Multiple solutions for a nonlinear and non-homogeneous problem in Orlicz–Sobolev spaces

Mihai Mihăilescu [a,b,*], Dušan Repovš [c,d]

[a] Department of Mathematics, Central European University, 1051 Budapest, Hungary
[b] Department of Mathematics, University of Craiova, Craiova 200585, Romania
[c] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia
[d] Faculty of Education, University of Ljubljana, Kardeljeva ploščad 16, Ljubljana 1000, Slovenia

ARTICLE INFO	ABSTRACT
	We study a non-homogeneous boundary value problem in a smooth bounded domain in $\mathbb{R}^N$. We prove the existence of at least two non-negative and non-trivial weak solutions. Our approach relies on Orlicz–Sobolev spaces theory combined with adequate variational methods and a variant of Mountain Pass Lemma. © 2011 Elsevier Inc. All rights reserved.

1. Introduction and preliminary results

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geqslant 3$) with smooth boundary $\partial\Omega$. Assume that $a : (0, \infty) \to \mathbb{R}$ is a function such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(t) = \begin{cases} a(|t|)t, & \text{for} \quad t \neq 0, \\ 0, & \text{for} \quad t = 0, \end{cases}$$

is an odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. This paper studies a nonlinear boundary value problem of the type

$$\begin{cases} -\text{div}(a(|\nabla u|)\nabla u) = \lambda f(x, u), & \text{for} \quad x \in \Omega, \\ u = 0, & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ is a positive parameter.

In order to go further we introduce the functional space setting where problem (1) will be discussed. In this context we note that the operator in the divergence form is not homogeneous and thus, we introduce an Orlicz–Sobolev space setting for problems of this type.

The first general existence results using the theory of monotone operators in Orlicz–Sobolev spaces were obtained by Donaldson [6] and Gossez [8]. Other recent works that put the problem into this framework are contained in Clément et al. [4,5], García-Huidobro et al. [7], Gossez and Manàsevich [9], Le and Schmitt [11], etc. In these papers, the existence results are obtained by means of variational techniques, monotone operator methods, or fixed point and degree theory arguments. The goal of our paper is to present a new multiplicity result for equations involving non-homogeneous operators. Thus, it supplements the aforementioned results in the aspect that most of the papers guarantee existence but not multiplicity of solutions.

We start by recalling some basic facts about Orlicz spaces. For more details we refer to the books by Adams and Hedberg [1], Adams [2], Rao and Ren [17] and the papers by Clément et al. [4,5], García-Huidobro et al. [7] and Gossez [8].

---

* Corresponding author.
E-mail addresses: mmihailes@yahoo.com (M. Mihăilescu), dusan.repovs@guest.arnes.si (D. Repovš).

For $\varphi : \mathbb{R} \to \mathbb{R}$ introduced at the start of the paper, we define

$$\Phi(t) = \int_0^t \varphi(s)\,ds, \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\,ds, \quad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a *Young function*, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{x\to\infty} \Phi(x) = +\infty$. Furthermore, since $\Phi(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0} \Phi(x)/x = 0$, and $\lim_{x\to\infty} \Phi(x)/x = +\infty$, then Φ is called an *N*-function. The function $\Phi^\star$ is called the *complementary* function of Φ, and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s);\ s \geqslant 0\}, \quad \text{for all} \quad t \geqslant 0.$$

We also observe that $\Phi^\star$ is also an *N*-function and the Young's inequality holds

$$st \leqslant \Phi(s) + \Phi^\star(t), \quad \text{for all} \quad s, t \geqslant 0.$$

The Orlicz space $L_\Phi(\Omega)$ defined by the *N*-function Φ (see [1,2,4]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{ \int_\Omega uv\,dx;\ \int_\Omega (\Phi)^\star(|v|)\,dx \leqslant 1 \right\} < \infty.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{ k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)dx \leqslant 1 \right\}.$$

For Orlicz spaces Hölder's inequality reads as follows (see [17, Inequality 4, p. 79]):

$$\int_\Omega uv\,dx \leqslant 2\|u\|_{L_\Phi}\|v\|_{L_{(\Phi)^\star}} \quad \text{for all} \quad u \in L_\Phi(\Omega) \quad \text{and} \quad v \in L_{(\Phi)^\star}(\Omega).$$

