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Higher nonlocal problems with bounded potential

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A R T I C L E I N F O

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A B S T R A C T

The aim of this paper is to study a class of nonlocal fractional Laplacian equations depending on two real parameters. More precisely, by using an appropriate analytical context on fractional Sobolev spaces due to Servadei and Valdinoci, we establish the existence of three weak solutions for nonlocal fractional problems exploiting an abstract critical point result for smooth functionals. We emphasize that the dependence of the underlying equation from one of the real parameters is not necessarily of affine type.

1. Introduction

This paper is devoted to the following two-parameter nonlocal problem, namely $(P^{\mu,\lambda,h}_{M,K,f})$:

$$\begin{cases} -M\left(\|u\|^2_{X_0}\right)\mathcal{L}_K u = \mu h\left(\displaystyle\int_\Omega \left(\int_0^{u(x)} f(x,t)dt\right)dx - \lambda\right) f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Here and in the sequel, Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with $n > 2s$ (where $s \in (0,1)$), smooth (Lipschitz) boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth, λ and μ are real parameters, M, h are two suitable continuous functions and

$$\|u\|^2_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \left|u(x) - u(y)\right|^2 K(x-y)dxdy.$$

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Further, $\mathcal{L}_K$ is a nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \big(u(x+y)+u(x-y)-2u(x)\big)K(y)dy \quad \big(x \in \mathbb{R}^n\big)$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function with the properties that:

(k_1) $\gamma K \in L^1(\mathbb{R}^n)$, *where* $\gamma(x) := \min\{|x|^2, 1\}$;
(k_2) *there exists* $\beta > 0$ *such that*

$$K(x) \geq \beta |x|^{-(n+2s)},$$

for any $x \in \mathbb{R}^n \setminus \{0\}$;
(k_3) $K(x) = K(-x)$, *for any* $x \in \mathbb{R}^n \setminus \{0\}$.

A typical example of the kernel K is given by $K(x) := |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n.$$

Problem $(P^{\mu,\lambda,h}_{M,K,f})$ is clearly highly nonlocal due to the presence of the fractional operator $\mathcal{L}_K$ and to the map M as well as in the source term f. In our context, to avoid some additional technical difficulties originated by the presence of the term

$$M\bigg(\int_{\mathbb{R}^n\times\mathbb{R}^n} \big|u(x)-u(y)\big|^2 K(x-y)dxdy\bigg),$$

we impose some restrictions on the behavior of M (see Section 3).

This setting includes the Kirchhoff-type problem of the form

$$\begin{cases} -\big(a + b\|u\|^2_{X_0}\big)\mathcal{L}_K u = v(\mu,\lambda,h,f) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $a, b > 0$ and

$$v(\mu,\lambda,h,f) := \mu h\Bigg(\int_\Omega \bigg(\int_0^{u(x)} f(x,t)dt\bigg)dx - \lambda\Bigg) f(x,u),$$

see Remark 3.3.

For completeness, in the vast literature on this subject, we refer the reader to some interesting recent results (in the non-fractional setting) obtained by Autuori and Pucci in [1–3] studying Kirchhoff equations by using different approaches.

We also mention that the same authors studied in [5] the existence and multiplicity of solutions for elliptic equations in $\mathbb{R}^n$, driven by a nonlocal integro-differential operator whose standard prototype is the fractional Laplacian (this work is related to the results on general quasilinear elliptic problems given in [4]).

We seek conditions on the data for which problem $(P^{\mu,\lambda,h}_{M,K,f})$ possesses at least three weak solutions. It is worth pointing out that the variational approach to attack such problems is not often easy to perform;

indeed due to the presence of the nonlocal term, variational methods do not to work when applied to these classes of equations.

Fortunately, our approach here is realizable by checking that the associated energy functional (see Section 3) given by

$$J_K(u) := \frac{1}{2}\widehat{M}\left(\|u\|_{X_0}^2\right) - \mu H\left(\int_\Omega F\left(x, u(x)\right)dx - \lambda\right),$$

satisfies the assumptions requested by a recent critical point theorem (see Theorem 2.1 below) obtained by Ricceri in [13, Theorem 1.6] and thanks to a suitable framework developed in [16].

We emphasize that in [13, Theorem 1.6] Ricceri established a theorem tailor-made for a class of nonlocal problems involving nonlinearities with bounded primitive. This result follows from [12, Theorem 3] and the main novelty obtained in the most recent paper [13] is that, in contrast with a large part of the existing literature, the abstract energy functional does not depend on the parameter λ in an affine way.

