q \in \widehat{\Omega}_0} K(q) > 0. \tag{3.4}$$

Furthermore, we can find $(z_0, t_0) \in \Omega_\psi$ and

$$0 < R < 2|z_0|(\sqrt{2} - 1), \tag{3.5}$$

such that

$$A_R = \{q \in \mathbb{H}^n : q = (z,t) \text{ with } ||z| - |z_0|| \leq R, \ |t - t_0| \leq R\} \subset \Omega_0. \tag{3.6}$$

Of course, for every $\varrho \in (0, 1]$, it follows that

$$A_{\varrho R} \subseteq A_R \subset \Omega_0,$$

and $\mu(A_{\varrho R}) > 0$.

Set $\varrho \in (0, 1)$ and $c_0 \in \mathbb{R}$. Let us consider the function $v_\varrho^{c_0} \in HW_{0,\mathrm{cyl}}^{1,2}(\Omega_\psi) \subseteq HW_{0,T}^{1,2}(\Omega_\psi)$ given by

$$v_\varrho^{c_0}(q) = \frac{c_0}{1-\varrho} \left(1 - \max \left(\frac{||z| - |z_0||}{R}, \frac{||t| - |t_0||}{R}, \varrho \right) \right)_+, \quad q = (z,t) \in \Omega_\psi$$

where $\ell_+ := \max\{0, \ell\}$. With the above notation, we have

(i_1) $\mathrm{supp}(v_\varrho^{c_0}) = A_R$;
(i_2) $\|v_\varrho^{c_0}\|_\infty \leq |c_0|$;
(i_3) $v_\varrho^{c_0}(q) = c_0$ for every $q \in A_{\varrho R}$.

By (f_1), there exists $(\eta_0, \zeta_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that $F(\eta_0, \zeta_0) > 0$. Moreover,

$$\int_{\Omega_\psi} K(q) F(v_\varrho^{\eta_0}(q), v_\varrho^{\zeta_0}(q)) \, d\mu(q) = \int_{A_{\varrho R}} K(q) F(v_\varrho^{\eta_0}(q), v_\varrho^{\zeta_0}(q)) \, d\mu(q)$$

$$+ \int_{A_R \setminus A_{\varrho R}} K(q) F(v_\varrho^{\eta_0}(q), v_\varrho^{\zeta_0}(q)) \, d\mu(q).$$

It then follows that

$$
\int_{\Omega_\psi} K(q) F(v_\varrho^{\eta_0}(q), v_\varrho^{\zeta_0}(q)) \, \mathrm{d}\mu(q) \geq \inf_{q \in A_R} K(q) F(\eta_0, \zeta_0) \mu(A_{\varrho R})
$$
$$
- \|K\|_\infty \max_{(|\eta|,|\zeta|) \in [0,|\eta_0|] \times [0,|\zeta_0|]} |F(\eta, \zeta)| \mu(A_R \setminus A_{\varrho R}).
$$
(3.7)

Since $\mu(A_R \setminus A_{\varrho R}) \to 0$, as $\varrho \to 1^-$, we of course, get

$$
\|K\|_\infty \max_{(|\eta|,|\zeta|) \in [0,|\eta_0|] \times [0,|\zeta_0|]} |F(\eta, \zeta)| \mu(A_R \setminus A_{\varrho R}) \to 0,
$$

as $\varrho \to 1^-$. Moreover,

$$
\mu(A_{\varrho R}) \to \mu(A_R)
$$

as $\varrho \to 1^-$. Thus, there exists $\varrho_0 > 0$ such that

$$
\inf_{q \in A_R} K(q) F(\eta_0, \zeta_0) \mu(A_{\varrho_0 R}) > \|K\|_\infty \max_{(|\eta|,|\zeta|) \in [0,|\eta_0|] \times [0,|\zeta_0|]} |F(\eta, \zeta)| \mu(A_R \setminus A_{\varrho_0 R}).
$$

Hence (3.3) can be proved by choosing

$$
u_0(q) = v_{\varrho_0}^{\eta_0}(q) = \frac{\eta_0}{1 - \varrho_0} \left(1 - \max\left(\frac{||z| - |z_0||}{R}, \frac{||t| - |t_0||}{R}, \varrho_0\right)\right)_+,
$$

and

$$
v_0(q) = v_{\varrho_0}^{\zeta_0}(q) = \frac{\zeta_0}{1 - \varrho_0} \left(1 - \max\left(\frac{||z| - |z_0||}{R}, \frac{||t| - |t_0||}{R}, \varrho_0\right)\right)_+,
$$

for every $q = (z, t) \in \Omega_\psi$.