Next, we introduce the Orlicz–Sobolev spaces. We denote by $W^1 L_\Phi(\Omega)$ the Orlicz–Sobolev space defined by

$$W^1 L_\Phi(\Omega) := \left\{ u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\quad i = 1, \ldots, N \right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

We also define the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1 L_\Phi(\Omega)$. By [8, Lemma 5.7] we may consider on $W_0^1 L_\Phi(\Omega)$ the equivalent norm

$$\|u\| := \||\nabla u|\|_\Phi.$$

For an easier manipulation of the spaces defined above, we define

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}.$$

In this paper we assume that we have

$$1 < \varphi_0 \leqslant \frac{t\varphi(t)}{\Phi(t)} \leqslant \varphi^0 < \infty, \quad \forall t \geqslant 0. \tag{2}$$

The above relation implies that Φ satisfies the Δ_2-condition, i.e.

$$\Phi(2t) \leqslant K\Phi(t) \quad \forall t \geqslant 0, \tag{3}$$

where K is a positive constant (see [15, Proposition 2.3]).

On the other hand, the following relations hold:

$$\|u\|^{\varphi^0} \leqslant \int_\Omega \Phi(|\nabla u|)\,dx \leqslant \|u\|^{\varphi_0} \quad \forall u \in W_0^1 L_\Phi(\Omega), \quad \|u\| < 1, \tag{4}$$

$$\|u\|^{\varphi_0} \leqslant \int_\Omega \Phi(|\nabla u|)\,dx \leqslant \|u\|^{\varphi^0} \quad \forall u \in W_0^1 L_\Phi(\Omega), \quad \|u\| > 1, \tag{5}$$

(see, e.g. [14, Lemma 1]).

Furthermore, in this paper we shall assume that the function Φ satisfies the following condition:

$$\text{the function } [0, \infty) \ni t \to \Phi(\sqrt{t}) \quad \text{is} \quad \text{convex}. \tag{6}$$

Conditions (3) and (6) assure that the Orlicz space $L_\Phi(\Omega)$ is a uniformly convex space and thus, a reflexive Banach space (see [15, Proposition 2.2]). This fact implies that also the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$ is a reflexive Banach space.

Remark. We point out certain examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and satisfy conditions (2) and (6). For more details the reader can consult [5, Examples 1–3, p. 243].

1) Let

$$\varphi(t) = p|t|^{p-2}t \quad \forall \ t \in \mathbb{R}$$

with $p > 1$. For this function it can be proved that

$$\varphi_0 = \varphi^0 = p.$$

Furthermore, in this particular case the corresponding Orlicz space $L_\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ while the Orlicz–Sobolev space $W_0^1 L_\Phi(\Omega)$ is the classical Sobolev space $W_0^{1,p}(\Omega)$. We will use the classical notations to denote the Orlicz–Sobolev spaces in this particular case.

2) Consider

$$\varphi(t) = \log(1 + |t|^s)|t|^{p-2}t \quad \forall \ t \in \mathbb{R},$$

with $p, s > 1$. In this case it can be proved that

$$\varphi_0 = p, \quad \varphi^0 = p + s.$$

3) Let

$$\varphi(t) = \frac{|t|^{p-2}t}{\log(1 + |t|)}, \quad \text{if} \quad t \neq 0, \quad \varphi(0) = 0,$$

with $p > 2$. In this case we have

$$\varphi_0 = p - 1, \quad \varphi^0 = p.$$

2. The main result

In this paper we study problem (1) in the special case when

$$f(x, t) = t^{p-1} - t^{q-1},$$

with

$$1 < q < p < \varphi_0 \tag{7}$$

and $t \geqslant 0$.

More precisely, we consider the degenerate boundary value problem

$$\begin{cases} -\text{div}(a(|\nabla u|)\nabla u) = \lambda(u^{p-1} - u^{q-1}), & \text{for} \quad x \in \Omega, \\ u = 0, & \text{for} \quad x \in \partial\Omega \\ u \geqslant 0, & \text{for} \quad x \in \Omega. \end{cases} \tag{8}$$

We say that $u \in W_0^1 L_\Phi(\Omega)$ is a *weak solution* of problem (8) if $u \geqslant 0$ a.e. in Ω and

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \, dx - \lambda \int_\Omega u^{p-1} v \, dx + \lambda \int_\Omega u^{q-1} v \, dx = 0$$

for all $v \in W_0^1 L_\Phi(\Omega)$.

