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Multiple solutions of *p*-biharmonic equations with Navier boundary conditions

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In this article, exploiting variational methods, the existence of multiple weak solutions for a class of elliptic Navier boundary problems involving the *p*-biharmonic operator is investigated. Moreover, a concrete example of an application is presented.

Keywords: three weak solutions; *p*-biharmonic type operators; Navier boundary value problem; variational methods

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1. Introduction

Motivated by the fact that such kinds of problems are used to describe a large class of physical phenomena, many authors have looked for multiple solutions of elliptic equations involving biharmonic and *p*-biharmonic type operators (see, for instance, [1–5]). In this work we are interested in the existence of multiple weak solutions for the following nonlinear elliptic Navier boundary value problem involving the *p*-biharmonic operator:

$$\left(H_\lambda^f\right)\quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of IR^N with a smooth enough boundary $\partial\Omega$, $p > \max\{1, N/2\}$, Δ is the usual Laplace operator, λ is a positive parameter and f is a suitable continuous function defined on the set $\bar{\Omega} \times \mathrm{IR}$.

For $p=2$, the linear operator $\Delta^2 u := \Delta(\Delta u)$ is the iterated Laplace which multiplied with a positive constant often occurs in Navier–Stokes equations as a viscosity coefficient. Moreover, its reciprocal operator denoted by $(\Delta^2 u)^{-1}$ is the celebrated Green operator [6].

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In [4], a Navier boundary value problem is treated where the left-hand side of the equation involves an operator that is more general than the p-biharmonic. Meanwhile in [7], a concrete example of application of such mathematical model to describe a physical phenomena is also pointed out.

Further, by using the abstract and technical approach developed in [8–10], the authors are interested in looking for the existence of infinitely many weak solutions of perturbed p-biharmonic equations.

Here, requiring a suitable growth of the primitive of f, we are able to establish suitable intervals of values of the parameter λ for which the problem (H_λ^f) admits at least three weak solutions.

More precisely, the main result ensures the existence of two real intervals of parameters Λ_1 and Λ_2 such that, for each $\lambda \in \Lambda_1 \cup \Lambda_2$, the problem (H_λ^f) admits at least three weak solutions whose norms are uniformly bounded with respect to every $\lambda \in \Lambda_2$ (Theorem 3.1).

Our method is mostly based on a useful critical point theorem given in [11, Theorem 3.1] (Theorem 2.1). We also cite a recent monograph by Kristály et al. [12] as a general reference on variational methods adopted here.

The obtained results are related to some recent contributions from [2, Theorem 1] where, by using a critical point result from [13], the existence of at least three weak solutions has been obtained (see also [14, Theorem 1]). We emphasize that, in our cases, on the contrary of the above-mentioned works, we give a qualitative analysis of the real intervals Λ_i $(i = 1, 2)$ for which problem (H_λ^f) admits multiple weak solutions (see, for details, Remarks 2.2 and 2.2).

As an example, we present a special case of our results (see Theorem 3.5 and Remark 3.6 for more details) on the existence of two nontrivial weak solutions.

Theorem 1.1 *Let $p > \max\{1, N/2\}$ and f: $\mathrm{IR} \to [0, +\infty[$ be a continuous function. Hence, consider the following autonomous problem*:

$$\left(G_\lambda^f\right) \quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that there exist real positive constants γ and δ such that

$$F(\gamma) < \kappa_\Omega F(\delta),$$

for some $1 \leq s \leq p$, where κ_Ω is a precise constant depending on the geometry of the open set Ω. Further, we require that

$$\lim_{t \to +\infty} \frac{f(t)}{|t|^{s-1}} = 0,$$

for some $1 \leq s \leq p$.

Then there exist two real intervals of parameters Λ_1' and Λ_2' such that: for every $\lambda \in \Lambda_1'$ problem (G_λ^f) admits two distinct nontrivial weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and, moreover, for each $\lambda \in \Lambda_2'$ there are two distinct nontrivial weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ uniformly bounded in norm with respect to the parameter λ.

See Remarks 2.2 and 3.6 for more details on the intervals Λ_1' and Λ_2' as well as for a concrete expression of the constant κ_Ω.

For completeness, we refer the reader interested in fourth-order two-point boundary value problems to papers [15–18] and references therein.

The plan of this article is as follows. Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main results and their consequences in the autonomous case. A concrete example of an application is then presented (Example 3.7).

