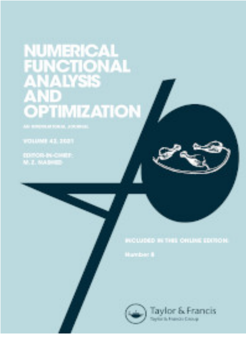

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Existence Results for Integro-Differential Equations with Reflection

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ABSTRACT

We prove several important results concerning existence and uniqueness of pseudo almost automorphic (paa) solutions with measure for integro-differential equations with reflection. We use the properties of almost automorphic functions with measure and the Banach fixed point theorem, and we discuss two linear and nonlinear cases. We conclude with an example and some observations.

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1. Introduction

Many authors have studied problems of existence of periodic, almost periodic and automorphic solutions for different kinds of differential and integral equations (cf. Adivar and Koyuncuoğlu [1], Baskakov et al. [2], Bochner [3], N'Guérékata [4], and Papageorgiou et al. [5, 6]). For example, the function

$$t \longrightarrow \sin t + \sin \sqrt{2}t$$

is almost periodic but not periodic on $\mathbb{R}$, whereas the function

$$t \longrightarrow \sin \left(\frac{1}{2 + \cos t + \cos \sqrt{2}t} \right)$$

is almost automorphic but not uniformly continuous, hence not almost periodic on $\mathbb{R}$.

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Recently, these research directions have taken various generalizations (cf. Ait Dads et al. [7–9], Ben-Salah et al. [10], Blot et al. [11], Chérif and Miraoui [12], Diagana et al. [13], Li [14], Miraoui [15, 16], Miraoui et al. [17], Miraoui and Yaakobi [18], and Zhang [19]), as well as various applications (cf. e.g. Kong and Nieto [20], and the references therein).

Let μ be positive measure on $\mathbb{R}$ and X a Banach space. A continuous function $f : \mathbb{R} \mapsto X$ is said to be *measure paa* (cf. Ait Dads et al. [7] and Papageorgiou et al. [6]), if f can be written as a sum of an almost periodic function g_1 and an ergodic function φ_1 satisfying

$$\lim_{z\to\infty} \frac{1}{\mu([-z,z])} \int_{-z}^{z} \|\varphi_1(y)\| d\mu(y) = 0,$$

where

$$\mu([-z,z]) := \int_{-z}^{z} d\mu(t).$$

Diagana [21] defined the network of weighted pseudo almost periodic functions, which generalizes the pseudo almost periodicity in Gupta [22].

Motivated by above mentioned work, we investigate in the present paper measure paa solutions of differential equations involving reflection of the argument. This type of differential equations has applications in the study of stability of differential-difference equations, cf. e.g. Sharkovskii [23], and such equations show very interesting properties by themselves. Therefore several authors have worked on this category of equations.

Aftabizadeh et al. [24], Papageorgiou et al. [25]. and Gupta [26] studied the existence of unique bounded solution of equation

$$u'(y) = f(y, u(y), u(-y)), y \in \mathbb{R}.$$

They proved that $u(y)$ is almost periodic by assuming the existence of bounded solution. Piao [27, 28] studied the following equations

$$u'(y) = au(y) + bu(-y) + g(y), b \neq 0, y \in \mathbb{R}, \tag{1}$$

and

$$u'(y) = au(y) + bu(-y) + f(y, u(y), u(-y)), b \neq 0, y \in \mathbb{R}. \tag{2}$$

Xin and Piao [29] obtained some results of weighted pseudo almost periodic solutions for equations (1) and (2). Recently, Miraoui [30] has studied the pseudo almost periodic (pap) solutions with two measures of equations (1) and (2).

Throughout this paper, we shall assume the following hypothesis:

(M0): There exists a continuous and strictly increasing function $\beta : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$, we have $x \circ \beta \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$.

The key goal of our paper is to study equations which are more general than equations (1) and (2), and are given by the following expression

$$u'(y) = au(y) + bu(-y) + f(y, u(\beta(y)), u(\beta(-y))) \tag{3}$$
$$+ \int_{y}^{+\infty} K(s-y)h(s, u(\beta(s)), u(\beta(-s)))ds$$
$$+ \int_{-y}^{+\infty} K(s+y)h(s, u(s), u(-s))ds, y \in \mathbb{R}, \tag{4}$$

where $a \in \mathbb{R}, b \in \mathbb{R}^*, f, h : \mathbb{R}^3 \to \mathbb{R}$, and $K : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions.

Let $\mathbb{X}$ be Banach space. We begin by defining the notion of a measure pseudo almost automorphic function.