Our main result asserts that problem (8) has at least two non-trivial weak solutions provided that $\lambda > 0$ is large enough. More precisely, we prove.

Theorem 1. *Assume that condition (7) is fulfilled and*

$$\varphi^0 < \min\left\{ N, \frac{N\varphi_0}{N - \varphi_0} \right\}. \tag{9}$$

Then there exists $\lambda^\star > 0$ such that for all $\lambda > \lambda^\star$ problem (8) has at least two distinct non-negative, non-trivial weak solutions.

Remark. We point out that our result was inspired by Perera [16, Theorem 1.2], where a related property was proved in the case of the p-Laplace operators. The extension from p-Laplace operator to the differential operators involved in (8) is not trivial, since the new operators have a more complicated structure than the p-Laplace operator, for example they are non-homogeneous.

Finally, we mention that a similar study regarding the existence and multiplicity of solutions for equations involving the $p(x)$-Laplace operator can be found in Mihăilescu and Rădulescu [12].

3. Proof of Theorem 1

Let E denote the generalized Sobolev space $W_0^1 L_\Phi(\Omega)$.
Define the energy functional $I : E \to \mathbb{R}$ by

$$I(u) = \int_\Omega \Phi(|\nabla u|)\, dx - \frac{\lambda}{\gamma} \int_\Omega u_+^p\, dx + \frac{\lambda}{\beta} \int_\Omega u_+^q\, dx,$$

where $u_+(x) = \max\{u(x), 0\}$.
We remember that $u \in E$ implies $u_+, u_- \in E$ and

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u \leqslant 0] \\ \nabla u, & \text{if } [u > 0], \end{cases} \quad \nabla u_- = \begin{cases} 0, & \text{if } [u \geqslant 0], \\ \nabla u, & \text{if } [u < 0], \end{cases}$$

where $u_\pm = \max\{\pm u(x), 0\}$ for all $x \in \Omega$ (see, e.g. [4, p. 52]). That fact and some standards arguments assure that functional I is well-defined on E and $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), v \rangle = \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v\, dx - \lambda \int_\Omega u_+^{p-1} v\, dx + \lambda \int_\Omega u_+^{q-1} v\, dx,$$

for all $u, v \in E$.

Remark. We point out that if u is a critical point of I then using the above information and condition (2) we have

$$0 = \langle I'(u), u_- \rangle = \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla u_-\, dx - \lambda \int_\Omega (u_+)^{p-1} u_-\, dx + \lambda \int_\Omega (u_+)^{q-1} u_-\, dx = \int_\Omega a|\nabla u|)\nabla u \cdot \nabla u_-\, dx$$

$$= \int_\Omega a(|\nabla u_-|)|\nabla u_-|^2\, dx \geqslant \varphi_0 \int_\Omega \Phi(|\nabla u_-|)\, dx.$$

By the above estimates and relation (4) we deduce that $u \geqslant 0$. It follows that the non-trivial critical points of I are non-negative solutions of (8).

The above remark shows that we can prove Theorem 1 using the critical point theory. More exactly, we first show that for $\lambda > 0$ large enough, the functional I has a global minimizer $u_1 \geqslant 0$ such that $I(u_1) < 0$. Next, by means of the Mountain Pass Theorem, a second critical point u_2 with $I(u_2) > 0$ is obtained.

Lemma 1. *There exists $\lambda_1 > 0$ such that*

$$\lambda_1 = \inf_{u \in E, \|u\| > 1} \frac{\int_\Omega \Phi(|\nabla u|)\, dx}{\int_\Omega |u|^{\varphi_0}\, dx}.$$

Proof. First, we note that by condition (2) we can deduce that E is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_0}(\Omega)$. Consequently, E is continuously embedded in the classical Lebesgue space $L^{\varphi_0}(\Omega)$. It follows that there exists $C > 0$ such that

$$\|u\| \geqslant C\|u\|_{L^{\varphi_0}(\Omega)} \quad \forall u \in E.$$

On the other hand, by (5) we have

$$\int_\Omega \Phi(|\nabla u|)\, dx \geqslant \|u\|^{\varphi_0}, \quad \forall u \in E \text{ with } \|u\| > 1.$$

Combining the above inequalities we obtain

$$\int_\Omega \Phi(|\nabla u|)\, dx \geqslant C^{\varphi_0} \int_\Omega |u|^{\varphi_0}\, dx \quad \forall u \in E \text{ with } \|u\| > 1.$$

The proof of Lemma 1 is complete. □

Proposition 1. *(i) The functional I is bounded from below and coercive.*
(ii) The functional I is weakly lower semi-continuous.

