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On the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity

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ABSTRACT

In this paper, we consider the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity

$$\begin{cases} \varepsilon^{2s} M([u]^2_{s,A_\varepsilon})(-\Delta)^s_{A_\varepsilon} u + V(x)u = |u|^{2^*_s-2}u + h(x,|u|^2)u, & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

where $(-\Delta)^s_{A_\varepsilon}$ is the fractional magnetic operator with $0 < s < 1$, $2^*_s = 2N/(N-2s)$, $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ is a continuous nondecreasing function, $V : \mathbb{R}^N \to \mathbb{R}^+_0$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ are the electric and magnetic potentials, respectively. By using the fractional version of the concentration compactness principle and variational methods, we show that the above problem: (i) has at least one solution provided that $\varepsilon < \mathcal{E}$; and (ii) for any $m^* \in \mathbb{N}$, has m^* pairs of solutions if $\varepsilon < \mathcal{E}_{m^*}$, where $\mathcal{E}$ and $\mathcal{E}_{m^*}$ are sufficiently small positive numbers. Moreover, these solutions $u_\varepsilon \to 0$ as $\varepsilon \to 0$.

1. Introduction

The main purpose of this paper is to study the existence and multiplicity of solutions for the fractional Schrödinger–Kirchhoff equations with external magnetic operator and critical nonlinearity

$$\begin{cases} \varepsilon^{2s} M([u]^2_{s,A_\varepsilon})(-\Delta)^s_{A_\varepsilon} u + V(x)u = |u|^{2^*_s-2}u + h(x,|u|^2)u, \ x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a positive parameter, $N > 2s$, $0 < s < 1$,

$$[u]^2_{s,A_\varepsilon} := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dxdy,$$

where $2^*_s = \frac{2N}{N-2s}$ is the critical Sobolev exponent, $V \in C(\mathbb{R}^N, \mathbb{R}^+_0)$ is the electric potential, $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic potential, and $A_\varepsilon(x) := \varepsilon^{-1}A(x)$. Further assumptions for the functions $V(x)$, $M(x)$ and $h(x)$ will be given in Section 3. If A is a smooth function, the fractional operator $(-\Delta)^s_A$, which up to normalization constants can be defined on smooth functions u as

$$(-\Delta)^s_A u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

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has recently been introduced in [1]. Hereafter, $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and of radius $\varepsilon > 0$. As stated in [2], up to correcting the operator by the factor $(1-s)$, it follows that $(-\Delta)^s_A u$ converges to $-(\nabla u - iA)^2 u$ as $s \to 1$. Thus, up to normalization, the nonlocal case can be seen as an approximation of the local one. The motivation for its introduction is described in more detail in [1,2] and replies essentially on the Lévy–Khintchine formula for the generator of a general Lévy process. If the magnetic field $A \equiv 0$, then the operator $(-\Delta)^s_{A_\varepsilon}$ can be reduced to the fractional Laplacian operator $(-\Delta)^s$, which is defined as

$$(-\Delta)^s u := P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $P.V.$ stands for the principal value. It may be viewed as the infinitesimal generator of a Lévy stable diffusion processes [3]. This operator arises in the description of various phenomena in applied sciences, such as phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics and so on, see [4] and references therein for more detailed introduction. Indeed, the study of fractional and nonlocal operators of elliptic type has recently attracted more attention. For example, for the case in which bounded domains and the entire space are involved, we refer the readers to [5–10] and the references therein for more related results.

The main driving force for the study of problem (1.1) arises in the following time-dependent Schrödinger equation when $s = 1$:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla + A(x))^2\psi + P(x)\psi - \rho(x, |\psi|)\psi, \tag{1.2}$$

where $\hbar$ is the Planck constant, m is the particle mass, $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential, $P : \mathbb{R}^N \to \mathbb{R}^N$ is the electric potential, ρ is the nonlinear coupling, and ψ is the wave function representing the state of the particle. This equation arises in quantum mechanics and describes the dynamics of the particle in a non-relativistic setting, see for example [11,12]. Clearly, the form $\psi(x, t) = e^{-i\omega t\hbar^{-1}} u(x)$ is a standing wave solution of (1.2) if and only if $u(x)$ satisfies the following stationary equation:

$$(-i\varepsilon\nabla + A)^2 u + V(x)u = f(x, |u|)u, \tag{1.3}$$

where $\varepsilon = \hbar$, $V(x) = 2m(P(x) - \omega)$ and $f = 2m\rho$, see [13–16] and the references cited therein for recent results in this direction. When $A \equiv 0$, problem (1.3) becomes the classical Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Similarly, we can deduce the following fractional Schrödinger equation:

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.5}$$

Felmer, Quaas and Tan [1] studied the existence and regularity of positive solutions for problem (1.5) with $\varepsilon = 1$ when f has subcritical growth and satisfies the Ambrosetti–Rabinowitz condition. Secchi [17] obtained the existence of ground state solutions of (1.5) when $V(x) \to \infty$ as $|x| \to \infty$ and the Ambrosetti–Rabinowitz condition holds. Dong, Xu and Wei [18] obtained the existence of infinitely many weak solutions for (1.5) by a variant of the fountain theorem when f has subcritical growth. For the case of critical growth, Shang and Zhang [19] studied the existence and multiplicity of solutions for the critical fractional Schrödinger equation:

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{2^*_s-2}u + \lambda f(u) \quad x \in \mathbb{R}^N. \tag{1.6}$$

Based on variational methods, they showed that problem (1.6) has a nonnegative ground state solution for all sufficiently large λ and small ε. Moreover, Shen and Gao [20] proved the existence of nontrivial solutions for problem (1.6) under various assumptions on f and potential function $V(x)$, among which they also assumed the well-known Ambrosetti–Rabinowitz condition. See also recent papers [21,22,17,23] for more results. Teng and He [24] were concerned with the following fractional Schrödinger equation involving a critical nonlinearity

$$\varepsilon^{2s}(-\Delta)^s u + u = Q(x)|u|^{2^*_s-2}u + P(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $2 < p < 2^*_s$ and potential functions $P(x)$ and $Q(x)$ satisfy certain hypotheses. Using the s-harmonic extension technique of Caffarelli and Silvestre [25], the concentration-compactness principle of Lions [26] and methods of Brézis and Nirenberg [27], they proved the existence of ground state solutions. On the other hand, Feng [28] investigated the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda|u|^p u, \quad x \in \mathbb{R}^N, \tag{1.8}$$

where $2 < p < 2^*_s$ and $V(x)$ is a positive continuous function. By using the fractional version of concentration compactness principle of Lions [26], he obtained the existence of ground state solutions to problem (1.8) for some $\lambda > 0$. By applying another fractional version of concentration compactness principle and radially decreasing rearrangements, Zhang et al. [29]

proved the existence of a ground state solutions for problem (1.6) with $V(x) = 1$ for large enough $\lambda > 0$, see [30] for related result with application of the same method.

Another important reason for studying problem (1.1) lies in the following feature of the Kirchhoff problems. More precisely, *Kirchhoff* proposed the following model in 1883

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{\lambda} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.9}$$

as a generalization of the well-known D'Alembert's wave equation for free vibrations of elastic strings. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial tension. Essentially, Kirchhoff's model takes into account the changes in the length of the string produced by transverse vibrations. For recent results in this direction, for example, we refer the reader to [31,32] and references therein. Recently, Fiscella and Valdinoci [33] first deduced a stationary fractional Kirchhoff model which considered the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see the Appendix of [33] for more details. Moreover, they investigated in [33] also the following Kirchhoff type problem involving critical exponent:

$$\begin{cases} M([u]_s^2)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.10}$$

where Ω is an open bounded domain in $\mathbb{R}^N$. By using the mountain pass theorem and the concentration compactness principle, together with a truncation technique, they obtained the existence of non-negative solutions for problem (1.10), see for example [34–36] for more recent results. For the results on the entire space, see for instance [37–39].