Definition 1.1. (Bochner [3]) Let $f \in \mathcal{C}(\mathbb{R}, \mathbb{X})$. Then f is said to be almost automorphic, $f \in \mathcal{AA}(\mathbb{R}, \mathbb{X})$, if for every real sequence (s_n), there exists a subsequence (s_{n_k}), such that the following limits

$$\lim_{n_k\to\infty} f(t + s_{n_k}) = f(t) \text{ and } \lim_{n_k\to\infty} g(t-s_{n_k}) = f(t),$$

exist for every $t \in \mathbb{R}$.

Definition 1.2. (Blot et al. [11]) Let $\mathcal{B}$ is the Lebesque σ-field of $\mathbb{R}$ and μ a positive measure on $\mathcal{B}$. Then $\mu \in \mathcal{M}$ if the following conditions are satisfied

- $\mu([a, b]) < \infty$, for all $a \leq b \in \mathbb{R}$; and
- $\mu(\mathbb{R}) = +\infty$.

In this paper we shall be working with a positive measure satisfying the following two important hypotheses:

(M1) For every $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta\mu(A), \text{ whenever } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

(M2) There exist $m, n > 0$ such that for all $A \in \mathcal{B}$,

$$\mu(-A) \leq m + n\mu(A).$$

Definition 1.3. (Diagana et al. [13]) Suppose that $\mu \in \mathcal{M}$. Then $f \in \mathcal{BC}(\mathbb{R}, \mathbb{X})$ is said to be μ-ergodic, $f \in \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu)$, if the following condition is satisfied:

$$\lim_{z\to\infty} \frac{1}{\mu([-z, z])} \int_{[-z,z]} \|f(y)\| d\mu(y) = 0.$$

Definition 1.4. (Diagana et al. [13]) Suppose that $\mu \in \mathcal{M}$. Then $f \in \mathcal{C}(\mathbb{R},\mathbb{X})$ is said to be μ-paa, $f \in PAA(\mathbb{R},\mathbb{X},\mu)$, if

$$f = g + h,$$

where $g \in AA(\mathbb{R},\mathbb{X})$ and the function h is μ-ergodic.

In the sequel, we shall also need the following hypotheses

(h0) There exists a continuous, strictly increasing function $\lambda : \mathbb{R} \to \mathbb{R}^+$ such that $d\mu_\beta(t) \le \lambda(t)d\mu(t)$, where $\mu \in \mathcal{M}, \mu_\beta(O) = \mu(\beta^{-1}(O))$, for all $O \in \mathbb{B}(\mathbb{R})$ and

$$\limsup \frac{\mu[-T(r),T(r)]}{\mu[-r,r]} S(T(r)) < +\infty,$$

where $T(r) = |\beta(r)| + |\beta(-r)|$ and $S(T(r)) = \sup_{t\in[-T(r),T(r)]\lambda(T)}$.

(h1) Given $\lambda := \sqrt{a^2-b^2}$, where $a > b$, the following holds

$$P_1(\lambda,\mu) := \sup_{z>0}\left\{\int_{-z}^{z} \exp(-\lambda(t+z))d\mu(t)\right\} < \infty \textit{ and}$$
$$P_2(\lambda,\mu) := \sup_{z>0}\left\{\int_{-z}^{z} \exp(-\lambda(-t+z))d\mu(t)\right\} < \infty.$$

(h2) There exists $L_f > 0$, such that $f : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ satisfies the Lipschitz condition

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \le L_f(|x_1-x_2| + |y_1-y_2|), \text{for all } (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2.$$

(h3) There exists $L_h > 0$ such that

$$|h(t,u_1,u_2)-h(t,v_1,v_2)| < L_h(|u_1-v_1|+|u_2-v_2|), \text{for all } u_1,u_2,v_1,v_2 \in \mathbb{R}.$$

(h4) There exists $K : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$c := \int_0^{+\infty} K(y)dy < \infty.$$

Our first main result of the paper treats the case when L_f and L_h are constant.