2. Preliminaries

Here, and in the sequel, Ω is an open bounded subset of $\mathbb{R}^N$, $p > \max\{1, N/2\}$, while X denotes a separable and reflexive real Banach space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\Delta u(x)|^p dx\right)^{1/p}, \quad \forall\, u \in X. \tag{1}$$

The Rellich–Kondrachov theorem assures that X is compactly imbedded in $C^0(\bar{\Omega})$, whenever

$$k := \sup_{u \in X \setminus \{0\}} \frac{\|u\|_{C^0(\bar{\Omega})}}{\|u\|} < +\infty, \tag{2}$$

where $\|u\|_{C^0(\bar{\Omega})} := \sup_{x \in \bar{\Omega}} |u(x)|$, for every $u \in X$.

Moreover, if $N \geq 3$, $\partial\Omega$ is of class $C^{1,1}$ and $p \in]N/2, +\infty[$, due to Theorem 2 and [19, Remark 1], one has the following upper bound:

$$k \leq \operatorname{meas}(\Omega)^{\frac{2}{N}+\frac{1}{p'}-1} \frac{\Gamma(1+N/2)^{2/N}}{N(N-2)\pi} \left[\frac{\Gamma(1+p')\Gamma(N/(N-2)-p')}{\Gamma(N/(N-2))}\right]^{1/p'},$$

where Γ is the Gamma function, p' the conjugate exponent of p and 'meas(Ω)' denotes the Lebesgue measure of Ω.

For our aim, the main tool is a critical points theorem contained in [11, Theorem 3.1], which we recall here for the reader's convenience.

THEOREM 2.1 *Let X be a separable and reflexive real Banach space; Φ: $X \to \mathbb{R}$ a nonnegative, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Ψ: $X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists an $u_0 \in X$ such that*

$$\Phi(u_0) = \Psi(u_0) = 0,$$

and that

(i) $\lim_{\|u\|\to\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$, *for all* $\lambda \in [0, +\infty[$. *Further, assume that there are $r > 0$ and $\bar{u} \in X$ such that:*
(ii) $r < \Phi(\bar{u})$;
(iii) $\displaystyle\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u) < \frac{r}{r + \Phi(\bar{u})} \Psi(\bar{u}).$

Then, for each

$$\lambda \in \Lambda_1 := \left] \frac{\Phi(\bar{u})}{\Psi(\bar{x}) - \sup\limits_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u)}, \frac{r}{\sup\limits_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u)} \right[,$$

the equation

$$\Phi'(u) - \lambda \Psi'(u) = 0 \tag{3}$$

has at least three distinct solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subset \left[0, \frac{hr}{r\dfrac{\Psi(\bar{u})}{\Phi(\bar{u})} - \sup\limits_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u)} \right],$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, Equation (8) *has at least three solutions in X whose norms are less than σ.*

Note that, in the above result, the symbol $\overline{\Phi^{-1}(] - \infty, r[)}^w$ denotes the weak closure of the sublevel $\Phi^{-1}(]-\infty, r[)$. For completeness, given an operator S: $X \to X^*$, we say that S admits a continuous inverse on X^* if there exists a continuous operator T: $X^* \to X$ such that $T(S(x)) = x$ for all $x \in X$.

Remark 2.2 As observed in [11, Remark 2.1], the real intervals Λ_1 and Λ_2 in Theorem 2.1 are such that either

$$\Lambda_1 \cap \Lambda_2 = \emptyset$$

or

$$\Lambda_1 \cap \Lambda_2 \neq \emptyset.$$

In the first case, we actually obtain two distinct open intervals of positive real parameters for which Equation (8) admits two nontrivial solutions; otherwise, we achieve only one interval of positive real parameters, precisely $\Lambda_1 \cup \Lambda_2$, for which Equation (8) admits three solutions and, in addition, the subinterval Λ_2 for which the solutions are uniformly bounded. We also observe that if the two intervals are disjoint, we do not have information about the number of solutions of Equation (8) within the gap interval.