Now, fix $\eta \in \mathbb{R}$ such that

$$
0 < \eta < \mathfrak{F}(u_0, v_0) \left(\frac{\|u_0\|^2 + \|v_0\|^2}{2}\right)^{-1}.
$$

By (3.2), there exists $\xi_0 \in \left(0, \frac{\|u_0\|^2 + \|v_0\|^2}{2}\right)$ such that $f(\xi_0) < \eta \xi_0$.

Let $\rho_0 > 0$ such that

$$
f(\xi_0) < \rho_0 < \xi_0 \mathfrak{F}(u_0, v_0) \left(\frac{\|u_0\|^2 + \|v_0\|^2}{2}\right)^{-1}.
$$
(3.8)

Due to the choice of ξ_0, one has $\rho_0 < \mathfrak{F}(u_0, v_0)$.

We are now in position to prove that the following strict inequality holds:

$$\sup_{\lambda \in D} \inf_{(u,v) \in Y_T} \Psi((u,v), \lambda) < \inf_{(u,v) \in Y_T} \sup_{\lambda \in D} \Psi((u,v), \lambda),$$

i.e. that condition (Ψ_3) of Theorem 6 is satisfied.

Indeed, the real function

$$\lambda \mapsto \inf_{(u,v) \in Y_T} \Psi((u,v), \lambda)$$

is upper semicontinuous on D and

$$\lim_{\lambda \to \infty} \inf_{(u,v) \in Y_T} \Psi((u,v), \lambda) \leq \lim_{\lambda \to \infty} \Psi((u_0, v_0), \lambda) = -\infty.$$

Consequently (see [39, Chapter I]), there exists $\bar{\lambda} \in D$ such that

$$\sup_{\lambda \in D} \inf_{(u,v) \in Y_T} \Psi((u,v), \lambda) = \inf_{(u,v) \in Y_T} \Psi((u,v), \bar{\lambda}). \tag{3.9}$$

For each $(u,v) \in \Phi^{-1}((-\infty, \xi_0])$, we have

$$\mathfrak{F}(u,v) \leq f(\xi_0) < \rho_0$$

and hence

$$\xi_0 \leq \inf \left\{ \Phi(u,v) : \mathfrak{F}(u,v) \geq \rho_0 \right\}. \tag{3.10}$$

On the other hand, we also have

$$\inf_{(u,v) \in Y_T} \sup_{\lambda \in D} \Psi((u,v), \lambda) = \inf_{(u,v) \in Y_T} \left(\Phi(u,v) + \sup_{\lambda \in D} (\lambda (\rho_0 - \mathfrak{F}(u,v))) \right)$$

$$= \inf_{(u,v) \in Y_T} \left\{ \Phi(u,v) : \mathfrak{F}(u,v) \geq \rho_0 \right\},$$

and therefore

$$\xi_0 \leq \inf_{(u,v) \in Y_T} \sup_{\lambda \in D} \Psi((u,v), \lambda). \tag{3.11}$$

There are two distinct cases.

If $0 \leq \bar{\lambda} < \xi_0/\rho_0$, it follows that

$$\inf_{(u,v) \in Y_T} \Psi((u,v), \bar{\lambda}) \leq \Phi(0,0) - \bar{\lambda}\mathfrak{F}(0,0) + \bar{\lambda}\rho_0 = \bar{\lambda}\rho_0 < \xi_0,$$

and inequality (Ψ_3) is verified.