Theorem 1.1. *Suppose that $f, h \in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$ and that hypotheses **(h0)–(h4)** and **(M0)–(M2)** hold. Then equation (3) has a unique μ−paa solution if and only if*

$$\frac{|\lambda-a| + |\lambda+a| + 2|b|}{\lambda^2}(L_f + 2cL_h) < 1.$$

For the second main result of this paper we shall need the following hypotheses for the case when L_f and L_h are not constant.
(h'2) $\mu \in \mathcal{M}$ and $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \le L_f(t)(|x_1-x_2|+|y_1-y_2|), \text{for all} (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2,$$

where $p>1, L_f \in \mathcal{L}^p(\mathbb{R},\mathbb{R},dx) \cap \mathcal{L}^p(\mathbb{R},\mathbb{R},d\mu)$, and $\frac{1}{p}+\frac{1}{q}=1$.
(h'3) $\mu \in \mathcal{M}$ and $h : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$|h(t,x_1,y_1)-h(t,x_2,y_2)| \le L_h(t)(|x_1-x_2|+|y_1-y_2|), \text{for all} (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2,$$

where $p>1, L_h \in \mathcal{L}^p(\mathbb{R},\mathbb{R},dx) \cap \mathcal{L}^p(\mathbb{R},\mathbb{R},d\mu)$ and $\frac{1}{p}+\frac{1}{q}=1$.
(h'4) There exists $K : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$\int_0^{+\infty} (K(y))^{\tau} dy < +\infty, \text{for all } \tau > 1.$$

Theorem 1.2 . *Suppose that $f, h \in PAA(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}, \mu)$ and that hypotheses **(h0)-(h1), (h'2)–(h'4) and (M0)–(M2)** hold. Then equation (3) has a unique μ-paa solution if and only if*

$$\|L_f\|_{\mathcal{L}^p(\mathbb{R},\mathbb{R},dx)} + 2\left(\int_0^{+\infty} (K(y))^q\right)^{\frac{1}{q}} \|L_h\|_{\mathcal{L}^p(\mathbb{R},\mathbb{R},dx)} < \frac{\lambda(q\lambda)^{\frac{1}{q}}}{|\lambda - a| + |\lambda + a| + 2|b|}.$$

We conclude the introduction by description of the structure of the paper. In Section 2, we collect some basic results needed for the proofs of the main results of this paper. In section 3, we prove both main results (Theorems 1.1 and 1.2). In Section 4, we give an application of the measure paa, in connection with integro-differential equations with reflection and delay. In Section 5 we discuss the results and their applications.

2. Preliminaries

Theorem 2.1. *(Diagana et al. [13]) Suppose that $\mu \in \mathcal{M}$ satisfies hypothesis **(M1)**. Then $PAA(\mathbb{R}, \mathbb{X}, \mu)$ is translation invariant and $(PAA(\mathbb{R}, \mathbb{X}, \mu), \|.\|_\infty)$ is a Banach space.*

Lemma 2.1. *(Miraoui [30]) Suppose that $g \in PAA(\mathbb{R}, \mathbb{X}, \mu)$ and that hypothesis **(M2)** holds. Then*

$$[t \to g(-t)] \in PAA(\mathbb{R}, \mathbb{X}, \mu).$$

Lemma 2.2. *(Miraoui [30]) If $\mu \in \mathcal{M}$ satisfies hypothesis **(M1)**, then for all $p \ge 1$,*

$$L^p(\mathbb{R}, \mathbb{X}, d\mu) \subset \mathcal{E}(\mathbb{R}, \mathbb{X}, \mu).$$

Lemma 2.3. *(Ben Salah et al. [10]) Suppose that hypotheses* **(h0)** *and* **(M0)** *hold. If $v \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, then $[t \mapsto v(\beta(t))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$.*

Lemma 2.4. *Suppose that hypotheses* **(h0),(h2)** *and* **(M0)–(M2)** *hold. If $f \in \mathcal{PAA}(\mathbb{R}^3, \mathbb{R}, \mu)$, and $v \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, then $[t \mapsto f(t, v(\beta(t)), v(\beta(-t)))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$.*

Proof. Let $f \in \mathcal{PAA}(\mathbb{R}^3, \mathbb{R}, \mu)$. Then f can be written as $f = h + \varphi$, where $h \in \mathcal{AAU}(\mathbb{R}^3, \mathbb{R}), \varphi \in \mathcal{EU}(\mathbb{R}^3, \mathbb{R}, \mu)$ (see [7]). We set $V(t) = v(\beta(t)$, for all $t \in \mathbb{R}$. By Lemma 2.3, we can conclude that $V \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, hence $V = V_1 + V_2$, where $V_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R}), V_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$, and so we have

$$\begin{aligned} f(t, V(t), V(-t)) = \varphi_1(t, V_1(t), V_1(-t)) + f(t, V(t), V(-t)) \\ - f(t, V_1(t), V_1(-t)) + \varphi_2(t, V_1(t), V_1(-t)). \end{aligned}$$