3. Main results

Let

$$\tau := \sup_{x \in \Omega} \text{dist}(x, \partial\Omega).$$

Simple calculations show that there is an $x_0 \in \Omega$ such that $B(x_0, \tau) \subseteq \Omega$, where $B(x_0, \tau)$ denotes the open ball with centre x_0 and radius τ. Now, fix $\delta > 0$ and consider the function $u_\delta \in X$ defined by

$$u_\delta(x) := \begin{cases} 0 & \text{if } x \in \bar{\Omega} \setminus B(x^0, \tau) \\ 16\dfrac{l^2}{\tau^4}(\tau - l)^2\delta & \text{if } x \in B(x^0, \tau) \setminus B(x^0, \tau/2) \\ \delta & \text{if } x \in B(x^0, \tau/2), \end{cases}$$

where $l := \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$.

At this point, let

$$F(x, \xi) := \int_0^\xi f(x, t)\mathrm{d}t, \quad \forall\ (x, \xi) \in \bar{\Omega} \times \mathrm{IR},$$

and substitute

$$R_F(\tau, \delta) := \int_{B(x^0,\tau)\setminus B(x^0,\tau/2)} F(x, u_\delta(x))\,\mathrm{d}x.$$

Moreover, set

$$\sigma_{p,N}(\tau) := \int_{\tau/2}^{\tau} |2(N+2)s^2 - 3(N+1)\tau s + N\tau^2|^p s^{N-1}\mathrm{d}s.$$

Finally, let us denote

$$K_{p,N}(\tau) := \frac{\tau^{4p}\Gamma(N/2)}{2^{5p+1}\pi^{N/2}k^p\sigma_{p,N}(\tau)},$$

and, for $\gamma > 0$, define

$$\eta(\gamma, \delta) := \frac{\tau^{4p}\Gamma(N/2)\gamma^p}{\tau^{4p}\Gamma(N/2)\gamma^p + k^p 2^{5p+1}\pi^{N/2}\delta^p\sigma_{p,N}(\tau)}.$$

With the above notations, the main result reads as follows.

Theorem 3.1 *Let $f \in C^0(\bar{\Omega} \times \mathrm{IR})$ and substitute*

$$F(x, \xi) := \int_0^\xi f(x, t)\mathrm{d}t, \quad \forall\ (x, \xi) \in \bar{\Omega} \times \mathrm{IR}.$$

Assume that there exist two positive constants γ and δ such that

(h_1) $\delta > K_{p,N}(\tau)^{1/p}\gamma$;

(h_2) *The following inequality holds*:

$$\int_\Omega \max_{|\xi|\leq\gamma} F(x, \xi)\,\mathrm{d}x < \eta(\gamma, \delta)\left(R_F(\tau, \delta) + \int_{B(x^0,\tau/2)} F(x, \delta)\,\mathrm{d}x\right).$$

Further, require that

(h_3) *There exist a function* $\alpha \in L^1(\Omega)$ *and a positive constant* s *with* $s < p$ *such that*

$$F(x, \xi) \leq \alpha(x)(1 + |\xi|^s),$$

for almost every $x \in \Omega$ *and for every* $\xi \in IR$.

Then, for each

$$\lambda \in \Lambda_1 :=]\lambda_1, \lambda_2[,$$

where

$$\lambda_1 := \frac{2^{5p+1}\pi^{N/2}\sigma_{p,N}(\tau)\delta^p}{\tau^{4p}\Gamma(N/2)\,p\left(R_F(\tau,\delta) + \int_{B(x^0,\tau/2)} F(x,\delta)\,\mathrm{d}x - \int_\Omega \max_{|\xi|\leq\gamma} F(x,\xi)\,\mathrm{d}x\right)},$$

and

$$\lambda_2 := \frac{\gamma^p}{pk^p \int_\Omega \max_{|\xi|\leq\gamma} F(x,\xi)\,\mathrm{d}x},$$

problem (H_λ^f) *has at least three distinct solutions in* X *and, moreover, for each* $h > 1$, *there exists an open interval*

$$\Lambda_2 \subset \left[0, \lambda_{3,h}\right],$$

where

$$\lambda_{3,h} := \frac{h\gamma^p/(pk^p)}{\dfrac{\gamma^p\left(R_F(\tau,\delta) + \int_{B(x^0,\tau/2)} F(x,\delta)\,\mathrm{d}x\right)\tau^{4p}\Gamma(N/2)}{2^{5p+1}k^p\pi^{N/2}\sigma_{p,N}(\tau)\delta^p} - \int_\Omega \max_{|\xi|\leq\gamma} F(x,\xi)\,\mathrm{d}x},$$

and a positive real number $\sigma > 0$ *such that, for each* $\lambda \in \Lambda_2$, *problem* (H_λ^f) *has at least three solutions in* X *whose norms are less than* σ.

