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Existence and localization of solutions for nonlocal fractional equations

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Abstract. This work is devoted to the study of the existence of at least one weak solution to nonlocal equations involving a general integro-differential operator of fractional type. As a special case, we derive an existence theorem for the fractional Laplacian, finding a nontrivial weak solution of the equation

$$\begin{cases} (-\Delta)^s u = h(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $h \in L^\infty_+(\Omega) \setminus \{0\}$ and $f : \mathbb{R} \to \mathbb{R}$ is a suitable continuous function. These problems have a variational structure and we find a nontrivial weak solution for them by exploiting a recent local minimum result for smooth functionals defined on a reflexive Banach space. To make the nonlinear methods work, some careful analysis of the fractional spaces involved is necessary.

Keywords: fractional equations, multiple solutions, critical points results

1. Introduction

The aim of this paper is to prove some existence results for fractional Laplacian problems whose prototype is

$$\begin{cases} (-\Delta)^s u = h(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad (D_h^f)$$

where $(-\Delta)^s$ denotes the fractional Laplacian operator with $s \in (0,1)$, and Ω is an open bounded set with Lipschitz boundary of $\mathbb{R}^n$, requiring that $n > 2s$. Moreover, $h \in L^\infty(\Omega) \setminus \{0\}$ is a nonnegative map and $f : \mathbb{R} \to \mathbb{R}$ represents a subcritical continuous function.

The existence of weak solutions for such type of problems has been intensively studied under different assumptions on the nonlinearities (see, for instance, [6,8,14,16,18] and references therein).

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Moreover, the existence and multiplicity of solutions for elliptic equations in $\mathbb{R}^n$, driven by a nonlocal integro-differential operator, whose standard prototype is the fractional Laplacian, have been studied, very recently, by Autuori and Pucci in [2] (this work is related to the results on general quasilinear elliptic problems given in [1]).

Motivated by this wide interest, we prove in the present note some existence results (see Theorem 3.2 and its consequences) for fractional equations assuming that f has a suitable behaviour at zero together with some global properties formulated by the means of an auxiliary function ψ.

The strategy for proving Theorem 3.2 is based on the fact that our problem can be seen as the Euler–Lagrange equation of a suitable functional defined in a Sobolev space X_0.

Hence, the solutions of (D_h^f) or more generally of problem $(P_K^{h,f})$ defined in the sequel, can be found as critical points of this functional: for this purpose, along the paper, we will exploit a critical point result due to Ricceri (see Theorem 2.1 and Proposition 3.1).

Exploiting this result, a key point is to prove the existence of a suitable $\sigma > 0$ such that

$$\frac{\sup_{\|u\|_{X_0} \leqslant \sigma} \int_\Omega h(x)(\int_0^{u(x)} f(t)\,\mathrm{d}t)\,\mathrm{d}x}{\sigma} < \frac{1}{2}.$$

One of the main novelties here is that, in contrast with several known results (see references contained in [11]), we obtain the above inequality without continuous embedding of the ambient space in $C^0(\bar{\Omega})$.

In the nonlocal framework, denoting by $\lambda_{1,s}$ the first eigenvalue of the problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

the simplest example we can deal with is given by the fractional Laplacian, according to the following result.

Theorem 1.1. *Let $s \in (0,1)$, $n > 2s$ and let Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Moreover, let $f : [0,+\infty) \to [0,+\infty)$ be a continuous function satisfying the following hypothesis*:

(h_1) *for some $q \in [1, \frac{2n}{n-2s})$ the function*

$$t \mapsto \frac{f(t)}{t^{q-1}}$$

is strictly decreasing in $(0,+\infty)$ and $\lim_{t\to+\infty} \frac{f(t)}{t^{q-1}} = 0$.

Further, suppose that

$$\lim_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \liminf_{\xi\to0^+} \frac{F(\xi)}{\xi^2} > 0.$$

Then, for every

$$\alpha > \frac{\lambda_{1,s}}{2\liminf_{\xi\to0^+} \frac{F(\xi)}{\xi^2}},$$

the parametric nonlocal problem

$$\begin{cases} (-\Delta)^s u = \alpha f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

admits at least one nonnegative and nonzero weak solution $u_\alpha \in H^s(\mathbb{R}^n)$, *such that* $u_\alpha = 0$ *a.e. in* $\mathbb{R}^n \setminus \Omega$.