On the one hand, we shall prove that $[t \to \varphi_1(t, V_1(t), V_1(-t))] \in \mathcal{AAU}(\mathbb{R}^3, \mathbb{R})$. Let $H(t) = \varphi_1(t, V_1(t), V_1(-t))$. If $\{s_n\}$ is a sequence of real numbers, then we can extract a subsequence $\{\tau_n\}$ of $\{s_n\}$ such that

1. $\lim_{n\to\infty} \varphi_1(t + \tau_n, v, u) = \phi(t, v, u)$, for all $t, v, u \in \mathbb{R}$;
2. $\lim_{n\to\infty} \phi(t - \tau_n, v, u) = \varphi_1(t, v, u)$, for all $t, v, u \in \mathbb{R}$;
3. $\lim_{n\to\infty} V_1(t + \tau_n, v, u) = U_1(t, v, u)$, for all $t, v, u \in \mathbb{R}$;
4. $\lim_{n\to\infty} U_1(t - \tau_n, v, u) = V_1(t, v, u)$, for all $t, v, u \in \mathbb{R}$.

If $\Phi(t) : \mathbb{R} \to \mathbb{R}$ by $\Phi(t) = \phi(t, V_1(t), U_1(t))$, then we can show that

$$\lim_{n\to\infty} H(t + \tau_n) = \Phi(t); \lim_{n\to\infty} \Phi(t - \tau_n) = H(t), \text{for all } t \in \mathbb{R}$$

and we get

$$\begin{aligned} ||H(t + \tau_n) - \Phi(t)|| \le ||\varphi_1(t + \tau_n, V_1(t + \tau_n), V_1(-t + \tau_n)) - \varphi_1(t + \tau_n, U_1(t), U_1(-t))|| \\ + ||\varphi_1(t + \tau_n, U - 1(t), U_1(-t)) - \phi(t, U_1(t), U_1(-t))||. \end{aligned}$$

Since $V_1(t)$ is almost automorphic, it follows that $V_1(t)$, and $U_1(t)$ are bounded. Therefore there exists a bounded subset $K \subset \mathbb{R}$. From (3) and (**h2**), we see that $\varphi_1(t, V_1(t), V_1(-t))$ are uniformly continuous on every bounded subset $K \subset \mathbb{R}$, hence

$$\lim_{n\to\infty} ||\varphi_1(t + \tau_n, V_1(t + \tau_n), V_1(-t + \tau_n)) - \varphi_1(t + \tau_n, U_1(t), U_1(-t))|| = 0$$

therefore

$$\lim_{n\to\infty} \Phi(t-\tau_n) = H(t), \quad \text{for all t } \in \mathbb{R}.$$

This proves that H is an almost automorphic function. On the other hand, we shall show that $[t \to f(t, V(t), V(-t)) - f(t, V_1(t), V_1(-t))] \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$.

We consider now the following function $\Phi(t) = f(t, V(t), V(-t)) - f(t, V_1(t), V_1(-t))$. Clearly, $\Phi(t) \in \mathcal{BC}(\mathbb{R}, \mathbb{R})$. Since

$$||f(t, u_1, u_2) - f(t, v_1, v_2)|| \le L_f(||u_1 - v_1|| + ||u_2 - v_2||),$$

we have

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}||\Phi(t)||d\mu(t) &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}||f(t,V(t),V(-t)) - f(t,V_1(t),V_1(-t))||d\mu(t)\\ &\le \frac{1}{\mu([-r,r])}\int_{-r}^{r}L_f^1||V(t)-V_1(t)|| + L_f^2||V(-t)-V_1(-t)||d\mu(t)\\ &\le \frac{L_f}{\mu([-r,r])}\int_{-r}^{r}||V_2(t)||d\mu(t) + \frac{L_f}{\mu([-r,r])}\int_{-r}^{r}||V_2(-t)||d\mu(t),\end{aligned}$$

so by Lemma 2.1,

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}||\Phi(t)||d\mu(t) = 0.$$

Therefore $[t \to f(t, V(t), V(-t)) = f(t, v(\beta(t)), v(\beta(-t)))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$. □

Lemma 2.5. *Suppose that hypotheses* ***(h0),(h2), (h4)*** *and* ***(M0)–(M2)*** *hold. Then for every* $h \in \mathcal{PAA}(\mathbb{R}^3, \mathbb{R}, \mu), v \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$,

$$\left[t \mapsto \int_t^{+\infty} K(s-t)h(s, v(\beta(s)), v(\beta(-s)))ds\right] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu).$$