Proof For each $u \in X$, let Φ, Ψ: $X \to \mathrm{IR}$ defined by setting

$$\Phi(u) := \frac{\|u\|^p}{p}, \quad \Psi(u) := \int_\Omega F(x, u(x))\mathrm{d}x.$$

It is easy to verify that Φ: $X \to \mathrm{IR}$ is a nonnegative, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*. Meanwhile, Ψ is continuously Gâteaux differentiable with compact derivative and, moreover, $\Phi(u_0) = \Psi(u_0) = 0$, where u_0 is the identically zero function in X. In particular, one has

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\mathrm{d}x,$$

and

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\mathrm{d}x,$$

for every u, $v \in X$.

Now, fixing $\lambda > 0$, if we recall that a weak solution of problem (H_λ^f) is a function $u \in X$ such that

$$\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\mathrm{d}x = \lambda \int_\Omega f(x, u(x))v(x)\mathrm{d}x,$$

for every $v \in X$, it is obvious that our goal is to find critical points of the energy functional $J_\lambda := \Phi - \lambda\Psi$.

Thanks to hypothesis (h_3) and bearing in mind (2), one has

$$\int_\Omega F(x, u(x))\mathrm{d}x \le \|\alpha\|_{L^1(\Omega)}(1 + k^s\|u\|^s).$$

Hence

$$J_\lambda(u) \ge \frac{\|u\|^p}{p} - \lambda\|\alpha\|_{L^1(\Omega)}(1 + k^s\|u\|^s).$$

Therefore, due to $s < p$, the following relation holds:

$$\lim_{\|u\|\to\infty} J_\lambda(u) = +\infty$$

for every $\lambda > 0$.

Since J_λ is coercive for every positive parameter λ, condition (i) is verified. Next, consider the function $u_\delta \in X$. Since

$$\sum_{i=1}^N \frac{\partial^2 u_\delta(x)}{\partial x_i^2} = 32d\left(\frac{2(N+2)l^2 - 3\tau(N+1)l + N\tau^2}{\tau^4}\right),$$

for every $x \in B(x^0, \tau)\setminus B(x^0, \tau/2)$ and

$$\sum_{i=1}^N \frac{\partial^2 u_\delta(x)}{\partial x_i^2} = 0, \quad \forall x \in (\bar{\Omega} \setminus B(x^0, \tau)) \cup B(x^0, \tau/2),$$

one has

$$\Phi(u_\delta) = \frac{\|u_\delta\|^p}{p} = \frac{2^{5p+1}\pi^{N/2}\delta^p}{\tau^{4p}\Gamma(N/2)\,p}\sigma_{p,N}(\tau). \tag{4}$$

Substitute

$$r := \frac{\gamma^p}{pk^p}.$$

Now, it follows from $\delta > K_{p,N}(\tau)^{1/p}\gamma$ that $\Phi(u_\delta) > r$. We explicitly observe that, in view of (2), one has

$$\Phi^{-1}(]-\infty, r]) \subseteq \{u \in C^0(\bar{\Omega}) : \ \|u\|_\infty \le \gamma\}. \tag{5}$$

Moreover, taking (5) into account, a direct computation ensures that

$$\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(u) = \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \le \int_\Omega \max_{|\xi|\le\gamma} F(x, \xi)\,\mathrm{d}x. \tag{6}$$

At this point, by the definition of u_δ, we can clearly write

$$\int_\Omega F(x, u_\delta(x))\,\mathrm{d}x = R_F(\tau,\delta) + \int_{B(x^0,\tau/2)} F(x,\delta)\,\mathrm{d}x. \tag{7}$$

By using hypothesis (h_2), from (6) and (7), we also have

$$\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(u) < \frac{r}{r+\Phi(u_\delta)}\Psi(u_\delta),$$

taking into account that

$$\frac{r}{r+\Phi(u_\delta)} = \frac{\tau^{4p}\Gamma(N/2)\gamma^p}{\tau^{4p}\Gamma(N/2)\gamma^p + k^p 2^{5p+1}\pi^{N/2}\delta^p\sigma_{p,N}(\tau)} = \eta(\gamma,\delta).$$