Proof

(i) Since $1 < q < p < \varphi_0$ we have

$$\lim_{t \to \infty} \frac{\frac{1}{p} t^p - \frac{1}{q} t^q}{t^{\varphi_0}} = 0.$$

Then for any $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$\lambda \left(\frac{1}{p} t^p - \frac{1}{q} t^q \right) \leqslant \frac{\lambda_1}{2} t^{\varphi_0} + C_\lambda \quad \forall t \geqslant 0,$$

where λ_1 was defined in Lemma 1.
The above inequality and condition (5) show that for any $u \in E$ with $\|u\| > 1$ we have

$$I(u) \geqslant \int_\Omega \Phi(|\nabla u|)\, dx - \frac{\lambda_1}{2} \int_\Omega |u|^{\varphi_0}\, dx - C_\lambda \mu(\Omega) \geqslant \frac{1}{2} \int_\Omega \Phi(|\nabla u|)\, dx - C_\lambda \mu(\Omega) \geqslant \frac{1}{2} \|u\|^{\varphi_0} - C_\lambda \mu(\Omega).$$

This shows that I is bounded from below and coercive.

(ii) Similar arguments as those used in the proof of [13, Theorem 2] (see also [15, Lemma 4.3]) show that the functional $I_0 : E \to \mathbb{R}$ defined by

$$I_0(u) = \int_\Omega \Phi(|\nabla u|)\, dx \tag{10}$$

is weakly lower semi-continuous. We justify that I is weakly lower semi-continuous. Let $(u_n) \subset E$ be a sequence which converges weakly to u in E. Since I_0 is weakly lower semi-continuous we have

$$I_0(u) \leqslant \liminf_{n \to \infty} I_0(u_n). \tag{11}$$

On the other hand, since E is compactly embedded in $L^p(\Omega)$ and $L^q(\Omega)$ it follows that (u_{n+}) converges strongly to u_+ both in $L^p(\Omega)$ and in $L^q(\Omega)$. (The compact embedding of E into $L^p(\Omega)$ and $L^q(\Omega)$ is a direct consequence of the fact that E is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_0}(\Omega)$ combined with condition (7).) This fact together with relation (11) imply

$$I(u) \leqslant \liminf_{n \to \infty} I(u_n).$$

Therefore, I is weakly lower semi-continuous. The proof of Proposition 1 is complete. $\square$

From Proposition 1 and Theorem 1.2 in [18] we deduce that there exists $u_1 \in E$ a global minimizer of I. The following result implies that $u_1 \neq 0$, provided that λ is sufficiently large.

Proposition 2. *There exists $\lambda^\star > 0$ such that $\inf_E I < 0$.*

Proof. Let $\Omega_1 \subset \Omega$ be a compact subset, large enough and $u_0 \in E$ be such that $u_0(x) = t_0$ in Ω_1 and $0 \leqslant u_0(x) \leqslant t_0$ in $\Omega \setminus \Omega_1$, where $t_0 > 1$ is chosen such that

$$\frac{1}{p} t_0^p - \frac{1}{q} t_0^q > 0.$$

We have

$$\frac{1}{p} \int_\Omega u_0^p\, dx - \frac{1}{q} \int_\Omega u_0^q\, dx \geqslant \frac{1}{p} \int_{\Omega_1} u_0^p\, dx - \frac{1}{q} \int_{\Omega_1} u_0^q\, dx - \frac{1}{q} \int_{\Omega \setminus \Omega_1} u_0^q\, dx \geqslant \frac{1}{p} \int_{\Omega_1} u_0^p\, dx - \frac{1}{q} \int_{\Omega_1} u_0^q\, dx - \frac{1}{q} t_0^q \mu(\Omega \setminus \Omega_1) > 0$$

and thus $I(u_0) < 0$ for $\lambda > 0$ large enough. The proof of Proposition 2 is complete. $\square$