Proof. By Lemma 2.4, we know that $[t \mapsto h(t, v(\beta(t)), v(\beta(-t)))] \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, so

$$h(t, v(\beta(t)), v(\beta(-t))) = h_1(t) + h_2(t),$$

where $h_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$ and $h_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. Set

$$\Theta(t) = \int_t^{+\infty} K(s-t)h(s, v(\beta(s)), v(\beta(-s)))ds.$$

Then

$$\Theta(t) = \int_t^{+\infty} K(s-t)h_1(s)ds + \int_t^{+\infty} K(s-t)h_2(s)ds = \theta_1(t) + \theta_2(t),$$

where

$$\theta_1(t) = \int_t^{+\infty} K(s-t)h_1(s)ds \text{ and } \theta_2(t) = \int_t^{+\infty} K(s-t)h_2(s)ds.$$

Since $u_1 \in \mathcal{AA}(\mathbb{R}, \mathbb{R})$, it follows that for every sequence $(\tau'_n)_{n\in\mathbb{N}}$ there exists a subsequence (τ_n) such that

$$h_1(t) = \lim_{n\to\infty} u_1(t+\tau_n) \tag{5}$$

is well-defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} h_1(t-\tau_n) = u_1(t), \text{for each } t \in \mathbb{R}. \tag{6}$$

Let $M(t) = \int_t^{+\infty} K(s-t)u_1(s)ds$. Then

$$\begin{aligned} |\theta_1(t)-M(t+s_n)| &= \left| \int_t^{+\infty} K(s-t)h_1(s)ds - \int_{t+s_n}^{+\infty} K(s-t-s_n)u_1(s)ds \right| \\ &= \left| \int_t^{+\infty} K(s-t)(h_1(s)-u_1(s+s_n))ds \right|. \end{aligned}$$

Using Eq. (5), hypotheses **(h4)** and the LDC Theorem, it follows that

$$\left|\left| \int_t^{+\infty} K(s-t)(h_1(s)-u_1(s+s_n))ds \right|\right| \to 0, \text{as } n\to\infty, t\in\mathbb{R}.$$

Therefore, we have

$$\theta_1(t) = \lim_{n\to\infty} M(t+\tau_n), \text{for all } \ t\in\mathbb{R}.$$

Using the same argument, we also obtain

$$\lim_{n\to\infty} h_1(t-\tau_n) = u_1(t).$$

Therefore, $\theta_1 \in AA(\mathbb{R}, \mathbb{R})$. To prove that $\Theta(t)PAA(\mathbb{R}, \mathbb{R}, \mu)$, we need to show that $\theta_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. We know that

$$\begin{aligned} \lim_{r\to+\infty} \frac{1}{\mu[-r,r]} \int_{-r}^{r} ||\theta_2(t)||d\mu(t) &= \lim_{r\to+\infty} \frac{1}{\mu[-r,r]} \int_{-r}^{r} \int_t^{+\infty} ||K(s-t)h_2(s)ds||d\mu(t) \\ &\leq \lim_{r\to+\infty} \frac{1}{\mu[-r,r]} \int_{-r}^{r} \int_t^{+\infty} ||K(s-t)||||h_2(s)||dsd\mu(t) \\ &\leq \lim_{r\to+\infty} \frac{1}{\mu[-r,r]} \int_{-r}^{r} \int_0^{+\infty} ||K(y)||||h_2(y+t)||dyd\mu(t) \\ &= \lim_{r\to+\infty} \int_0^{+\infty} \frac{K(y)}{\mu[-r,r]} \int_{-r}^{r} ||h_2(y+t)||d\mu(t)dy. \end{aligned}$$

By the LDC Theorem and Theorem 2.1, we have

$$\lim_{r\to+\infty}\frac{1}{\mu[-r,r]}\int_{-r}^{r}||\theta_2(t)||d\mu(t)\leq\int_0^{+\infty}K(y)\lim_{r\to+\infty}\frac{1}{\mu[-r,r]}\int_{-r}^{r}||h_2(y+t)||d\mu(t)dy=0.$$

It follows that $\left[t\mapsto\int_t^{+\infty}K(s-t)h(s,v(\beta(s)),v(\beta(-s)))ds\right]\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$. □

Remark 2.1. We have shown that

$$\left[t\to\int_t^{+\infty}K(s-t)h(s,v(\beta(s)),v(\beta(-s)))ds\right]\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu). \tag{7}$$

From (**M2**) and Eq. (7) we can also obtain

$$\left[t\to\int_{-t}^{+\infty}K(s+t)h(s,v(\beta(s)),v(\beta(-s)))ds\right]\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu).$$