If $\bar{\lambda} \in (\xi_0/\rho_0, \infty)$ it is easy to note that

$$\inf_{(u,v)\in Y_T} \Psi((u, v), \bar{\lambda}) \leq \Psi((u_0, v_0), \bar{\lambda}) \leq \Psi((u_0, v_0), \xi_0/\rho_0) < \xi_0.$$

Hence, also in this case inequality (Ψ_3) is satisfied.

Therefore, fixing $\zeta > \sup_{\lambda \in D} \inf_{(u,v)\in Y_T} \Psi((u, v), \lambda)$, Theorem 6 assures the existence of a nonempty open set $\Lambda \subseteq D$ with the following property:

If $\lambda \in \Lambda$ and $G : \Omega_\psi \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies (g_1)–(g_3), then there exists $\delta > 0$ such that, for each $\mu \in (0, \delta)$, the functional

$$\mathcal{E}_{\lambda,\mu}(u, v) = \Psi((u, v), \lambda) + \mu\Theta(u, v), \quad \text{for every } (u, v) \in Y_T$$

has at least two local minima in

$$\{(u, v) \in Y_T : \Psi((u, v), \lambda) < \zeta\},$$

say $(u_{\lambda,\mu}^{(j)}, v_{\lambda,\mu}^{(j)})$, with $j \in \{1, 2\}$.

Here $\Theta : Y_T \to \mathbb{R}$ is the functional defined by

$$\Theta(u, v) = -\int_{\Omega_\psi} G(q, u(q), v(q)) d\mu(q).$$

Notice that, similarly to $\mathfrak{F}$, the functional Θ is sequentially weakly continuous on Y_T thanks to assumptions (g_1)–(g_3). Now $K(\cdot)$ and $G(\cdot, \eta, \zeta)$ are symmetric functions (respectively, by (h_K) and (g_4)), and the action $\pi_\sharp : T \times X \to X$ given by

$$\pi_\sharp(\widehat{\tau}, (u, v)) = (\widehat{\tau}\sharp u, \widehat{\tau}\sharp v),$$

for every $\widehat{\tau} \in T$ and $(u, v) \in X$, is isometric. Thus, the functional $I_{\lambda,\mu} : X \to \mathbb{R}$

$$I_{\lambda,\mu}(u, v) = \frac{1}{2}\left(\|u\|^2 + \|v\|^2\right) - \lambda \int_{\Omega_\psi} K(q)F(u(q), v(q)) d\mu(q)$$

$$-\mu \int_{\Omega_\psi} G(q, u(q), v(q)) d\mu(q),$$

for every $(u, v) \in X$, is T-invariant, i.e.

$$I_{\lambda,\mu}(\pi_\sharp(\widehat{\tau}, (u, v))) = I_{\lambda,\mu}(u, v),$$

for every $(u, v) \in X$, see Sect. 2.2 for details.

Moreover,

$$I_{\lambda,\mu}|_{Y_T}(u, v) = \mathcal{E}_{\lambda,\mu}(u, v) - \lambda\rho_0 = \Psi((u, v), \lambda) + \mu\Theta(u, v) - \lambda\rho_0,$$

for every $(u, v) \in Y_T$.

By Theorem 5, $(u_{\lambda,\mu}^{(j)}, v_{\lambda,\mu}^{(j)}) \in Y_T$, with $j \in \{1, 2\}$, turn out also to be critical points of $I_{\lambda,\mu}$ and hence weak solutions to (1.2).

Finally, to estimate the norm of $(u_{\lambda,\mu}^{(j)}, v_{\lambda,\mu}^{(j)}) \in Y_T$, with $j \in \{1, 2\}$, we take a nondegenerate compact interval $[a, b] \subset \Lambda$. Notice that one has

$$\bigcup_{\lambda \in [a,b]} \{(u, v) \in Y_T : \Psi((u, v), \lambda) \leq \zeta\}$$

$$\subseteq \{(u, v) \in Y_T : \Psi((u, v), a) \leq \zeta\} \cup \{(u, v) \in Y_T : \Psi((u, v), b) \leq \zeta\}$$

and hence the set

$$S := \bigcup_{\lambda \in [a,b]} \{(u, v) \in Y_T : \Psi((u, v), \lambda) \leq \zeta\}$$

is bounded. In conclusion, the local minima of the energy functional $\mathcal{E}_{\lambda,\mu}$ (defined on Y_T) have norm at most equal to $\sigma = \sup_{(u,v) \in S} \|(u, v)\|$. This concludes the proof. $\square$

A direct application of Theorem 9 is given below.