The nonlocal analysis (see Section 2) that we perform here in order to use Theorem 2.1 is quite general and has been successfully exploited for other goals in several recent contributions; see [15–18] and [9] for an elementary introduction to this topic and for a list of related references.

In the nonlocal framework, the simplest example we can deal with is given by the fractional Laplacian, according to the following result.

Theorem 1.1. *Let $s \in (0,1)$, $n > 2s$ and let Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero continuous function such that*

$$\sup_{\xi\in\mathbb{R}}\left|F(\xi)\right| < +\infty,$$

where $F(\xi) := \int_0^\xi f(t)\,dt$, for every $\xi \in \mathbb{R}$. Further, let

$$h : \left(-|\Omega| \operatorname{osc}_{\xi\in\mathbb{R}} F(\xi), |\Omega| \operatorname{osc}_{\xi\in\mathbb{R}} F(\xi)\right) \to \mathbb{R}$$

be a continuous and non-decreasing function such that $h^{-1}(0) = \{0\}$.

Then, fixing $a, b > 0$, for each μ sufficiently large, there exists an open interval

$$\Lambda \subseteq \left(|\Omega| \inf_{\xi\in\mathbb{R}} F(\xi), |\Omega| \sup_{\xi\in\mathbb{R}} F(\xi)\right)$$

and a number $\rho > 0$ such that, for every $\lambda \in \Lambda$, the following equation

$$\begin{aligned}\left(a + b\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy\right)&\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy\\ &= \mu h\left(\int_\Omega F\left(u(x)\right)dx - \lambda\right)\int_\Omega f\left(u(x)\right)\varphi(x)dx,\end{aligned}$$

for every

$$\varphi \in H^s\left(\mathbb{R}^n\right) \quad \text{such that } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega,$$

has at least three distinct weak solutions $\{u_j\}_{j=1}^3 \subset H^s(\mathbb{R}^n)$, such that $u_j = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, and

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dxdy < \rho^2,$$

for every $j \in \{1,2,3\}$.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework and preliminaries. Successively, in Section 3 we give the main result; see Theorem 3.1. Finally, a concrete example of an application is presented in Example 3.5.

We cite the monograph [10] for related topics on variational methods adopted in this paper and [6–8] for recent nice results in the fractional setting.

2. Variational framework

In this section we briefly recall the definition of the functional space X_0, firstly introduced in [15,16]. The reader familiar with this topic may skip this section and go directly to the next one. The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\big((x,y)\mapsto \big(g(x)-g(y)\big)\sqrt{K(x-y)}\,\big) \in L^2\big(\big(\mathbb{R}^n\times\mathbb{R}^n\big)\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega), dxdy\big),$$

where $\mathcal{C}\Omega := \mathbb{R}^n\setminus\Omega$. We denote by X_0 the following linear subspace of X

$$X_0 := \big\{g\in X : g = 0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega\big\}.$$

We remark that X and X_0 are non-empty, since $C_0^2(\Omega)\subseteq X_0$ by [15, Lemma 5.1].

Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left(\int\limits_Q \big|g(x)-g(y)\big|^2 K(x-y)dxdy\right)^{1/2},$$

where $Q := (\mathbb{R}^n\times\mathbb{R}^n)\setminus\mathcal{O}$ and $\mathcal{O} := (\mathcal{C}\Omega)\times(\mathcal{C}\Omega)\subset\mathbb{R}^n\times\mathbb{R}^n$. It is easily seen that $\|\cdot\|_X$ is a norm on X; see [16].

By [16, Lemmas 6 and 7] in the sequel we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} := \left(\int\limits_Q \big|v(x)-v(y)\big|^2 K(x-y)dxdy\right)^{1/2} \tag{1}$$

as a norm on X_0. Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u,v\rangle_{X_0} := \int\limits_Q \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,K(x-y)dxdy,$$

see [16, Lemma 7].

Note that in (1) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^n\times\mathbb{R}^n$, since $v\in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^n\setminus\Omega$).

While for a general kernel K satisfying conditions from (k_1) to (k_3) we have that $X_0\subset H^s(\mathbb{R}^n)$, in the model case $K(x) := |x|^{-(n+2s)}$ the space X_0 consists of all the functions of the usual fractional Sobolev space $H^s(\mathbb{R}^n)$ which vanish a.e. outside Ω; see [18, Lemma 7].

Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} + \left(\int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2}.$$

Before concluding this subsection, we recall the embedding properties of X_0 into the usual Lebesgue spaces; see [16, Lemma 8]. The embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^n)$ is continuous for any $\nu \in [1, 2^*]$, while it is compact whenever $\nu \in [1, 2^*)$, where $2^* := 2n/(n - 2s)$ denotes the *fractional critical Sobolev exponent.*

For further details on the fractional Sobolev spaces we refer to [9] and to the references therein, while for other details on X and X_0 we refer to [15], where these functional spaces were introduced, and also to [14,16–18], where various properties of these spaces were proved.

Finally, our abstract tool for proving the main result of the present paper is [13, Theorem 1.6] that we recall here for reader's convenience.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a separable and reflexive real Banach space and let $\eta, J : E \to \mathbb{R}$ be two C^1-functionals with compact derivative and $J(0_E) = \eta(0_E) = 0$. Assume also that J is bounded and non-constant, and that η is bounded above. Then, for every sequentially weakly lower semicontinuous and coercive C^1-functional $\psi : E \to \mathbb{R}$ whose derivative admits a continuous inverse on E^* and with $\psi(0_E) = 0$, for every convex C^1-function*

$$\varphi : \big(-\operatorname{osc}_{u\in E} J(u), \operatorname{osc}_{u\in E} J(u)\big) \to [0, +\infty),$$

with $\varphi^{-1}(0) = \{0\}$, for which the number

$$\theta^\star := \inf_{u \in J^{-1}((\inf_{u\in E} J(u), \sup_{u\in E} J(u)) \setminus \{0\})} \frac{\psi(u) - \eta(u)}{\varphi(J(u))}$$

is non-negative, and for every $\mu > \theta^\star$ there exists an open interval

$$\Lambda \subseteq \Big(\inf_{u\in E} J(u), \sup_{u\in E} J(u)\Big)$$

and a number $\rho > 0$ such that, for each $\lambda \in \Lambda$, the equation

$$\psi'(u) = \mu\varphi'\big(J(u) - \lambda\big) J'(u) + \eta'(u)$$

has at least three distinct solutions whose norms are less than ρ.

Remark 2.2. Note that, for a generic function $\psi : E \to \mathbb{R}$, the symbol $\operatorname{osc}_{u\in E} \psi(u)$ denotes the number (possibly infinite) given by

$$\operatorname{osc}_{u\in E} \psi(u) := \sup_{u\in E} \psi(u) - \inf_{u\in E} \psi(u).$$

Moreover, if ψ is a C^1-functional, we say that the derivative ψ' admits a continuous inverse on E^* provided that there exists a continuous operator $T : E \to E^*$ such that

$$T\big(\psi'(u)\big) = u,$$

for every $u \in E$.

3. The main result

Let $\mathcal{M}$ be the class of continuous functions $M : [0, +\infty) \to \mathbb{R}$ such that:

(C_M^1) $\inf_{t \geq 0} M(t) > 0$;
(C_M^2) *there exists a continuous function* $v_M : [0, +\infty) \to \mathbb{R}$ *such that*

$$v_M\big(tM\big(t^2\big)\big) = t,$$

for every $t \in [0, +\infty)$.

Further, if $M \in \mathcal{M}$, set

$$\widehat{M}(t) := \int_0^t M(s)ds,$$

for every $t \in [0, +\infty)$.

Denote by $\mathcal{A}$ the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{(x,t) \in \Omega \times \mathbb{R}} \frac{|f(x,t)|}{1 + |t|^{q-1}} < +\infty,$$

for some $q \in [1, 2^*)$. Further, if $f \in \mathcal{A}$ we put

$$F(x, \xi) := \int_0^\xi f(x,t)\,dt,$$

for every $(x, \xi) \in \Omega \times \mathbb{R}$.

We recall that a *weak solution* of problem $(P_{M,K,f}^{\mu,\lambda,h})$ is a function $u \in X_0$ such that

$$M\big(\|u\|_{X_0}^2\big) \int_Q \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)dxdy = \mu h\bigg(\int_\Omega F\big(x, u(x)\big)dx - \lambda\bigg) \int_\Omega f\big(x, u(x)\big)\varphi(x)dx,$$

for every $\varphi \in X_0$.

For the proof of our result, we observe that problem $(P_{k,K,f}^{\mu,\lambda,h})$ has a variational structure, indeed it is the Euler–Lagrange equation of the functional $J_K : X_0 \to \mathbb{R}$ defined as follows

$$J_K(u) := \frac{1}{2}\widehat{M}\big(\|u\|_{X_0}^2\big) - \mu H\bigg(\int_\Omega F\big(x, u(x)\big)dx - \lambda\bigg),$$

where

$$H(\xi) := \int_0^\xi h(t)dt,$$

for every $\xi \in \mathbb{R}$.