So conditions (ii) and (iii) are verified by taking $\bar{u} := u_\delta$. Thus, we can apply Theorem 2.1 bearing in mind that

$$\frac{\Phi(u_\delta)}{\Psi(u_\delta) - \sup\limits_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(u)} \leq \lambda_1$$

and

$$\frac{r}{\sup\limits_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(u)} \geq \lambda_2,$$

as well as

$$\frac{hr}{r\dfrac{\Psi(u_\delta)}{\Phi(u_\delta)} - \sup\limits_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} \Psi(u)} \leq \lambda_{3,h}.$$

The proof is complete. ■

Remark 3.2 Assume that $N \geq 4$ and let f be a global Lipschitz continuous function (with constant L) uniformly with respect to the first variable such that $f(x,0) = 0$, for almost every $x \in \bar{\Omega}$. Then, for every

$$0 \leq \lambda < \lambda^\star := \frac{1}{k^2 L\,\mathrm{meas}(\Omega)},$$

problem (H_λ^f) has no solutions u with $\|u\| \geq 1$. Indeed, since

$$\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\mathrm{d}x = \lambda\int_\Omega f(x,u(x))v(x)\mathrm{d}x,$$

for every $v \in X$, choosing $v = u$, one clearly has (arguing by contradiction suppose $\|u\| \geq 1$)

$$\begin{aligned}\int_\Omega |\Delta u(x)|^p\mathrm{d}x &= \lambda\int_\Omega f(x,u(x))u(x)\mathrm{d}x \\ &\leq \lambda k^2 L\,\mathrm{meas}(\Omega)\int_\Omega |\Delta u(x)|^p\mathrm{d}x.\end{aligned}$$

Hence, since $\lambda < \lambda^*$, the last estimates give that $u = 0$.

Remark 3.3 Assuming that

(j_1) $F(x, \xi) \geq 0$ for every $(x, \xi) \in (B(x^0, \tau) \backslash B(x^0, \tau/2)) \times [0, \delta]$;
(j_2) For every $|\xi| \leq \gamma$ one has

$$\int_\Omega \max_{|\xi| \leq \gamma} F(x, \xi)\,\mathrm{d}x < \eta(\gamma, \delta) \int_{B(x^0, \tau/2)} F(x, \delta)\,\mathrm{d}x,$$

it follows that hypothesis (h_1) in Theorem 3.1 automatically holds.

Remark 3.4 We point out that hypothesis (h_2) in Theorem 3.1 can be stated in a more general form. Precisely, fix $x^0 \in \Omega$ and pick $r_1, r_2 \in \mathrm{IR}$ with $r_2 > r_1 > 0$, such that $B(x^0, r_1) \subset B(x_0, r_2) \subseteq \Omega$. Moreover, set

$$\sigma_{p,N}(r_1, r_2) := \int_{r_1}^{r_2} |(N+2)s^2 - (N+1)(r_1 + r_2)s + Nr_1r_2|^p s^{N-1}\mathrm{d}s,$$

and denote

$$K_{p,N}(r_1, r_2) := \frac{(r_2 - r_1)^{3p}(r_1 + r_2)^p \Gamma(N/2)}{2^{2p+1} 3^p \pi^{N/2} k^p \sigma_{p,N}(r_1, r_2)}.$$

At this point, let v_δ be the function defined as follows:

$$v_\delta(x) := \begin{cases} 0 & \text{if } x \in \bar{\Omega} \setminus B(x^0, r_2) \\ \dfrac{\delta(3(l^4 - r_2^4) - 4(r_1 + r_2)(l^3 - r_2^3) + 6r_1r_2(l^2 - r_2^2))}{(r_2 - r_1)^3(r_1 + r_2)} & \text{if } x \in B(x^0, r_2) \setminus B(x^0, r_1) \\ \delta & \text{if } x \in B(x^0, r_1), \end{cases}$$

where $l := \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$.

If δ and γ in Theorem 3.1 satisfy $\delta > K(r_1, r_2)^{1/p}\gamma$, instead of ($h_1$), hypothesis ($h_2$) can be replaced by the following assumption, namely ($h_2^\star$):

$$\int_\Omega \max_{|\xi| \leq \gamma} F(x, \xi)\,\mathrm{d}x < \frac{r}{r + \Phi(v_\delta)} \left(R_F(r_1, r_2, \delta) + \int_{B(x^0, r_1)} F(x, \delta)\,\mathrm{d}x \right),$$

where

$$r := \frac{\gamma^p}{pk^p},$$

$$R_F(r_1, r_2, \delta) := \int_{B(x^0, r_2) \backslash B(x^0, r_1)} F(x, v_\delta(x))\,\mathrm{d}x,$$