Lemma 2.6. *(Ben Salah et al. [10]) Let $\mu\in\mathcal{M},g\in PAA(\mathbb{R},\mathbb{R}^2,\mu)$, $h\in PAAU(\mathbb{R}\times\mathbb{R}^2,\mathbb{R},\mu)$, and suppose that hypotheses **(M1)** and **(h'3)** hold. Then*

$$[t\mapsto h(t,g(t))]\in PAA(\mathbb{R},\mathbb{R},\mu).$$

3. Proofs of main results

3.1. Proof of Theorem 1.1

Proof. By Aftabizadeh and Wiener [25], for any $f,h\in PAA(\mathbb{R},\mathbb{R},\mu)$, a particular solution of equation (1) is as follows

$$\begin{aligned}\Gamma x(t)=&-\frac{1}{2\lambda}\left[\exp(\lambda t)\int_t^{\infty}\exp(-\lambda y)((\lambda-a)f(y,x(y),x(-y))+bf(-y,x(-y),x(y)))dy\right]\\&+\frac{1}{2\lambda}\left[\exp(-\lambda t)\int_{-\infty}^{t}\exp(\lambda y)((\lambda+a)f(y,x(y),x(-y))-bf(-y,x(-y),x(y)))dy\right]\\&-\frac{1}{2\lambda}\left[\exp(\lambda t)\int_t^{\infty}\exp(-\lambda y)((\lambda-a)g(y)+bg(-y))dy\right]\\&+\frac{1}{2\lambda}\left[\exp(-\lambda t)\int_{-\infty}^{t}\exp(\lambda y)((\lambda+a)g(y)-bg(-y))dy\right],\end{aligned}\tag{8}$$

where

$$g(y)=\int_y^{+\infty} K(s-y)h(s,u(\beta(s)),u(\beta(-s)))ds$$
$$+\int_{-y}^{+\infty} K(s+y)h(s,u(\beta(s)),u(\beta(-s)))ds.$$

According to Lemmas 2.1, 2.4, and 2.5, we can conclude

$$\left[y\mapsto \int_y^{+\infty} K(s-y)h(s,u(\beta(s)),u(\beta(-s)))ds\right]\in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu).$$

Also, by Lemma 2.1, we have

$$\left[y\mapsto \int_{-y}^{+\infty} K(s+y)h(s,u(\beta(s)),u(\beta(-s)))ds\right]\in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu).$$

Therefore $g\in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$. (We can also use the parity of g to see that $g\in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$.)

So, using lemmas from Section 2, we can deduce that Γ is a mapping of $\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$ into itself. Set

$$\begin{aligned}F(t,v(\beta(t)),v(\beta(-t))) = {} & f(t,v(\beta(t)),v(\beta(-t)))\\ & +\int_y^{+\infty} K(s-y)h(s,u(\beta(s)),u(\beta(-s)))ds\\ & +\int_{-y}^{+\infty} K(s+y)h(s,u(\beta(s)),u(\beta(-s)))ds. \qquad (9)\end{aligned}$$

It remains to show that $\Gamma:\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)\to \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$ is a strict contraction. Since by hypothesis **(M0)**, $\beta:\mathbb{R}\to\mathbb{R}$ is bijective, it follows that for all $u,v\in \mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$,

$$\begin{aligned}&|F(t,v(\beta(t)),v(\beta(-t)))-F(t,u(\beta(t)),u(\beta(-t)))|\\ &\quad = |f(t,v(\beta(t)),v(\beta(-t)))-f(t,u(\beta(t)),u(\beta(-t)))|\\ &\qquad +\int_t^{+\infty} K(s-t)(h(s,v(\beta(s)),v(\beta(-s)))-h(s,u(\beta(s)),u(\beta(-s))))ds\\ &\qquad +\int_{-t}^{+\infty} K(t+s)(h(s,v(\beta(s)),v(\beta(-s)))-h(s,u(\beta(s)),u(\beta(-s))))ds\\ &\quad \le |f(t,v(\beta(t)),v(\beta(-t)))-f(t,u(\beta(t)),u(\beta(-t)))|\\ &\qquad +\int_0^{+\infty} K(s)(h((s+t),v(\beta(s+t)),v(\beta(-(s+t))))-h((s+t,u(\beta(s+t)),u(\beta(-(s+t)))))ds\\ &\qquad +\int_0^{+\infty} K(s)(h(s-t,v(\beta(s-t)),v(\beta(-(s-t))))-h(s-t,u(\beta(s-t)),u(\beta(-(s-t)))))ds\\ &\quad \le 2(L_f+2cL_h)||v-u||_\infty,\end{aligned}$$

therefore

$$|\Gamma v(t) - \Gamma u(t)| \leq \frac{|\lambda - a| + |\lambda + a| + 2|b|}{\lambda^2}(L_f + 2cL_h)||v - u||_\infty.$$