Mingqi et al. [40] first studied the following Schrödinger–Kirchhoff type equation involving the fractional p-Laplacian and the magnetic operator

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x,|u|)u \quad \text{in } \mathbb{R}^N, \tag{1.11}$$

where the right-hand term in (1.11) satisfies the subcritical growth. By using variational methods, they obtained several existence results for problem (1.11). Following similar methods, for $M(t) = a + bt$ with $a \in \mathbb{R}_0^+$ and $p = 2$, Wang and Xiang [41] proved the existence of two solutions and infinitely many solutions for fractional Schrödinger–Choquard–Kirchhoff type equations with external magnetic operator and critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. Binlin et al. [42] first considered the following fractional Schrödinger equations:

$$\varepsilon^{2s}(-\Delta)_{A_\varepsilon}^s u + V(x)u = f(x,|u|)u + K(x)|u|^{2_\alpha^*-2}u \qquad \text{in } \mathbb{R}^N, \tag{1.12}$$

where $V(x)$ satisfies the assumption (V) which will be introduced in Section 3. By using variational methods, they proved the existence of ground state solution (mountain pass solution) u_ε which tends to the trivial solution as $\varepsilon \to 0$. Moreover, they proved the existence of infinite many solutions and sign-changing solutions for problem (1.12) under some additional assumptions.

Inspired by the above works, in particular by [42,43,40,30], we consider in this article the existence and multiplicity of semiclassical solutions of the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity in $\mathbb{R}^N$. It is worthwhile to remark that in the arguments developed in [42,43], one of the key points is to prove the $(PS)_c$ condition. Here we use the fractional version of Lions' second concentration compactness principle to prove that the $(PS)_c$ condition holds, which is different from methods used in [42,43]. Some difficulties arise when dealing with this problem, because of the appearance of the magnetic field and the critical frequency, and of the nonlocal nature of the fractional Laplacian. Therefore, we need to develop new techniques to overcome difficulties induced by these new features. As far as we know, this is the first time that the fractional version of the concentration compactness principle and variational methods have been combined to get the multiplicity of solutions for the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity. We believe that the ideas used here can be applied in other situations to deal with similar potentials.

The paper is organized as follows. In Section 2, we will introduce the working space and give some necessary definitions and properties, which will be used in the sequel. In Section 3, we will give an equivalent form of problem (1.1). In Section 4, we will use the fractional version of Lions' second concentration compactness principle to prove that the $(PS)_c$ condition holds true. In Section 5, using the critical point theory, we will prove the main result (see Section 3).

2. Preliminaries

For the convenience of the reader, we recall in this part some definitions and basic properties of fractional Sobolev spaces $H_{A_\varepsilon}^s(\mathbb{R}^N, \mathbb{C})$. For a deeper treatment of the (magnetic) fractional Sobolev spaces and their applications to fractional Laplacian problems of elliptic type, we refer to [42,4,44,40,45–47] and the references therein.

For any $s \in (0,1)$, the fractional Sobolev space $H_{A_\varepsilon}^s(\mathbb{R}^N, \mathbb{C})$ is defined by

$$H_{A_\varepsilon}^s(\mathbb{R}^N, \mathbb{C}) = \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) : [u]_{s,A_\varepsilon} < \infty\right\},$$

where $[u]_{s,A_\varepsilon}$ denotes the so-called Gagliardo semi-norm, that is

$$[u]_{s,A_\varepsilon} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}$$

and $H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$ is endowed with the norm

$$\|u\|_{H^s_{A_\varepsilon}(\mathbb{R}^N,\mathbb{C})} = \left([u]^2_{s,A_\varepsilon} + \|u\|^2_{L^2}\right)^{\frac{1}{2}}.$$

If $A = 0$, then $H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$ reduces to the well-known space $H^s(\mathbb{R}^N)$ with the norm $[u]_s := [u]_{s,0}$. The space $H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$ is also a Hilbert space with the real scalar product

$$\langle u, v\rangle_{s,A_\varepsilon} := \langle u, v\rangle_{L^2} + \mathrm{Re} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y))\overline{(v(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}} dxdy,$$

for any $u, v \in H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$. The operator $((-\Delta)^s_{A_\varepsilon}) : H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C}) \to H^{-s}_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$ is defined by

$$\langle(-\Delta)^s_{A_\varepsilon} u, v\rangle := \mathrm{Re} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y))\overline{(v(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}} dxdy,$$

via duality.

We recall the following embedding theorem:

Proposition 2.1 (*See [1, Lemma 3.5]*). *Let $A \in C(\mathbb{R}^N, \mathbb{R}^N)$. Then the embedding*

$$H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^\theta(\mathbb{R}^N, \mathbb{C}),$$

is continuous for any $\theta \in [2, 2^*_s]$. Moreover, the embedding

$$H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow\hookrightarrow L^\theta_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$$

is compact for any $\theta \in [1, 2^*_s)$.