and

$$\Phi(v_\delta) = \frac{\delta^p}{pk^p K_{p,N}(r_1, r_2)}.$$

Then for each

$$\lambda \in \Lambda_1^\star := \left]\frac{\Phi(v_\delta)}{\Psi(v_\delta) - \int_\Omega \max_{|\xi|\leq\gamma} F(x,\xi)\,\mathrm{d}x}, \frac{\gamma^p}{pk^p \int_\Omega \max_{|\xi|\leq\gamma} F(x,\xi)\,\mathrm{d}x}\right[,$$

the equation

$$J_\lambda(u) = \Phi'(u) - \lambda\Psi'(u) = 0, \tag{8}$$

has at least three distinct solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2^\star \subset \left[0, \frac{hr}{r\dfrac{\Psi(v_\delta)}{\Phi(v_\delta)} - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi(x)}\right],$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, Equation (8) has at least three solutions in X whose norms are less than σ. It is clear that if $r_1 = \tau/2$ and $r_2 = \tau$, condition ($\mathrm{h}_2^\star$) coincides with (h_2).

Now, for completeness, we analyse the autonomous case

$$\left(G_\lambda^f\right) \quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \mathrm{IR} \to \mathrm{IR}$ is a continuous function. With the above notations, let us define

$$G_F(\tau, \delta) := \int_{B(x^0,\tau)\setminus B(x^0,\tau/2)} F(u_\delta(x))\,\mathrm{d}x.$$

Finally, the symbol 'meas($B(x^0, \tau/2)$)' denotes the Lebesgue measure of the ball $B(x^0, \tau/2)$.

THEOREM 3.5 *Let $f \in C^0(\mathrm{IR})$ and substitute*

$$F(\xi) := \int_0^\xi f(t)\mathrm{d}t, \quad \forall\, \xi \in R.$$

Assume that there exist two positive constants γ and δ such that condition (h_1) *holds in addition to*

(h_2') $F(\xi) < \dfrac{\eta(\gamma,\delta)}{\mathrm{meas}(\Omega)}\big(G_F(\tau,\delta) + \mathrm{meas}(B(x^0,\tau/2))F(\delta)\big)$, *for every* $|\xi| \leq \gamma$. *Moreover…*

(h_3') *There exist two positive constants b and s with $s < p$ such that*

$$F(\xi) \leq b(1 + |\xi|^s).$$

Then, for each

$$\lambda \in \Lambda_1' :=]\lambda_1', \lambda_2'[,$$

where

$$\lambda_1' := \frac{2^{5p+1}\pi^{N/2}\sigma_{p,N}(\tau)\delta^p / \operatorname{meas}(\Omega)}{\tau^{4p}\Gamma(N/2)\, p\left(\frac{G_F(\tau,\delta)+\operatorname{meas}(B(x^0,\tau/2))F(\delta)}{\operatorname{meas}(\Omega)} - \max_{|\xi|\le\gamma} F(\xi)\right)}$$

and

$$\lambda_2' := \frac{\gamma^p}{pk^p \operatorname{meas}(\Omega) \max_{|\xi|\le\gamma} F(\xi)},$$

problem (G_λ^f) *has at least three distinct solutions in* X *and, moreover, for each* $h > 1$, *there exists an open interval*

$$\Lambda_2' \subset \left[0, \lambda_{3,h}'\right],$$

where

$$\lambda_{3,h}' := \frac{h\gamma^p/(p \operatorname{meas}(\Omega)k^p)}{\frac{\gamma^p\left(G_F(\tau,\delta)+\operatorname{meas}(B(x^0,\tau/2))F(\delta)\right)\tau^{4p}\Gamma(N/2)}{2^{5p+1}k^p\pi^{N/2}\sigma_{p,N}(\tau)\operatorname{meas}(\Omega)\delta^p} - \max_{|\xi|\le\gamma} F(\xi)},$$

and a positive real number $\sigma > 0$ *such that, for each* $\lambda \in \Lambda_2'$, *problem* (G_λ^f) *has at least three solutions in* X *whose norms are less than* σ.

Remark 3.6 The following two conditions

(j_1') $G_F(\tau, \delta) \ge 0$;
(j_2') For every $|\xi| \le \gamma$ one has

$$F(\xi) < \eta(\gamma,\delta)\frac{\operatorname{meas}(B(x^0,\tau/2))}{\operatorname{meas}(\Omega)}F(\delta),$$

which implies hypotheses (h_2') in Theorem 3.5.