Since Proposition 2 holds it follows that $u_1 \in E$ is a non-trivial weak solution of problem (8).
Fix $\lambda \geqslant \lambda^\star$. Set

$$g(x, t) = \begin{cases} 0, & \text{for} \quad t < 0, \\ t^{p-1} - t^{q-1}, & \text{for} \quad 0 \leqslant t \leqslant u_1(x), \\ u_1(x)^{p-1} - u_1(x)^{q-1}, & \text{for} \quad t > u_1(x) \end{cases}$$

and

$$G(x, t) = \int_0^t g(x, s)\, ds.$$

Define the functional $J : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \Phi(|\nabla u|)\,dx - \lambda \int_\Omega G(x, u)\,dx.$$

The same arguments as those used for functional I imply that $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(u), v \rangle = \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v\,dx - \lambda \int_\Omega g(x, u)v\,dx,$$

for all $u,\, v \in E$.

On the other hand, we point out that if $u \in E$ is a critical point of J then $u \geqslant 0$. The proof can be carried out as in the case of functional I.

Next, we prove.

Lemma 2. *If u is a critical point of J then $u \leqslant u_1$.*

Proof. We have

$$0 = \langle J'(u) - I'(u_1), (u - u_1)_+ \rangle = \int_\Omega (a(|\nabla u|)\nabla u - a(|\nabla u_1|)\nabla u_1) \cdot \nabla(u - u_1)_+\,dx - \lambda \int_\Omega [g(x, u) - (u_1^{p-1} - u_1^{q-1})](u - u_1)_+\,dx$$

$$= \int_{[u > u_1]} (a(|\nabla u|)\nabla u - a(|\nabla u_1|)\nabla u_1) \cdot \nabla(u - u_1)\,dx.$$

Notice that since φ is increasing in $\mathbb{R}$ we have for each ξ and $\psi \in \mathbb{R}^N$

$$(\varphi(|\xi|) - \varphi(|\psi|))(|\xi| - |\psi|) \geqslant 0,$$

with equality if and only if $\xi = \psi$. Thus, we can deduce that

$$(a(|\xi|)|\xi| - a(|\psi|)|\psi|)(|\xi| - |\psi|) \geqslant 0,$$

for all $\xi, \psi \in \mathbb{R}^N$, with equality if and only if $\xi = \psi$. On the other hand, some simple computations show that

$$(a(|\xi|)\xi - a(|\psi|)\psi) \cdot (\xi - \psi) \geqslant (a(|\xi|)|\xi| - \varphi(|\psi|)|\psi|)(|\xi| - |\psi|),$$

for all $\xi, \psi \in \mathbb{R}^N$. Consequently, we conclude that

$$(a(|\xi|)\xi - a(|\psi|)\psi) \cdot (\xi - \psi) \geqslant 0,$$

for all $\xi, \psi \in \mathbb{R}^N$, with equality if and only if $\xi = \psi$.

Using the above pieces of information we deduce that the above equality holds if and only if $\nabla u = \nabla u_1$. It follows that $\nabla u(x) = \nabla u_1(x)$ for all $x \in \omega := \{y \in \Omega;\ u(y) > u_1(y)\}$. Hence

$$\int_\omega \Phi(|\nabla(u - u_1)|)\,dx = 0$$

and thus

$$\int_\Omega \Phi(|\nabla(u - u_1)_+|)\,dx = 0.$$

By relation (4) we obtain

$$\|(u - u_1)_+\| = 0.$$

We obtain that $(u - u_1)_+ = 0$ in Ω, that is, $u \leqslant u_1$ in Ω. The proof of Lemma 2 is complete. $\square$

In the following we determine a critical point $u_2 \in E$ of J such that $J(u_2) > 0$ via the Mountain Pass Theorem. By the above lemma we will deduce that $0 \leqslant u_2 \leqslant u_1$ in Ω. Therefore

$$g(x, u_2) = u_2^{p-1} - u_2^{q-1} \quad \text{and} \quad G(x, u_2) = \frac{1}{p}u_2^p - \frac{1}{q}u_2^q$$

and thus

$$J(u_2) = I(u_2) \quad \text{and} \quad J'(u_2) = I'(u_2).$$

More precisely, we find

$$I(u_2) > 0 = I(0) > I(u_1) \qquad \text{and} \quad I'(u_2) = 0.$$

This shows that u_2 is a weak solution of problem (8) such that $0 \leqslant u_2 \leqslant u_1$, $u_2 \neq 0$ and $u_2 \neq u_1$.