Example 10 Let $\Omega_\psi \subset \mathbb{H}^n$ be as in (1.1), with $O = (0, 0) \in \Omega_\psi$ and let $V, K : \Omega_\psi \to \mathbb{R}$ be potentials satisfying, respectively, (h_V) and (h_K). Furthermore, let us fix $\alpha \in (2, 2^*)$ and let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-function defined by

$$F(\eta, \zeta) = \sin\left(|\eta|^\alpha + |\zeta|^\alpha\right),$$

for every $(\eta, \zeta) \in \mathbb{R}^2$. Then, if $\nu \in [0, C_V^{-1} n^2)$, there exist by Theorem 9 a number $\sigma > 0$ and a nonempty open set $\Lambda \subset (0, \infty)$ such that, for every $\lambda \in \Lambda$, the following singular subelliptic system

$$\begin{cases} -\Delta_{\mathbb{H}^n} u - \nu V(q)u + u = \alpha\lambda K(q)|u|^{\alpha-2}u\cos(|u|^\alpha + |v|^\alpha) & \text{in } \Omega_\psi \\ -\Delta_{\mathbb{H}^n} v - \nu V(q)v + v = \alpha\lambda K(q)|v|^{\alpha-2}v\cos(|u|^\alpha + |v|^\alpha) & \text{in } \Omega_\psi \\ u = v = 0 \text{ on } \partial\Omega_\psi, \end{cases}$$

has at least two weak solutions $(u_{\lambda,\mu}^{(j)}, v_{\lambda,\mu}^{(j)}) \in HW_{0,\text{cyl}}^{1,2}(\Omega_\psi) \times HW_{0,\text{cyl}}^{1,2}(\Omega_\psi)$, with $j \in \{1, 2\}$, lying in the ball

$$\left\{(u, v) \in HW_{0,\text{cyl}}^{1,2}(\Omega_\psi) \times HW_{0,\text{cyl}}^{1,2}(\Omega_\psi) : \|(u, v)\| \leq \sigma\right\}.$$

In other words,

$$\left(\left\|u_{\lambda,\mu}^{(j)}\right\|_{HW_0^{1,2}(\Omega_\psi)}^2 - \nu\int_{\Omega_\psi} V(q)\left|u_{\lambda,\mu}^{(j)}(q)\right|^2 d\mu(q)\right)^{1/2} \leq \sigma,$$

and

$$\left(\left\| v_{\lambda,\mu}^{(j)} \right\|_{HW_0^{1,2}(\Omega_\psi)}^2 - \nu \int_{\Omega_\psi} V(q) \left| v_{\lambda,\mu}^{(j)}(q) \right|^2 \mathrm{d}\mu(q) \right)^{1/2} \leq \sigma,$$

for $j \in \{1, 2\}$.

Remark 11 For the sake of completeness, we point out that the results presented in this paper could be also investigated for a larger class of elliptic equations where the leading term is governed by some differential operators such as the ones considered in [3–5]. However, in these cases some different technical approaches need to be adopted in order to get analogous existence results for this wider class of energies. We will consider these interesting cases in our future investigations.