Furthermore, assumption (j_1') is verified by requiring that $F(\xi) \ge 0$ for every $\xi \in [0,\delta]$. Moreover, if f is nonnegative, condition (j_1') automatically holds and (j_2') attains a more simple form

$$F(\gamma) < \kappa_\Omega F(\delta),$$

where

$$\kappa_\Omega := \eta(\gamma,\delta)\frac{\operatorname{meas}(B(x^0,\tau/2))}{\operatorname{meas}(\Omega)}.$$

Hence, Theorem 1.1 is a direct consequence of the above observations. Indeed, we observe that if

($h_3^\star$) $\lim_{|t|\to\infty} \frac{f(t)}{|t|^{s-1}} = 0$, for some $1 \leq s \leq p$, the functional J_λ is coercive. We give just some computations in the case $s = p$; analogous conclusion holds for $s \in [1, p[$. So, fix $\lambda > 0$ and pick $\varepsilon < 1/(\lambda k^p \text{meas}(\Omega))$.

Now, by our assumption at infinity, there exists a $c(\varepsilon) > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + c(\varepsilon), \quad \forall\, t \in \mathrm{IR}.$$

Then the previous inequality gives

$$F(\xi) \leq \frac{\varepsilon}{p}|\xi|^p + c(\varepsilon)|\xi|, \quad \forall\, \xi \in \mathrm{IR},$$

and, consequently, taking into account (2), one has

$$\Psi(u) \leq \left(\frac{\varepsilon k^p}{p}\|u\|^p + c(\varepsilon)k\|u\|\right)\text{meas}(\Omega), \quad \forall\, u \in X.$$

Since, for every $u \in X$, the following inequality holds:

$$J_\lambda(u) \geq \left(\frac{1}{p} - \lambda\frac{\varepsilon k^p}{p}\text{meas}(\Omega)\right)\|u\|^p - \lambda c(\varepsilon)k\|u\|\,\text{meas}(\Omega),$$

the functional J_λ is coercive.

Thus, all the assumptions (with $(\mathrm{h}_3^\star)$ instead of (h_3')) of Theorem 3.5 are verified and the conclusion follows.

At the end we exhibit a concrete application of our results.

Example 3.7 Let Ω be a nonempty bounded open subset of the Euclidean space IR^3 with a smooth boundary $\partial\Omega$ and define $f : \mathrm{IR} \to \mathrm{IR}$ as follows:

$$f(t) := \begin{cases} 0 & \text{if } t < 2 \\ \sqrt{t-2} & \text{if } t \geq 2, \end{cases}$$

whose potential is given by

$$F(\xi) := \begin{cases} 0 & \text{if } \xi < 2 \\ \dfrac{2(\xi-2)^{3/2}}{3} & \text{if } \xi \geq 2. \end{cases}$$

Consider the following problem

$$\left(H_\lambda^f\right) \quad \begin{cases} \Delta^2 u = \lambda f(u) & \Omega \\ u = \Delta u = 0 & \partial\Omega. \end{cases}$$

We easily observe that there exist two positive constants $\gamma = 2$ and

$$\delta > 2\left\{1, K_{2,3}(\tau)^{1/p}\right\},$$

such that, taking into account Remark 3.6, all the conditions of Theorem 3.5 hold. Then, for each

$$\lambda \in \Lambda_1' := \left]\lambda_1^\star, +\infty\right[,$$

where

$$\lambda_1^\star := \frac{2^{10}\pi^{3/2}\sigma_{2,3}(\tau)\delta^2}{\tau^8\Gamma(3/2)\big(G_F(\tau,\delta) + (B(x^0,\tau/2))F(\delta)\big)},$$

problem (H_λ^f) has at least three distinct (two nontrivial) solutions in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2' \subset \Big[0, \lambda_{3,h}^\star\Big],$$

where

$$\lambda_{3,h}^\star := \frac{2^{10}\pi^{3/2}\sigma_{2,3}(\tau)\delta^2 h}{\tau^8\Gamma(3/2)\big(G_F(\tau,\delta) + (B(x^0,\tau/2))F(\delta)\big)} = h\lambda_1^\star,$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2'$, problem (H_λ^f) has at least three (two nontrivial) solutions in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ whose norms are less than σ.

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