In order to find u_2 described above we prove.

Lemma 3. *There exists $\rho \in (0, \|u_1\|)$ and $a > 0$ such that $J(u) \geqslant a$, for all $u \in E$ with $\|u\| = \rho$.*

Proof. Let $u \in E$ be fixed, such that $\|u\| < 1$. It is clear that

$$\frac{1}{p} t^p - \frac{1}{q} t^q \leqslant 0 \quad \forall t \in [0, 1].$$

Define

$$\Omega_u := \{ x \in \Omega; \ u(x) > \min\{1, u_1(x)\} \}.$$

If $x \in \Omega \setminus \Omega_u$ then $u(x) \leqslant \min\{1, u_1(x)\} \leqslant u_1(x)$ and we have

$$G(x, u) = \frac{1}{p} u_+^p - \frac{1}{q} u_+^q \leqslant 0.$$

If $x \in \Omega_u \cap \{ x \in \Omega; \ u_1(x) < u(x) < 1 \}$ then

$$G(x, u) = \frac{1}{p} u_1^p - \frac{1}{q} u_1^q + \left(u_1^{p-1} - u_1^{q-1} \right)(u - u_1) \leqslant 0.$$

Define

$$\Omega_{u,1} := \Omega_u \setminus \{ x \in \Omega, u_1(x) < u(x) < 1 \}.$$

Thus, provided that $\|u\| < 1$ by relation (4) we get

$$J(u) \geqslant \int_\Omega \Phi(|\nabla u|) \, dx - \lambda \int_{\Omega_{u,1}} G(x, u) \, dx \geqslant \|u\|^{\varphi^0} - \lambda \int_{\Omega_{u,1}} G(x, u) \, dx. \tag{12}$$

By relation (9) it follows that $\varphi^0 < \varphi_0^\star := \frac{N \varphi_0}{N - \varphi_0}$. On the other hand, as we already pointed out, by condition (2) we deduce that E is continuously embedded in the classical Sobolev space $W_0^{1, \varphi_0}(\Omega)$. Consequently, there exists $s \in (\varphi^0, \frac{N \varphi_0}{N - \varphi_0})$ such that E is continuously embedded in the classical Lebesgue space $L^s(\Omega)$. Thus, there exists a positive constant $C > 0$ such that

$$\|u\|_{L^s(\Omega)} \leqslant C \|u\|, \quad \forall \ u \in E.$$

Using the definition of G, Hölder's inequality and the above estimate, we obtain
Using the definition of G and the above estimate, we obtain

$$
\begin{aligned}
\lambda \int_{\Omega_{u,1}} G(x, u) \, dx &= \lambda \int_{\Omega_{u,1} \cap [u < u_1]} \left(\frac{1}{p} u_+^p - \frac{1}{q} u_+^q \right) dx + \lambda \int_{\Omega_{u,1} \cap [u > u_1]} \left(\frac{1}{p} u_1^p - \frac{1}{q} u_1^q \right) dx \\
&\quad + \lambda \int_{\Omega_{u,1} \cap [u > u_1]} \left(u_1^{p-1} - u_1^{q-1} \right)(u - u_1) dx \leqslant \frac{\lambda}{p} \int_{\Omega_{u,1} \cap [u < u_1]} u_+^p \, dx + \frac{\lambda}{q} \int_{\Omega_{u,1} \cap [u > u_1]} u_1^q \, dx \\
&\quad + \lambda \int_{\Omega_{u,1} \cap [u > u_1]} u_1^{p-1} u \, dx \leqslant \lambda D \int_{\Omega_{u,1}} u_+^p \, dx \leqslant \lambda D \int_{\Omega_{u,1}} u_+^s \, dx \leqslant \lambda D_1 \|u\|^s,
\end{aligned}
\tag{13}
$$

where D and D_1 are two positive constants. Combining inequalities (12) and (13) we find that for a $\rho \in (0, \min\{1, \|u_1\|\})$ small enough we have

$$J(u) \geqslant \left(1 - \lambda D_1 \|u\|^{s - \varphi^0} \right) \|u\|^{\varphi^0},$$

and taking into account that $s > \varphi^0$ we infer that the conclusion of Lemma 3 holds true. $\quad \square$