Since

$$\frac{|\lambda - a| + |\lambda + a| + 2|b|}{\lambda^2}(L_f + 2cL_h) < 1,$$

it follows that $\Gamma : \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu) \to \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ is indeed a strict contraction. Therefore Γ has a unique fixed point in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ and equation (3) has a unique measure paa solution. □

3.2. Proof of Theorem 1.2

Proof. We consider the function Γ defined in system (8). Using lemmas from Section 2 and paying attention to coefficients L_f and L_h which are not constants, we can deduce that Γ is a mapping of $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ into itself. It remains to show that Γ is a strict contraction. Indeed, knowing that F is given by (9), we have

$$\begin{aligned}
&|F(t, v(\beta(t)), v(\beta(-t))) - F(t, u(\beta(t)), u(\beta(-t)))| \\
&\quad \leq |f(t, v(\beta(t)), v(\beta(-t))) - f(t, u(\beta(t)), u(\beta(-t)))| \\
&\qquad + \int_0^{+\infty} K(s)(h((s+t), v(\beta(s+t)), v(\beta(-(s+t)))) - h((s+t, u(\beta(s+t)), u(\beta(-(s+t)))))ds \\
&\qquad + \int_0^{+\infty} K(s)(h(s-t, v(\beta(s-t)), v(\beta(-(s-t)))) - h(s-t, u(\beta(s-t)), u(\beta(-(s-t)))))ds \\
&\quad \leq \left[2L_f(t) + 4(\int_0^{+\infty} (K(y))^q dy)^{\frac{1}{q}} ||L_h||_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)}\right] ||v - u||_\infty,
\end{aligned}$$

where $u, v \in \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$, hence

$$\begin{aligned}
|\Gamma v(t) - \Gamma u(t)| \leq \frac{|\lambda - a| + |\lambda + a| + 2|b|}{\lambda (q\lambda)^{\frac{1}{q}}} &[||L_f||_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)} \\
&+ 2(\int_0^{+\infty} (K(y))^q dy)^{\frac{1}{q}} ||L_h||_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)}]||v - u||_\infty.
\end{aligned}$$

Since

$$\frac{|\lambda - a| + |\lambda + a| + 2|b|}{\lambda (q\lambda)^{\frac{1}{q}}} \left[||L_f||_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)} + 2\left(\int_0^{+\infty} (K(y))^q dy\right)^{\frac{1}{q}} ||L_h||_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)} \right] < 1,$$

the operator $\Gamma : \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu) \to \mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ is indeed a strict contraction. Therefore Γ has a unique fixed point in $\mathcal{PAA}(\mathbb{R}, \mathbb{R}, \mu)$ and equation (3) has a unique measure paa solution. □

4. Applications

Let a measure μ be defined by $d\mu(t) = \rho(t)dt$, where $\rho(t) = \exp(\sin t)$, $t \in \mathbb{R}$. Then $\mu \in \mathcal{M}$ satisfies hypothesis **(M1)**. Since $2 + \sin t \geq \sin(-t)$, it follows that if $I = [a, b]$, we have $1 + e^2\mu(I) \geq \mu(-I)$ and so hypothesis **(M2)** is also satisfied.

Consider the following integro-differential equations with reflection and delay.

$$\begin{aligned} x'(t) = \sqrt{2}x(t) + x(-t) + \frac{\exp(-|t|)}{9}\left[\sin x(t-p) + \cos x(-t+p)\right] \\ + \int_t^{+\infty} K(s-t)\frac{\exp(-|s|)}{9}\left[\sin x(s-p) + \cos x(-s+p)\right]ds \\ + \int_{-t}^{+\infty} K(s+t)\frac{\exp(-|s|)}{9}\left[\sin x(s-p) + \cos x(-s+p)\right]ds, \end{aligned} \tag{10}$$

where $K(s) = \exp(-s)$, for all $s \in \mathbb{R}^+$ and p is a strictly positive real number which denotes the delay. If we put $\beta(t) = t-p$, then hypothesis **(M0)** is satisfied, cf. Ben-Salah et al. [10]. Then equation (10) is a special case of equation (3) if we take

$$\begin{aligned} a = \sqrt{2}, b = 1, \lambda = \sqrt{a^2-b^2} = 1 \text{and} f(t,x,y) = h(t,x,y) \\ = \frac{\exp(-|t|)}{9}[\sin x + \cos y]. \end{aligned}$$