Note that the functional J_K is Fréchet differentiable in $u \in X_0$ and one has

$$\langle J_K'(u), \varphi\rangle = M\big(\|u\|_{X_0}^2\big) \int_Q \big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x - y)dxdy$$
$$- \mu h\bigg(\int_\Omega F\big(x, u(x)\big)dx - \lambda\bigg) \int_\Omega f\big(x, u(x)\big)\varphi(x)dx,$$

for every $\varphi \in X_0$.

Thus, critical points of J_K are solutions to problem $(P_{M,K,f}^{\mu,\lambda,h})$. In order to find these critical points, we will make use of Theorem 2.1.

Let us denote by

$$\alpha_f := \inf_{u\in X_0} \int_\Omega F\big(x, u(x)\big)dx, \qquad \beta_f := \sup_{u\in X_0} \int_\Omega F\big(x, u(x)\big)dx,$$

and

$$\omega_f := \beta_f - \alpha_f.$$

Finally, let

$$\mathcal{R} := \bigg\{u \in X_0, \int_\Omega F\big(x, u(x)\big)dx \in (\alpha_f, \beta_f) \setminus \{0\}\bigg\}.$$

With the above notations our result reads as follows.

Theorem 3.1. *Let $s \in (0, 1)$, $n > 2s$ and let Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ be a map satisfying* (k_1)–(k_3)*. Moreover, let $f \in \mathcal{A}$ be such that*

$$\sup_{(x,\xi)\in\Omega\times\mathbb{R}} \big|F(x, \xi)\big| < +\infty, \tag{2}$$

and

$$\sup_{u\in X_0} \bigg|\int_\Omega F\big(x, u(x)\big)dx\bigg| > 0.$$

Then, for every $M \in \mathcal{M}$ and every non-decreasing function $h : (-\omega_f, \omega_f) \to \mathbb{R}$ with $h^{-1}(0) = \{0\}$, for every

$$\mu > \inf_{u\in\mathcal{R}} \left\{\frac{\widehat{M}(\|u\|_{X_0}^2)}{2H(\int_\Omega F(x, u(x))dx)}\right\}$$

there exists an open interval $\Lambda \subseteq (\alpha_f, \beta_f)$ and a number $\rho > 0$ such that, for each $\lambda \in \Lambda$, the problem $(P_{k,K,f}^{\mu,\lambda,h})$ has at least three distinct weak solutions whose norms in X_0 are less than ρ.

Proof. Let us apply Theorem 2.1 by choosing $E := X_0$, $\eta = 0$, and

$$J(u) := \int_\Omega F\big(x, u(x)\big)\, dx, \qquad \psi(u) := \frac{1}{2}\widehat{M}\big(\|u\|_{X_0}^2\big)$$

for every $u \in E$.

Since $f \in \mathcal{A}$, the functional J is a C^1-functional with compact derivative (note that the embedding $j : E \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, 2^*)$). Furthermore (by (2)) J is clearly bounded.

Now, it is easy to see that ψ is a C^1-functional and, since $\widehat{M}$ is increasing, ψ is also sequentially weakly lower semicontinuous.

Let us prove that the derivative $\psi' : E \to E^*$ admits a continuous inverse. Since E is reflexive, we identify E with the topological dual E^*. For our goal, let $T : E \to E$ be the operator defined by

$$T(v) := \begin{cases} \frac{v_M(\|v\|_{X_0})}{\|v\|_{X_0}} v & \text{if } v \neq 0 \\ 0 & \text{if } v = 0, \end{cases}$$

where v_M appears in (C_M^2).

Thanks to the continuity of v_M and $v_M(0) = 0$, the operator T is continuous in E.

Moreover, for each $E \setminus \{0_E\}$, since $M(\|u\|_{X_0}^2) > 0$ (by (C_M^1)), one has

$$T\big(\psi'(u)\big) = T\big(M\big(\|u\|_{X_0}^2\big)u\big) = \frac{v_M(M(\|u\|_{X_0}^2)\|u\|_{X_0})}{M(\|u\|_{X_0}^2)\|u\|_{X_0}} M\big(\|u\|_{X_0}^2\big)u = u,$$

as desired.

Now, put

$$\gamma := \inf_{t \geq 0} M(t).$$

So, $\gamma > 0$ (by (C_M^1)) and

$$\widehat{M}(t) \geq \gamma t,$$

for every $t \in [0, +\infty)$. In particular, this implies that ψ is coercive.