Lemma 4. *The functional J is coercive.*

Proof. For each $u \in E$ with $\|u\| > 1$ by relation (5) and Hölder's inequality we have

$$
\begin{aligned}
J(u) &\geqslant \int_\Omega \Phi(|\nabla u|) \, dx - \lambda \int_{[u > u_1]} G(x, u) \, dx - \lambda \int_{[u < u_1]} G(x, u) \, dx \\
&\geqslant \|u\|^{\varphi_0} - \frac{\lambda}{p} \int_{[u > u_1]} u_1^p \, dx - \frac{\lambda}{p} \int_{[u > u_1]} u_1^{p-1} u \, dx - \frac{\lambda}{p} \int_{[u < u_1]} u_+^p \, dx \geqslant \|u\|^{\varphi_0} - \frac{\lambda}{p} \int_\Omega u_1^p \, dx - \frac{2\lambda}{p} \int_\Omega u_+^p \, dx \\
&\geqslant \|u\|^{\varphi_0} - \frac{\lambda}{p} [\mu(\Omega)]^{1 - p/\varphi_0} C_1 \|u\|^p - C_2 \geqslant \|u\|^{\varphi_0} - C_2 3 \|u\|^p - C_2,
\end{aligned}
$$

where C_1, C_2 and C_3 are positive constants. Since $p < \varphi_0$ the above inequality implies that $J(u) \to \infty$ as $\|u\| \to \infty$, that is, J is coercive. The proof of Lemma 4 is complete. $\quad \square$

The following result yields a sufficient condition which ensures that a weakly convergent sequence in E converges strongly, too.

Lemma 5. *Assume that the sequence (u_n) converges weakly to u in E and*

$$\limsup_{n\to\infty} \int_\Omega a(|\nabla u_n|)\nabla u_n \cdot (\nabla u_n - \nabla u)\,dx \leqslant 0.$$

Then (u_n) converges strongly to u in E.

Proof. Since u_n converges weakly to u in E implies that it follows that $(\|u_n\|)$ is a bounded sequence of real numbers. That fact and relations (4) and (5) imply thet the sequence $(I_0(u_n))$ is bounded, where I_0 is defined by relation (10). Then, up to a sub-sequence, we deduce that $I_0(u_n) \to c$. Furthermore, the weak lower semi-continuity of I_0 (pointed out above) implies

$$I_0(u) \leqslant \liminf_{n\to\infty} I_0(u_n) = c.$$

On the other hand, since I_0 is convex (because Φ is convex), we have

$$I_0(u) \geqslant I_0(u_n) + \langle I_0'(u_n), u - u_n\rangle.$$

Next, by the hypothesis $\limsup_{n\to\infty} \int_\Omega a(|\nabla u_n|)\nabla u_n \cdot (\nabla u_n - \nabla u)\,dx \leqslant 0$, we conclude that $I_0(u) = c$.

Taking into account that $(u_n + u)/2$ converges weakly to u in E and using again the weak lower semi-continuity of I_0 we find

$$c = I_0(u) \leqslant \liminf_{n\to\infty} I_0\left(\frac{u_n + u}{2}\right). \tag{14}$$

We assume by contradiction that u_n does not converge to u in E. Then by (4) it follows that there exist $\epsilon > 0$ and a subsequence (u_{n_m}) of (u_n) such that

$$I_0\left(\frac{u_{n_m} - u}{2}\right) \geqslant \epsilon \quad \forall\, m. \tag{15}$$

On the other hand, relations (3) and (6) enable us to apply [10, Lemma 2.1] in order to obtain

$$\frac{1}{2}I_0(u) + \frac{1}{2}I_0(u_{n_m}) - I_0\left(\frac{u + u_{n_m}}{2}\right) \geqslant I_0\left(\frac{u - u_{n_m}}{2}\right) \geqslant \epsilon \quad \forall\, m. \tag{16}$$

Letting $m \to \infty$ in the above inequality we obtain

$$c - \epsilon \geqslant \limsup_{m\to\infty} I_0\left(\frac{u + u_{n_m}}{2}\right)\,dx$$

and that is a contradiction with (14). It follows that u_n converges strongly to u in E and Lemma 5 is proved. $\square$

Proof of Theorem 1 completed. Using Lemma 3 and the Mountain Pass Theorem (see [3] with the variant given by Theorem 1.15 in [19]) we deduce that there exists a sequence $(u_n) \subset E$ such that

$$J(u_n) \to c > 0 \quad \text{and} \quad J'(u_n) \to 0, \tag{17}$$

where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0, \gamma(1) = u_1\}.$$