The plan of the paper is as follows; Section 2 is devoted to our abstract framework and preliminaries. Successively, in Section 3 we give the main result (see Theorem 3.2). Finally, the fractional Laplacian case is studied in the last section. A concrete example of an application is presented in Example 4.3.

2. Preliminaries

In this section we briefly recall the definition of the functional space X_0, first introduced in [14,15]. Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a function with the properties that:

(k_1) $\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) := \min\{|x|^2, 1\}$;
(k_2) there exists $\beta > 0$ such that

$$K(x) \geqslant \beta |x|^{-(n+2s)}$$

for any $x \in \mathbb{R}^n \setminus \{0\}$;
(k_3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\big((x, y) \mapsto \big(g(x) - g(y)\big)\sqrt{K(x - y)}\big) \in L^2\big(\big(\mathbb{R}^n \times \mathbb{R}^n\big) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), \mathrm{d}x\,\mathrm{d}y\big),$$

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. We denote by X_0 the following linear subspace of X

$$X_0 := \big\{g \in X\colon\ g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\big\}.$$

We remark that X and X_0 are nonempty, since $C_0^2(\Omega) \subseteq X_0$ by [15, Lemma 11]. Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left(\int_Q \big|g(x) - g(y)\big|^2 K(x - y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2},$$

where $Q := (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathcal{O}$ and $\mathcal{O} := (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^n \times \mathbb{R}^n$. It is easily seen that $\|\cdot\|_X$ is a norm on X; see, for instance, [14].

By [14, Lemmas 6 and 7] we can take in the sequel the function

$$X_0 \ni v \mapsto \|v\|_{X_0} := \left(\int_Q \big|v(x) - v(y)\big|^2 K(x - y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \tag{1}$$

as norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u, v\rangle_{X_0} := \int_Q \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)K(x - y)\,\mathrm{d}x\,\mathrm{d}y,$$

see [14, Lemma 7].

Note that in (1) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$).

While for a general kernel K satisfying conditions from (k_1) to (k_3) we have that $X_0 \subset H^s(\mathbb{R}^n)$, in the model case $K(x) := |x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω; see [18, Lemma 7].

Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

Before concluding this section, we recall the embedding properties of X_0 into the usual Lebesgue spaces; see [14, Lemma 8]. The embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*)$, where $2^* := 2n/(n - 2s)$ denotes the *fractional critical Sobolev exponent*.

For further details on the fractional Sobolev spaces we refer to [6] and to the references therein, while for other details on X and X_0 we refer to [15], where these functional spaces were introduced, and also to [13,14,16,18], where various properties of these spaces were proved.

Finally, our abstract tool for proving the main result of the present paper is the following local minimum result due to Ricceri (see [11] and [9]).

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals, with Ψ coercive and $\Phi(0_E) = \Psi(0_E) = 0$. Further, set*

$$J_\mu := \mu\Psi + \Phi.$$

Then, for each $\sigma > \inf_{u\in X}\Psi(u)$ and each μ satisfying

$$\mu > -\frac{\inf_{u\in\Psi^{-1}((-\infty,\sigma])}\Phi(u)}{\sigma}$$

the restriction of J_μ to $\Psi^{-1}((-\infty, \sigma))$ has a global minimum.

We cite the monograph [7] for related topics on variational methods adopted in this paper and [3–5] for recent nice results in the fractional setting.

3. The main result

Denote by $\mathcal{A}$ the class of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}\frac{|f(t)|}{1+|t|^{\gamma-1}}<+\infty$$

for some $\gamma\in[1,2^*)$. Further, if $f\in\mathcal{A}$ we put

$$F(\xi):=\int_0^\xi f(t)\,\mathrm{d}t$$

for every $\xi\in\mathbb{R}$.

Let $h\in L^\infty(\Omega)$ and $f\in\mathcal{A}$. Consider the fractional problem

$$\begin{cases}-\mathcal{L}_K u=h(x)f(u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases}\tag{$P_K^{h,f}$}$$

We recall that a *weak solution* of problem ($P_K^{h,f}$) is a function $u\in X_0$ such that

$$\int_Q\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega h(x)f\big(u(x)\big)\varphi(x)\,\mathrm{d}x$$

for every $\varphi\in X_0$.