In conclusion, let us take $\varphi := H$. By our assumptions on h, it follows that the function φ is non-negative, convex and $\varphi^{-1}(0) = \{0\}$.

Then, the assertion of Theorem 2.1 follows and the existence of three weak solutions to our problem is established. $\square$

Remark 3.2. Clearly, if the function M is non-decreasing in $[0, +\infty)$, with $M(0) > 0$, then the function $t \to tM(t^2)$ $(t \geq 0)$ is increasing and onto on $[0, +\infty)$, and so condition (C_M^2) is satisfied. Taking into account the previous observations one can conclude that Theorem 3.1 is the (non-perturbed) fractional analogous of [13, Theorem 1.3] in which a nonlocal Dirichlet problem, in the classical framework, was studied. If $g \in \mathcal{A}$, in analogy with the cited result, we point out that in Theorem 3.1, requiring that

$$\sup_{(x,\xi) \in \Omega \times \mathbb{R}} \max\left\{ \big|F(x,\xi)\big|, \int_0^\xi g(x,t)dt \right\} < +\infty,$$

instead of (2), we have that for every $M \in \mathcal{M}$ and every non-decreasing function $h : (-\omega_f, \omega_f) \to \mathbb{R}$ with $h^{-1}(0) = \{0\}$, for which the number

$$\theta^* := \inf_{u \in \mathcal{R}} \left\{ \frac{\widehat{M}(\|u\|_{X_0}^2) - 2\int_\Omega (\int_0^{u(x)} g(x,t)dt)dx}{2H(\int_\Omega F(x,u(x))dx)} \right\}$$

is non-negative, and for every $\mu > \theta^*$, there exists an open interval $\Lambda \subseteq (\alpha_f, \beta_f)$ and a number $\rho > 0$ such that, for each $\lambda \in \Lambda$, the perturbed problem

$$\begin{cases} -M\big(\|u\|^2_{X_0}\big)\mathcal{L}_K u = v(\mu,\lambda,h,f,g) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where

$$v(\mu,\lambda,h,f,g) := \mu h\left(\int_\Omega \left(\int_0^{u(x)} f(x,t)dt\right)dx - \lambda\right) f(x,u) + g(x,u)$$

has at least three distinct weak solutions whose norms in X_0 are less than ρ.

Remark 3.3. Fix $a, b > 0$ and take

$$M(t) := a + bt,$$

for every $t \in [0, +\infty)$. Clearly condition (C^1_M) and (C^2_M) hold. Thus, as claimed in Introduction and bearing in mind Remark 3.2, Theorem 3.1 produces the existence of multiple weak solutions for the following fractional Kirchhoff-type problem depending on two parameters:

$$\begin{cases} -\big(a + b\|u\|^2_{X_0}\big)\mathcal{L}_K u = v(\mu,\lambda,h,f,g) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Remark 3.4. Simple considerations explained in Section 2 prove that Theorem 1.1 in introduction is a consequence of Theorem 3.1.

In conclusion, we present a direct application of the main result.

Example 3.5. Let $s \in (0,1)$, $n > 2s$ and let Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be the non-zero continuous function belonging to $\mathcal{A}$, with

$$\sup_{\xi\in\mathbb{R}} \big|F(\xi)\big| < +\infty$$

and let $M \in \mathcal{M}$. Then, owing to Theorem 3.1, for a sufficiently large μ, there exists an open interval

$$\Lambda \subseteq \left(|\Omega| \inf_{\xi\in\mathbb{R}} F(\xi), |\Omega| \sup_{\xi\in\mathbb{R}} F(\xi)\right)$$

and a number $\rho > 0$ such that, for each $\lambda \in \Lambda$, the following equation

$$\begin{aligned} &M\left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dxdy\right) \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ &\quad = \mu \frac{(\int_\Omega F(u(x))dx - \lambda)\int_\Omega f(u(x))\varphi(x)dx}{(|\Omega| \operatorname{osc}_{\xi\in\mathbb{R}} F(\xi))^2 - (\int_\Omega F(u(x))dx - \lambda)^2}, \end{aligned}$$

for every

$$\varphi \in H^s\big(\mathbb{R}^n\big) \quad \text{such that } \varphi = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega,$$

has at least three distinct solutions $\{u_j\}_{j=1}^3 \subset H^s(\mathbb{R}^n)$, such that $u_j = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, and

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u_j(x) - u_j(y)|^2}{|x - y|^{n+2s}}\, dxdy < \rho^2,$$

for every $j \in \{1, 2, 3\}$.

Remark 3.6. We just observe that [11, Theorem 3.1] cannot be applied to the problem treated in the previous example.

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