By relation (17) and Lemma 4 we obtain that (u_n) is bounded and thus passing eventually to a subsequence, still denoted by (u_n), we may assume that there exists $u_2 \in E$ such that u_n converges weakly to u_2. Since E is compactly embedded in $L^i(\Omega)$ for any $i \in [1, \varphi_0]$, it follows that u_n converges strongly to u_2 in $L^i(\Omega)$ for all $i \in [1, \varphi_0]$. Hence

$$\langle I_0'(u_n) - I_0'(u_2), u_n - u_2\rangle = \langle J'(u_n) - J'(u_2), u_n - u_2\rangle + \lambda \int_\Omega [g(x, u_n) - g(x, u_2)](u_n - u_2)\,dx = o(1),$$

as $n \to \infty$, where I_0 is defined by relation (10). By Lemma 5 we deduce that u_n converges strongly to u_2 in E and using relation (17) we find

$$J(u_2) = c > 0 \quad \text{and} \quad J'(u_2) = 0.$$

Therefore, $J(u_2) = c > 0$ and $J'(u_2) = 0$. By Lemma 2 we deduce that $0 \leqslant u_2 \leqslant u_1$ in Ω. Therefore

$$g(x, u_2) = u_2^{p-1} - u_2^{q-1} \quad \text{and} \quad G(x, u_2) = \frac{1}{p}u_2^p - \frac{1}{q}u_2^q$$

and thus

$$J(u_2) = I(u_2) \quad \text{and} \quad J'(u_2) = I'(u_2).$$

We conclude that u_2 is a critical point of I and thus a solution of (8). Furthermore, $I(u_2) = c > 0$ and $I(u_2) > 0 > I(u_1)$. Thus u_2 is not trivial and $u_2 \neq u_1$. The proof of Theorem 1 is now complete. $\square$

Acknowledgements

References

[1] D.R. Adams, L.I. Hedberg, Function Spaces and Potential Theory, Grundlehren der Mathematischen Wissenschaften, vol. 314, Springer-Verlag, Berlin, 1996.
[2] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[3] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory, J. Funct. Anal. 14 (1973) 349–381.
[4] Ph. Clément, M. García-Huidobro, R. Manásevich, K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, Calc. Var. 11 (2000) 33–62.
[5] Ph. Clément, B. de Pagter, G. Sweers, F. de Thélin, Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces, Mediterr. J. Math. 1 (2004) 241–267.
[6] T.K. Donaldson, Nonlinear elliptic boundary value problems in Orlicz–Sobolev spaces, J. Differ. Equat. 10 (1971) 507–528.
[7] M. García-Huidobro, V.K. Le, R. Manásevich, K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev space setting, Nonlinear Differ. Equat. Appl. (NoDEA) 6 (1999) 207–225.
[8] J.P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, Trans. Am. Math. Soc. 190 (1974) 163–205.
[9] J.P. Gossez, R. Manàsevich, On a nonlinear eigenvalue problem in Orlicz–Sobolev spaces, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002) 891–909.
[10] J. Lamperti, On the isometries of certain function-spaces, Pacific J. Math 8 (1958) 459–466.
[11] V.K. Le, K. Schmitt, Quasilinear elliptic equations and inequalities with rapidly growing coefficients, J. London Math. Soc. 62 (2000) 852–872.
[12] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. Roy. Soc. A Math. Phys. Eng. Sci. 462 (2006) 2625–2641.
[13] M. Mihăilescu, V. Rădulescu, Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz–Sobolev space setting, J. Math. Anal. Appl. 330 (2007) 416–432.
[14] M. Mihăilescu, V. Rădulescu, Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces, Anal. Appl. 6 (1) (2008) 1–16.
[15] M. Mihăilescu, V. Rădulescu, Neumann problems associated to nonhomogeneous differential operators in Orlicz–Sobolev spaces, Ann. Inst. Fourier 58 (6) (2008) 2087–2111.
[16] K. Perera, Multiple positive solutions for a class of quasilinear elliptic boundary-value problems, Electron. J. Differ. Equat. 7 (2003) 1–5.
[17] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Marcel Dekker Inc., New York, 1991.
[18] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Heidelberg, 1996.
[19] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.