In this paper, we will use the following subspace of $H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$ defined by

$$E = \left\{ u \in H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty \right\}$$

with the norm

$$\|u\|_E := \left([u]^2_{s,A_\varepsilon} + \int_{\mathbb{R}^N} V(x)|u|^2 dx \right)^{\frac{1}{2}},$$

where V is non-negative. By the assumption (V) (see Section 3), we know that the embedding $E \hookrightarrow H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$ is continuous. Note that the norm $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|_\varepsilon$ defined by

$$\|u\|_\varepsilon := \left([u]^2_{s,A_\varepsilon} + \varepsilon^{-2s} \int_{\mathbb{R}^N} V(x)|u|^2 dx \right)^{\frac{1}{2}},$$

for each $\varepsilon > 0$. It is obvious that for each $\theta \in [2, 2^*_s]$, there is $c_\theta > 0$, independent of $0 < \varepsilon < 1$, such that

$$\|u\|_{L^\theta} \le c_\theta \|u\|_E \le c_\theta \|u\|_\varepsilon. \tag{2.1}$$

We have the following diamagnetic inequality:

Lemma 2.1. *For every $u \in H^s_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$, we get $|u| \in H^s(\mathbb{R}^N)$. More precisely,*

$$[|u|]_s \le [u]_{s,A_\varepsilon}.$$

Proof. The assertion follows directly from the pointwise diamagnetic inequality

$$||u(x)| - |u(y)|| \le |u(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y)|,$$

for a.e. $x, y \in \mathbb{R}^N$, see [1, Lemma 3.1, Remark 3.2]. □

By Proposition 3.6 in [4], we have

$$[u]_s = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2}$$

for any $u \in H^s(\mathbb{R}^N)$, i.e.

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy = \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}}u(x)\right|^2 dx.$$

Thus

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u(x)\cdot(-\Delta)^{\frac{s}{2}}v(x)dx.$$

3. The main result

Throughout the paper, without explicit mention, we suppose that the functions $V(x)$, $M(x)$ and $h(x)$ satisfy the following conditions:

- (V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\min_{x\in\mathbb{R}^N} V(x) = 0$ and there is $\tau_0 > 0$ such that the set $V^{\tau_0} = \{x \in \mathbb{R}^N : V(x) < \tau_0\}$ has finite Lebesgue measure;
- (M) (m_1) $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous nondecreasing function. Furthermore, there exists $\alpha_0 > 0$ such that and $M(t) \geq \alpha_0$ for all $t \in \mathbb{R}_0^+$;
  (m_2) there exists $\sigma \in (2/2_s^*, 1]$ satisfying $\widetilde{M}(t) \geq \sigma M(t)t$ for all $t \geq 0$, where $\widetilde{M}(t) = \int_0^t M(s)ds$;
- (H) (h_1) $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $h(x,t) = o(|t|)$ uniformly in x as $t \to 0$;
  (h_2) there exist $c_0 > 0$ and $q \in (2, 2_s^*)$ such that $|h(x,t)| \leq c_0(1 + t^{\frac{q-1}{2}})$;
  (h_3) there exist $l_0 > 0$, $2/\sigma < r$ and $2/\sigma < \mu < 2_s^*$ such that $H(x,t) \geq l_0|t|^{r/2}$ and $\mu H(x,t) \leq 2h(x,t)t$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $H(x,t) = \int_0^t h(x,s)ds$.

To obtain the solution of problem (1.1), we will use the following equivalent form

$$\begin{cases} M\left([u]_{s,A_\varepsilon}^2\right)(-\Delta)_{A_\varepsilon}^s u + \varepsilon^{-2s}V(x)u = \varepsilon^{-2s}|u|^{2_s^*-2}u + \varepsilon^{-2s}h(x,|u|^2)u, \ x \in \mathbb{R}^N, \\ u(x) \to 0, \ \ as \ \ |x| \to \infty, \end{cases} \tag{3.1}$$

for $\varepsilon \to 0$.