We observe that problem ($P_K^{h,f}$) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $J_K:X_0\to\mathbb{R}$ defined as follows

$$J_K(u):=\frac{1}{2}\|u\|_{X_0}^2-\int_\Omega h(x)F\big(u(x)\big)\,\mathrm{d}x.$$

Note that the functional J_K is Fréchet differentiable in $u\in X_0$ and one has

$$\big\langle J_K'(u),\varphi\big\rangle=\int_Q\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega h(x)f\big(u(x)\big)\varphi(x)\,\mathrm{d}x$$

for every $\varphi\in X_0$.

Thus, critical points of J_K are solutions to problem ($P_K^{h,f}$). In order to find these critical points, we will make use of Theorem 2.1.

Notations. Let $0\leqslant a<b\leqslant+\infty$. If $\lambda\in[a,b]$ and $\varphi,\psi:\mathbb{R}\to\mathbb{R}$ are two assigned functions, we set

$$g_\lambda^{\varphi,\psi}:=\lambda\psi-\varphi.$$

Denote

$$M(\varphi,\psi,\lambda):=\begin{cases}\text{the set of global minima of } g_\lambda^{\varphi,\psi} & \text{if } \lambda<+\infty,\\ \emptyset & \text{if } \lambda=+\infty,\end{cases}$$

$$\alpha(\varphi,\psi,b):=\max\Big\{\inf_{s\in\mathbb{R}}\psi(s),\ \sup_{s\in M(\varphi,\psi,b)}\psi(s)\Big\}$$

and

$$\beta(\varphi,\psi,a) := \min\Big\{\sup_{s\in\mathbb{R}}\psi(s), \inf_{s\in M(\varphi,\psi,b)}\psi(s)\Big\},$$

adopting the conventions $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$.

Further, if $q \in [1, 2^*)$, we denote by $\mathfrak{F}_q$ the family of all the lower semicontinuous functions $\psi : \mathbb{R} \to \mathbb{R}$ such that:

(i_1) $\sup_{s\in\mathbb{R}}\psi(s) > 0$;
(i_2) $\inf_{s\in\mathbb{R}}\frac{\psi(s)}{1+|s|^q} > -\infty$;
(i_3) $\gamma_\psi := \sup_{s\in\mathbb{R}\setminus\{0\}}\frac{\psi(s)}{|s|^q} < +\infty$.

The next result, that can be viewed as a special case of [10, Theorem 1], will be crucial in the proof of the main theorem.

Proposition 3.1. *Let $\varphi,\psi : \mathbb{R} \to \mathbb{R}$ be two functions such that, for each $\lambda \in (a,b)$, the function $g_\lambda^{\varphi,\psi}$ is lower semicontinuous, coercive and has a global minimum in $\mathbb{R}$. Assume that*

$$\alpha(\varphi,\psi,b) < \beta(\varphi,\psi,a).$$

Then, for each

$$r \in \big(\alpha(F,\psi,b), \beta(F,\psi,a)\big),$$

there exists $\lambda_r \in (a,b)$, such that the unique global minimum of $g_{\lambda_r}^{\varphi,\psi}$ lies in $\psi^{-1}(r)$.

Set

$$c_q := \sup_{u\in X_0\setminus\{0_{X_0}\}}\frac{\|u\|^q_{L^q(\Omega)}}{\|u\|^q_{X_0}}.$$

Note that, since $X_0 \hookrightarrow L^q(\Omega)$ continuously, clearly $c_q < +\infty$.

With the above notations our result reads as follows.

Theorem 3.2. *Let $s \in (0,1)$, $n > 2s$ and let Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary and $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a map satisfying* (k_1)–(k_3). *Moreover, let $f \in \mathcal{A}$ and $h \in L^\infty_+(\Omega)\setminus\{0\}$. Assume that there exists $\psi \in \mathfrak{F}_q$ such that, for each $\lambda \in (a,b)$, the function $g_\lambda^{F,\psi}$ is coercive and has a unique global minimum in $\mathbb{R}$. Further, suppose that there exists a number $r > 0$ satisfying*

$$r \in \big(\alpha(F,\psi,b), \beta(F,\psi,a)\big)$$

and

$$\sup_{\xi\in\psi^{-1}(r)} F(\xi) < \frac{r^{2/q}}{2(c_q\gamma_\psi \operatorname{esssup}_{x\in\Omega} h(x))^{2/q}\|h\|^{(q-2)/q}_{L^1(\Omega)}}. \tag{2}$$

Then, problem $(P_K^{h,f})$ *admits at least one weak solution which is a local minimum of the energy functional* J_K *and satisfies*

$$\int_Q \left|v(x)-v(y)\right|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y < \left(\frac{r\|h\|_{L^1(\Omega)}}{c_q\gamma_\psi \operatorname{esssup}_{x\in\Omega} h(x)}\right)^{2/q}.$$

Proof. Let us apply Theorem 2.1 by choosing $E := X_0$, and

$$\Phi(u) := -\int_\Omega h(x)F\bigl(u(x)\bigr)\,\mathrm{d}x, \qquad \Psi(u) := \|u\|_{X_0}^2$$

for every $u \in E$.