References

1. Balogh, Z.M., Kristály, A.: Lions-type compactness and Rubik actions on the Heisenberg group. Calc. Var. Part. Diff. Equ. **48**, 89–109 (1995)
2. Barndorff-Nielsen, O.E., Blæsild, P., Eriksen, P.S.: Decomposition and Invariance of Measures, and Statistical Transformation Models. Lecture Notes in Statistics, vol. 58. Springer, New York (1989)
3. Baroni, P., Kuusi, T., Mingione, G.: Borderline gradient continuity of minima. J. Fixed Point Theory Appl. **15**, 537–575 (2014)
4. Baroni, P., Colombo, M., Mingione, G.: Harnack inequalities for double phase functionals. Nonlinear Anal. Theory Methods Appl. **121**, 206–222 (2015)
5. Baroni, P., Colombo, M., Mingione, G.: Non-autonomous functionals, borderline cases and related function classes. St. Petersburg Math. J. **27**, 347–379 (2016)
6. Bisci, G.M., Ferrara, M.: Subelliptic and parametric equations on Carnot groups. Proc. Am. Math. Soc **144**, 3035–3045 (2016)
7. Bisci, G.M., Pucci, P.: Critical Dirichlet problems on $\mathcal{H}$ domains of Carnot groups. J. Diff. Equ. **25**, 179–196 (2017)
8. Bisci, G.M, Pucci, P.: Hypoelliptic critical Dirichlet problems in unbounded domains of the Heisenberg group. in preparation
9. Bisci, G.M., Repovš, D.: Yamabe-type equations on Carnot groups. Potential Anal. **46**, 369–383 (2017)
10. Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: Stratified Lie Groups and Potential Theory for Their Sub-Laplacians. Springer Monographs in Mathematics. Springer, Berlin (2007)
11. Bordoni, S., Pucci, P.: Schrödinger-Hardy systems involving two Laplacian operators in the Heisenberg group. Bull. Sci. Math. **146**, 50–88 (2018)
12. Bordoni, S., Filippucci, R., Pucci, P.: Existence of solutions in problems on Heisenberg groups involving Hardy and critical terms. Submitted for publication, p. 28
13. Bourbaki, N.: General Topology: Chapters 1–4, Translated from the French, Reprint of the 1989 English translation. Elements of Mathematics. Springer, Berlin (1998)
14. Bourbaki, N.: Integration, II: Chapters 7–9, Translated from the 1963 and 1969 French originals by Sterling K. Berberian. Elements of Mathematics. Springer, Berlin (2004)
15. Cammaroto, F., Vilasi, L.: Axially symmetric solutions to a fractional system on strip-like domains. Minimax Theory Appl. **3**, 23–34 (2018)
16. Caponi, M., Pucci, P.: Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations. Ann. Mat. Pura Appl. **195**, 2099–2129 (2016)
17. Chow, W.-L.: Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. Math. Ann. **117**, 98–105 (1939)