The energy functional $J_\varepsilon : E \to \mathbb{R}$ associated with problem (3.1)

$$J_\varepsilon(u) := \frac{1}{2}\widetilde{M}\left([u]_{s,A_\varepsilon}^2\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N} V(x)|u|^2dx - \frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx - \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N} H(x,|u|^2)dx$$

is well defined. Define the Nehari manifold

$$\mathcal{N} = \left\{u \in E : \langle J_\varepsilon'(u), u\rangle_E = 0\right\}.$$

Under the assumptions, it is easy to check that as shown in [48,49], $J_\varepsilon \in C^1(E, \mathbb{R})$ and its critical points are weak solutions of problem (3.1).

We recall that $u \in E$ is a weak solution of problem (3.1), if

$$M\left([u]_{s,A_\varepsilon}^2\right) \mathrm{Re} \iint_{\mathbb{R}^{2N}} \frac{(u(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y))\overline{(v(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}}dxdy$$
$$+ \varepsilon^{-2s}\mathrm{Re}\int_{\mathbb{R}^N} V(x)u\bar{v}dx = \varepsilon^{-2s}\mathrm{Re}\int_{\mathbb{R}^N}\left(|u|^{2_s^*-2}u + h(x,|u|^2)u\right)\bar{v}dx,$$

where $v \in E$.

The following is the main result of the present paper. It will be proved in Section 5.

Theorem 3.1. *Let the conditions (V), (M) and (H) be satisfied. Then the following statements hold:* (1) *For any $\kappa > 0$, there is $\mathcal{E}_\kappa > 0$ such that if $0 < \varepsilon < \mathcal{E}_\kappa$, then problem (3.1) has at least one solution u_ε satisfying*

$$\frac{\sigma\mu - 1}{2}\int_{\mathbb{R}^N} H(x,|u_\varepsilon|^2)dx + \left(\frac{2}{\sigma} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u_\varepsilon|^{2_s^*}dx \leq \kappa\varepsilon^N, \tag{3.2}$$

$$\left(\frac{\sigma}{2} - \frac{1}{\mu}\right)\alpha_0\varepsilon^{2s}[u_\varepsilon]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2dx \leq \kappa\varepsilon^N. \tag{3.3}$$

Moreover, $u_\varepsilon \to 0$ in E as $\varepsilon \to 0$.
(2) *For any $m \in \mathbb{N}$ and $\kappa > 0$, there is $\mathcal{E}_{m\kappa} > 0$ such that if $0 < \varepsilon < \mathcal{E}_{m\kappa}$, then problem (3.1) has at least m pairs of solutions $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \ldots, m$ which satisfy the estimates* (3.2) and (3.3)*. Moreover, $u_{\varepsilon,i} \to 0$ in E as $\varepsilon \to 0$, $i = 1, 2, \ldots, m$.*

4. Behavior of (PS) sequences

In this section, we recall the fractional version of concentration compactness principle in the fractional Sobolev space, see [50,51,29] for more details. Note that Prokhorov theorem (see Theorem 8.6.2 in [52]) ensures that bounded sequences $\{u_n\}_n$ are relatively sequentially compact in $H^s(\mathbb{R}^N)$ if and only if the sequence is *tight* in the sense that for any $\varepsilon > 0$, there exists a compact subset $\Omega \subseteq \mathbb{R}^N$ such that $\sup_n \int_{\mathbb{R}^N\setminus\Omega} |u_n|dx < \varepsilon$.

Lemma 4.1 *([50, Theorem 1.5]). Let $\Omega \subseteq \mathbb{R}^N$ be an open subset and let $\{u_n\}_n$ be a weakly convergent sequence in $H^s(\mathbb{R}^N)$, weakly converging to u as $n \to \infty$ and such that $|u_n|^{2_s^*} \rightharpoonup \nu$ and $|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup \eta$ in the sense of measures. Then either $u_n \to u$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or there exist a (at most countable) set of distinct points $\{x_j\}_{j\in I} \subseteq \overline{\Omega}$ and positive numbers $\{\nu_j\}_{j\in I}$ such that*

$$\nu = |u|^{2_s^*} + \sum_{j\in I} \delta_{x_j}\nu_j, \quad \nu_j > 0.$$

If, in addition, Ω is bounded, then there exist a positive measure $\widetilde{\eta} \in \mathcal{M}(\mathbb{R}^N)$ with $\operatorname{supp}\widetilde{\eta} \subseteq \overline{\Omega}$ and positive numbers $\{\eta_j\}_{j\in I}$ such that

$$\eta = |(-\Delta)^{\frac{s}{2}}u|^2 + \widetilde{\eta} + \sum_{j\in I} \delta_{x_j}\eta_j, \quad \eta_j > 0$$

and

$$\nu_j \le (S^{-1}\eta(\{x_j\}))^{\frac{2_s^*}{2}},$$

where S is the best Sobolev constant, i.e.