Set

$$\sigma := \left(\frac{r\|h\|_{L^1(\Omega)}}{c_q\gamma_\psi \operatorname{esssup}_{x\in\Omega} h(x)}\right)^{2/q}. \tag{3}$$

We claim that

$$\Psi^{-1}\bigl((-\infty,\sigma]\bigr) \subseteq D, \tag{4}$$

where

$$D := \left\{u \in L^q(\Omega)\colon \int_\Omega h(x)\psi\bigl(u(x)\bigr)\,\mathrm{d}x \leqslant r\|h\|_{L^1(\Omega)}\right\}.$$

Indeed, since $X_0 \hookrightarrow L^q(\Omega)$, it follows that

$$\Psi^{-1}\bigl((-\infty,\sigma]\bigr) \subseteq \bigl\{u \in L^q(\Omega)\colon \|u\|_{L^q(\Omega)} \leqslant c_q^{1/q}\sqrt{\sigma}\bigr\}. \tag{5}$$

On the other hand, taking into account that $\psi \in \mathfrak{F}_q$, one also has

$$\int_\Omega h(x)\psi\bigl(u(x)\bigr)\,\mathrm{d}x \leqslant \gamma_\psi \|u\|_{L^q(\Omega)}^q \operatorname*{esssup}_{x\in\Omega} h(x). \tag{6}$$

Hence, inclusion (4) follows from inequalities (5) and (6).

Now, for each parameter $\lambda \in (a,b)$ denote by $\xi_\lambda^\star$ the unique global minimum (in $\mathbb{R}$) of the real function $g_\lambda^{F,\psi}$.

By Lemma 3.1, since by assumption

$$r \in \bigl(\alpha(F,\psi,b), \beta(F,\psi,a)\bigr),$$

there exists $\lambda_r \in (a,b)$ such that $\psi(\xi_{\lambda_r}^\star) = r$.

Hence, since

$$g_{\lambda_r}^{F,\psi}\bigl(\xi_{\lambda_r}^\star\bigr) \leqslant g_{\lambda_r}^{F,\psi}(\xi) \tag{7}$$

for every $\xi \in \mathbb{R}$, it follows that

$$F\big(\xi^{\star}_{\lambda_r}\big) = \sup_{\xi \in \psi^{-1}(r)} F(\xi). \tag{8}$$

Bearing in mind that h is nonnegative, by (7) one has

$$g_{\lambda}^{F,\psi}\big(\xi^{\star}_{\lambda_r}\big)h(x) \leqslant h(x)g_{\lambda_r}^{F,\psi}(\xi) \tag{9}$$

for a.e. $x \in \Omega$. Hence, inequality (9) implies that

$$g_{\lambda_r}^{F,\psi}\big(\xi^{\star}_{\lambda_r}\big)\|h\|_{L^1(\Omega)} \leqslant \int_{\Omega} h(x)g_{\lambda_r}^{F,\psi}\big(u(x)\big)\,\mathrm{d}x \tag{10}$$

for every $u \in L^q(\Omega)$.

Exploiting (10), for every $u \in D$, one has

$$\int_{\Omega} h(x)F\big(u(x)\big)\,\mathrm{d}x \leqslant F\big(\xi^{\star}_{\lambda_r}\big)\|h\|_{L^1(\Omega)}.$$

Owing to (8), the above inequality assumes the form

$$\int_{\Omega} h(x)F\big(u(x)\big)\,\mathrm{d}x \leqslant \sup_{\xi \in \psi^{-1}(r)} F(\xi)\|h\|_{L^1(\Omega)} \tag{11}$$

for every $u \in D$.