18. D'Ambrosio, L., Mitidieri, E.: Quasilinear elliptic equations with critical potentials. Adv. Nonlinear Anal. **6**, 147–164 (2017)
19. de Morais Filho, D.C., Souto, M.A.S., do Ó, J.M.: A compactness embedding lemma, a principle of symmetric criticality and applications to elliptic problems. Proyecciones **19**, 1–17 (2000)
20. del Pino, M.A., Felmer, P.L.: Least energy solutions for elliptic equations in unbounded domains. Proc. R. Soc. Edinb. Sect. A **126**, 195–208 (1996)
21. Esteban, M., Lions, P.L.: A compactness lemma. Nonlinear Anal. **7**(4), 381–385 (1983)
22. Föglein, A.: Partial regularity results for sub-elliptic systems in the Heisenberg group. Calc. Var. Part. Diff. Equ. **32**, 25–51 (2008)
23. Folland, G.B.: Subelliptic estimates and function spaces on nilpotent Lie groups. Ark. Math. **13**, 161–207 (1975)
24. Folland, G.B., Stein, E.M.: Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. Comm. Pure Appl. Math. **27**, 429–522 (1974)
25. Gao, D., Niu, P., Wang, J.: Partial regularity for degenerate subelliptic systems associated with Hörmander's vector fields. Nonlinear Anal. **73**, 3209–3223 (2010)
26. Garofalo, N., Lanconelli, E.: Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation. Ann. Inst. Fourier **40**, 313–356 (1990)
27. Garofalo, N., Nhieu, D.-M.: Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces. Commun. Pure Appl. Math. **49**, 1081–1144 (1996)
28. Garofalo, N., Vassilev, D.: Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups. Math. Ann. **318**, 453–516 (2000)
29. Hörmander, L.: Hypoelliptic second order differential equations. Acta Math. **119**, 147–171 (1967)
30. Ivanov, S.P., Vassilev, D.N.: Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2011)
31. Kobayashi, J., Otani, M.: The principle of symmetric criticality for non-differentiable mappings. J. Funct. Anal. **214**(2), 428–449 (2004)
32. Kristály, A.: Existence of two non-trivial solutions for a class of quasilinear elliptic problems in strip-like domains. Proc. Edimb. Math. Soc. **48**, 465–477 (2005)
33. Kristály, A., Rădulescu, V.D., Varga, C.G.: Variational Principles in Mathematical Physics, Geometry, and Economics. Qualitative Analysis of Nonlinear Equations and Unilateral Problems, with a Foreword by Jean Mawhin. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (2010)
34. Lions, P.L.: Symétrie et compacité dans les espaces Sobolev. J. Funct. Anal. **49**, 315–334 (1982)
35. Maad, S.: Infinitely many solutions of a symmetric semilinear elliptic equation on an unbounded domain. Ark. Mat. **41**, 105–114 (2003)
36. Maad, S.: A semilinear problem for the Heisenberg Laplacian on unbounded domains. Can. J. Math. **57**, 1279–1290 (2005)
37. Magliaro, M., Mari, L., Mastrolia, P., Rigoli, M.: Keller–Osserman type conditions for differential inequalities with gradient terms on the Heisenberg group. J. Diff. Equ. **250**, 2643–2670 (2011)
38. Manfredi, J., Mingione, G.: Regularity results for quasilinear elliptic equations in the Heisenberg group. Math. Ann. **339**, 485–544 (2007)
39. Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences, 74. Springer, New York (1989)
40. Mingione, G., Zatorska-Goldstein, A., Zhong, X.: Gradient regularity for elliptic equations in the Heisenberg group. Adv. Math. **222**, 62–129 (2009)
41. Palais, R.S.: The principle of symmetric criticality. Commun. Math. Phys. **69**, 19–30 (1979)
42. Pucci, P.: Critical Schrödinger–Hardy systems in the Heisenberg group, to appear in Discrete Contin. Dyn. Syst. Ser. S, Special Issue on the occasion of the 60th birthday of V. Rădulescu, to appear
43. Rashevskiĭ, P.K.: About connecting two points of complete nonholonomic space by admissible curve. Uch. Zapiski Ped. Inst. K. Liebknecht **2**, 83–94 (1938)
44. Ricceri, B.: Minimax theorems for limits of parametrized functions having at most one local minimum lying in a certain set. Topol. Appl. **153**, 3308–3312 (2006)
45. Schindler, I., Tintarev, K.: Semilinear subelliptic problems without compactness on Lie groups. NoDEA Nonlinear Diff. Equ. Appl. **11**, 299–309 (2004)
46. Tintarev, K.: Semilinear elliptic problems on unbounded subsets of the Heisenberg group. Electron. J. Diff. Equ. **18**, 8 (2001)

47. Tintarev, K., Fieseler, K.-H.: Concentration Compactness. Functional-Analytic Grounds and Applications. Imperial College Press, London (2007)
48. Vassilev, D.: Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups. Pac. J. Math. **227**, 361–397 (2006)
49. Vittone, D.: Submanifolds in Carnot groups, Thesis, Scuola Normale Superiore, Pisa, 2008, (New Series) 7, Edizioni della Normale, xx+180 pp
50. Wang, J., Manfredi, J.: Partial Hölder continuity for nonlinear sub-elliptic systems with VMO-coefficients in the Heisenberg group. Adv. Nonlinear Anal. **7**(1), 97–116 (2018)
51. Wang, J., Niu, P.: Optimal partial regularity for weak solutions of nonlinear sub-elliptic systems in Carnot groups. Nonlinear Anal. **72**, 4162–4187 (2010)
52. Xu, C., Zuily, C.: Higher interior regularity for quasilinear subelliptic systems. Calc. Var. Partial Diff. Equ. **5**, 323–343 (1997)