$$S = \inf_{u\in H^s(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx}{\int_{\mathbb{R}^N} |u|^{2_s^*} dx},$$

$x_j \in \mathbb{R}^N$, δ_{x_j} are Dirac measures at x_j, and μ_j, ν_j are constants.

Remark 4.1. In the case $\Omega = \mathbb{R}^N$, the above principle of concentration compactness does not provide any information about the possible loss of mass at infinity. The following result expresses this fact in quantitative terms.

Lemma 4.2 *([29, Lemma 3.5]). Let $\{u_n\}_n \subset H^s(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly converges in $H^s(\mathbb{R}^N)$, $|u_n|^{2_s^*} \rightharpoonup \nu$ and $|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup \eta$ weakly-$*$ converges in $\mathcal{M}(\mathbb{R}^N)$ and define*

(i) $\eta_\infty = \lim_{R\to\infty}\limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} |(-\Delta)^{\frac{s}{2}}u_n|^2dx$,
(ii) $\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} |u_n|^{2_s^*}dx$.

Then the quantities ν_∞ and η_∞ exist and satisfy the following

(iii) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2dx = \int_{\mathbb{R}^N} d\eta + \eta_\infty$,
(iv) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2_s^*}dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty$,
(v) $\nu_\infty \le (S^{-1}\eta_\infty)^{\frac{2_s^*}{2}}$.

We recall that a C^1 functional J on Banach space X is said to satisfy the Palais–Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\}_n \subset X$ satisfying $\lim_{n\to\infty}J_\lambda(u_n) = c$ and $\lim_{n\to\infty}\|J_\lambda'(u_n)\|_{X^*} = 0$ has a convergent subsequence.

Lemma 4.3. *Suppose that conditions (V), (M) and (H) hold. Then any $(PS)_c$ sequence $\{u_n\}_n$ is bounded in E and $c \ge 0$.*

Proof. Let $\{u_n\}_n$ be a (PS) sequence in E. Then

$$\begin{aligned} c &= J_\varepsilon(u_n) \\ &= \frac{1}{2}\widetilde{M}\left([u_n]^2_{s,A_\varepsilon}\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N} V(x)|u_n|^2dx - \frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N} |u_n|^{2_s^*}dx - \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N} H(x,|u_n|^2)dx, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \langle J_\varepsilon'(u_n), v\rangle &= \mathrm{Re}\left\{ M\left([u_n]^2_{s,A_\varepsilon}\right) \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y))\overline{(v(x) - e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}}dxdy \right. \\ &\quad \left. + \varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)u_n\bar{v}dx - \varepsilon^{-2s}\int_{\mathbb{R}^N} |u_n|^{2_s^*-2}u_n\bar{v}dx - \varepsilon^{-2s}\int_{\mathbb{R}^N} h(x,|u_n|^2)u_n\bar{v}dx \right\} \\ &= o(1)\|u_n\|_\varepsilon. \end{aligned} \tag{4.2}$$

By (4.1), (4.2), (M) and condition (h_3), we have

$$\begin{aligned} c+o(1)\|u_n\|_\varepsilon &= J_\varepsilon(u_n)-\frac{1}{\mu}\langle J'_\varepsilon(u_n),u_n\rangle = \frac{1}{2}\widetilde{M}\left([u_n]^2_{s,A_\varepsilon}\right)-\frac{1}{\mu}M\left([u_n]^2_{s,A_\varepsilon}\right)[u_n]^2_{s,A_\varepsilon} \\ &\quad+\left(\frac{1}{2}-\frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2dx+\left(\frac{1}{\mu}-\frac{1}{2^*_s}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2^*_s}dx \\ &\quad+\varepsilon^{-2s}\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,|u_n|^2)u_n^2-\frac{1}{2}H(x,|u_n|^2)\right]dx \\ &\geq \left(\frac{\sigma}{2}-\frac{1}{\mu}\right)\alpha_0[u_n]^2_{s,A_\varepsilon}+\left(\frac{1}{2}-\frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2dx. \end{aligned} \tag{4.3}$$