Observing that

$$r = \frac{c_q\gamma_\psi\sigma^{q/2}\operatorname{esssup}_{x\in\Omega} h(x)}{\|h\|_{L^1(\Omega)}},$$

and since inclusion (4) holds, it follows that

$$\sup_{u \in \Psi^{-1}((-\infty,\sigma])} \int_{\Omega} h(x)F\big(u(x)\big)\,\mathrm{d}x \leqslant \sup_{\xi \in \psi^{-1}(r)} F(\xi)\|h\|_{L^1(\Omega)}. \tag{12}$$

Finally, relations (2) and (12) yield

$$\frac{\sup_{u\in\Psi^{-1}((-\infty,\sigma])}\int_{\Omega} h(x)F(u(x))\,\mathrm{d}x}{\sigma} < \frac{1}{2},$$

that is,

$$\frac{1}{2} > -\frac{\inf_{u\in\Psi^{-1}((-\infty,\sigma])}\Phi(u)}{\sigma}.$$

Then, the assertion of Theorem 2.1 follows and the existence of one weak solution $u \in \Psi^{-1}((-\infty,\sigma))$ to our problem is established. □

Remark 3.3. The above existence theorem extends to the nonlocal setting some results, already known in the literature in the case of the classical p-Laplace operator (see [12]).

4. The fractional Laplacian case

As observed in Section 2, by taking $K(x) := |x|^{-(n+2s)}$, the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish almost everywhere outside Ω; see [18, Lemma 7].

In this case $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

By [16, Proposition 9 and Appendix A], we a variational characterization of the first eigenvalue (denoted by $\lambda_{1,s}$) of the problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

as follows

$$\lambda_{1,s} = \min_{u \in X_0 \setminus \{0_{X_0}\}} \frac{\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega u(x)^2\,\mathrm{d}x}. \tag{13}$$

In the sequel it will be useful the following regularity result for the eigenvalues of $(-\Delta)^s$ proved in [19, Theorem 1]. See also [13, Proposition 2.4] for related topics.

Proposition 4.1. *Let* $e \in X_0$ *and* $\lambda > 0$ *be such that*

$$\langle e, \varphi \rangle_{X_0} = \lambda \int_\Omega e(x)\varphi(x)\,\mathrm{d}x$$

for every $\varphi \in X_0$. *Then* $e \in C^{0,\alpha}(\bar{\Omega})$, *for some* $\alpha \in (0,1)$, *i.e. the function* e *is Hölder continuous up to the boundary.*

Taking into account the above facts, a meaningful consequence of Theorem 3.2 is the following one.

Theorem 4.2. *Let* $s \in (0,1)$, $n > 2s$ *and* Ω *be an open bounded set of* $\mathbb{R}^n$ *with Lipschitz boundary. Moreover, let* $h \in L^\infty(\Omega) \setminus \{0\}$ *with* $\operatorname{ess\,inf}_{x\in\Omega} h(x) > 0$ *and let* $f : [0,+\infty) \to [0,+\infty)$ *be a continuous function satisfying the following hypotheses*:

(h_1) *for some* $q \in [1, 2^*)$ *the function*

$$t \mapsto \frac{f(t)}{t^{q-1}}$$

is strictly decreasing in $(0,+\infty)$ *and* $\lim_{t\to+\infty} \frac{f(t)}{t^{q-1}} = 0$;

(h_2) *one has*

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2}>\frac{\lambda_{1,s}}{2\operatorname{ess\,inf}_{x\in\Omega}h(x)};$$

(h_3) *there exists* $\xi_0>0$ *such that*

$$F(\xi_0)<\frac{\xi_0^2}{2(c_q\operatorname{esssup}_{x\in\Omega}h(x))^{2/q}\|h\|_{L^1(\Omega)}^{(q-2)/q}}.$$

Then, the problem

$$\int_{\mathbb{R}^{2n}}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega h(x)f\bigl(u(x)\bigr)\varphi(x)\,\mathrm{d}x$$

for every

$$\varphi\in H^s\bigl(\mathbb{R}^n\bigr)\quad\text{such that}\quad\varphi=0\ \text{a.e. in}\ \mathbb{R}^n\setminus\Omega,$$

admits at least one nonnegative and nonzero weak solution $u\in H^s(\mathbb{R}^n)$, *such that* $u=0$ *a.e. in* $\mathbb{R}^n\setminus\Omega$. *Moreover,* u *is a local minimum of the energy functional* J_K *and satisfies*

$$\int_{\mathbb{R}^{2n}}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y<\left(\frac{\|h\|_{L^1(\Omega)}}{c_q\operatorname{esssup}_{x\in\Omega}h(x)}\right)^{2/q}\xi_0^2.\tag{14}$$