Let $p = q = \frac{1}{2}$. Then

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq L_f(t)(|x_1-x_2| + |y_1-y_2|), \text{for all } (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2,$$

and

$$|h(t,x_1,y_1) - h(t,x_2,y_2)| \leq L_h(t)(|x_1-x_2| + |y_1-y_2|), \text{ for all } (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2,$$

where

$$\left[t \to L_f(t) = L_h(t) = \frac{\exp(-|t|)}{9}\right] \in \mathcal{L}^2(\mathbb{R},\mathbb{R},dx) \cap \mathcal{L}^2(\mathbb{R},\mathbb{R},d\mu),$$

since

$$\|L_f\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R},dx)} = \|L_h\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R},dx)} = \frac{1}{9} \text{ and } \|L_f\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R},d\mu)} = \|L_h\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R},d\mu)} \leq \frac{1}{9}\sqrt{e}.$$

This implies that hypothesis **(h3)** is satisfied. Since

$$\|L_f\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R},dx)} + 2\left(\int_0^{+\infty}(K(y))^2dy\right)^{\frac{1}{2}}\|L_h\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R},dx)} = \frac{\sqrt{2}+1}{9} < \frac{\lambda\sqrt{q\lambda}}{|\lambda-a|+|\lambda+a|+2|b|} = \frac{1}{\sqrt{2}+2},$$

we can deduce that all assumptions of Theorem 1.2 are satisfied and thus equation (10) has a unique μ-paa solution.

5. Epilogue

In practice, the purely periodic phenomena is negligible, which gives the idea to find other solutions and consider single measure paa oscillations. Based on composition, completeness, Banach fixed point theorem, and change of variables theorems, we proved two very important results concerning the existence and uniqueness of a single measure paa solution of a new scalar integro-differential system. Compared to previous works, this is first study of oscillations and dynamics of single measure paa solutions for certain integro-differential equations with reflection for the case when $\beta(t)\neq t$. Miraoui [30] studied pap solutions with two measures for our equation (3) for the case when $K=0$ or $h=0$ and $\beta(t)=t$. Ait Dads et al. [9] described equation (3) with matrix coefficients for the case when $\beta(t)=t$. On the other hand, we studied the impact of functions K, f, h and β on the uniqueness of the single measure paa solutions for equation (3).

Note that in the special case when $\beta(t)=t$, hypotheses **(M0)** and **(h0)** are satisfied, therefore the following new results can be deduced from Theorems 1.1 and 1.2.

Corollary 5.1. *Suppose that $f,h\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$ and that hypotheses **(h1)–(h4)** and **(M1)-(M2)** hold. Then the following equation*

$$u'(y) = au(y)+bu(-y)+f(y,u(y),u(-y))+\int_y^{+\infty}K(s-y)h(s,u(s),u(-s))ds + \int_{-y}^{+\infty}K(s+y)h(s,u(s),u(-s))ds, y\in\mathbb{R}, \tag{11}$$

has a unique $\mu-$paa solution if and only if

$$\frac{|\lambda-a|+|\lambda+a|+2|b|}{\lambda^2}(L_f+2cL_h)<1.$$

Corollary 5.2. *Suppose that $f\in\mathcal{PAA}(\mathbb{R},\mathbb{R},\mu)$ and that hypotheses **(h1)–(h2)** and **(M1)-(M2)** hold. Then the following equation*

$$u'(y) = au(y) + bu(-y) + f(y, u(y), u(-y)), y \in \mathbb{R}, \tag{12}$$

has a unique μ−paa solution if and only if

$$\frac{|\lambda - a| + |\lambda + a| + 2|b|}{\lambda^2} L_f < 1.$$

Corollary 5.3. *Suppose that $f, h \in PAA(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}, \mu)$ and that hypotheses* ***(h1), (h'2)–(h'4)*** *and* ***(M1)-(M2)*** *hold. Then equation (11) has a unique μ-paa solution if and only if*

$$\|L_f\|_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)} + 2\left(\int_0^{+\infty} (K(y))^q\right)^{\frac{1}{q}} \|L_h\|_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)} < \frac{\lambda(q\lambda)^{\frac{1}{q}}}{|\lambda - a| + |\lambda + a| + 2|b|}.$$

Corollary 5.4. *Suppose that $f \in PAA(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}, \mu)$ and that hypotheses* ***(h1), (h'2)****, and* ***(M1)-(M2)*** *hold. Then equation (2) has a unique μ-paa solution if and only if*

$$\|L_f\|_{\mathcal{L}^p(\mathbb{R}, \mathbb{R}, dx)} < \frac{\lambda(q\lambda)^{\frac{1}{q}}}{|\lambda - a| + |\lambda + a| + 2|b|}.$$

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