Proof. Let us define

$$\widetilde{f}(t):=\begin{cases}f(t) & \text{if } t\geqslant 0,\\ f(0) & \text{if } t<0,\end{cases}$$

and consider the following problem

$$\begin{cases}(-\Delta)^s u=h(x)\widetilde{f}(u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases}\tag{$D^h_{\widetilde{f}}$}$$

By [17, Lemma 6] every weak solution of problem ($D^h_{\widetilde{f}}$) is nonnegative in Ω. Furthermore, every nonnegative solution of problem ($D^h_{\widetilde{f}}$) also solves our initial problem. Taking $a:=0$ and $b:=+\infty$, and by exploiting Theorem 2.1 with

$$\psi(t):=|t|^q,\quad\forall t\in\mathbb{R},$$

substantially arguing as in [12], the existence of one weak solution of problem ($D^h_{\widetilde{f}}$) which is a local minimum of the associated energy functional (namely $\widetilde{J}_K$) satisfying (14) is established.

In conclusion, we prove that 0_{X_0} is not a local minimum of $\widetilde{J}_K$, i.e. the obtained solution is nonzero.

For this purpose, let us observe that the first eigenfunction $e_1 \in X_0$ is positive in Ω, see [17, Corollary 8], and it follows by (13) that

$$\|e_1\|_{X_0}^2 = \lambda_1 \int_\Omega e_1(x)^2 \, \mathrm{d}x. \tag{15}$$

Thanks to (h_2), there exists $\delta > 0$ such that

$$F(\xi) > \frac{\lambda_{1,s}}{2 \operatorname{ess\,inf}_{x\in\Omega} h(x)} \xi^2$$

for every $\xi \in (0, \delta)$.

Now, by Proposition 4.1, one has that $e_1 \in C^{0,\alpha}(\bar{\Omega})$. Hence, we can define $\theta_\eta(x) := \eta e_1(x)$, for every $x \in \Omega$, where

$$\eta \in \Lambda_\delta := \left(0, \frac{\delta}{\max_{x\in\Omega} e_1(x)}\right).$$

Taking into account (15), we easily get

$$\int_\Omega h(x) F\bigl(\theta_\eta(x)\bigr) \, \mathrm{d}x > \frac{\lambda_1 \int_\Omega h(x)\theta_\eta(x)^2 \, \mathrm{d}x}{2 \operatorname{ess\,inf}_{x\in\Omega} h(x)} \geqslant \frac{\lambda_1}{2} \int_\Omega \theta_\eta(x)^2 \, \mathrm{d}x = \frac{1}{2}\|\theta_\eta\|_{X_0}^2,$$

that is,

$$\widetilde{J}_K(\theta_\eta) = \frac{1}{2}\|\theta_\eta\|_{X_0}^2 - \int_\Omega h(x) F\bigl(\theta_\eta(x)\bigr) \, \mathrm{d}x < 0$$

for every $\eta \in \Lambda_\delta$.

The proof is thus complete. □

It is easy to see that Theorem 1.1 in the Introduction is a consequence of Theorem 4.2. A direct application of this result reads as follows.

Example 4.3. Let $s \in (0, 1)$, $n > 2s$ and let Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. By virtue of Theorem 4.2, for every $\alpha > \lambda_{1,s}$, the following nonlocal problem

$$\begin{cases} (-\Delta)^s u = \dfrac{\alpha u}{1 + u^2} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

admits at least one nonnegative and nonzero weak solution $u_\alpha \in H^s(\mathbb{R}^n)$, such that $u_\alpha = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

Remark 4.4. We observe that a special case of our results ensures that if $f:[0,+\infty)\to[0,+\infty)$ is any positive C^1-function such that $f(0)=0$, $f(t)/t$ is strictly decreasing in $(0,+\infty)$, $f(t)/t\to 0$ as $t\to+\infty$ and $f'(0)>\lambda_1$, then the following problem

$$\begin{cases}(-\Delta)^s u = f(u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega,\end{cases}$$

admits at least one nonnegative and nonzero weak solution $u\in H^s(\mathbb{R}^n)$, such that $u=0$ a.e. in $\mathbb{R}^n\setminus\Omega$.

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