Therefore, (4.3) implies that $\{u_n\}_n$ is bounded in E. Passing to the limit in (4.3) shows that $c\geq 0$. This completes the proof. □

The main result in this section is the following compactness result:

Theorem 4.1. *Suppose that conditions (V), (M) and (H) hold. Then for any $0<\varepsilon<1$, J_ε satisfies $(PS)_c$ condition, for all $c\in\left(0,\ \sigma_0\varepsilon^{N-2s}\right)$, where $\sigma_0:=\left(\frac{1}{\mu}-\frac{1}{2^*_s}\right)(\alpha_0 S)^{N/(2s)}$, that is, any $(PS)_c$-sequence $\{u_n\}_n\subset E$ has a strongly convergent subsequence in E.*

Proof. Let $\{u_n\}_n$ be a $(PS)_c$ sequence. By Lemma 4.3, $\{u_n\}_n$ is bounded in E. Hence, by diamagnetic inequality, $\{|u_n|\}_n$ is bounded in $H^s(\mathbb{R}^N)$. Then, for some subsequence, there is $u\in E$ such that $u_n\rightharpoonup u$ in E. We claim that as $n\to\infty$

$$\int_{\mathbb{R}^N}|u_n|^{2^*_s}dx\to\int_{\mathbb{R}^N}|u|^{2^*_s}dx. \tag{4.4}$$

In order to prove this claim, we invoke Prokhorov's Theorem (see Theorem 8.6.2 in [52]) to conclude that there exist $\eta,\nu\in\mathcal{M}(\mathbb{R}^N)$ such that

$$\begin{aligned} &|(-\Delta)^{\frac{s}{2}}u_n|^2\rightharpoonup\eta \quad \text{(weak*-sense of measures)},\\ &|u_n|^{2^*_s}\rightharpoonup\nu \quad \text{(weak*-sense of measures)}, \end{aligned}$$

where μ and ν are nonnegative bounded measures on $\mathbb{R}^N$. For this, we have to show the *tightness* of sequences $\{|(-\Delta)^{s/2}u_n|^2\}_n$ and $\{|u_n|^{2^*_s}\}_n$, which follows easily from the boundedness of $\{|u_n|\}_n$ in $H^s(\mathbb{R}^N)$ and absolute continuity of the Lebesgue integral. Then, in view of Lemma 4.1, we know that either $u_n\to u$ in $L^{2^*_s}_{loc}(\mathbb{R}^N)$ or $\nu=|u|^{2^*_s}+\sum_{j\in I}\delta_{x_j}\nu_j$, as $n\to\infty$, where I is a countable set, $\{\nu_j\}_j\subset[0,\infty)$, $\{x_j\}_j\subset\mathbb{R}^N$.

Take $\phi\in C_0^\infty(\mathbb{R}^N)$ such that $0\leq\phi\leq 1$; $\phi\equiv 1$ in $B(x_j,\rho)$, $\phi(x)=0$ in $\mathbb{R}^N\setminus B(x_j,2\rho)$. For any $\rho>0$, define $\phi_\rho=\phi\left(\frac{x-x_j}{\rho}\right)$, where $j\in I$. It follows that

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)\phi_\rho(x)-u_n(y)\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy \\ &\leq 2\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2\phi_\rho^2(y)}{|x-y|^{N+2s}}dxdy+2\iint_{\mathbb{R}^{2N}}\frac{|\phi_\rho(x)-\phi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy \\ &\leq 2\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dxdy+2\iint_{\mathbb{R}^{2N}}\frac{|\phi_\rho(x)-\phi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy. \end{aligned} \tag{4.5}$$

Similar to the proof of Lemma 3.4 in [30], we have

$$\iint_{\mathbb{R}^{2N}}\frac{|\phi_\rho(x)-\phi_\rho(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}dxdy\leq C\rho^{-2s}\int_{B(x_i,K\rho)}|u_n(x)|^2dx+CK^{-N}, \tag{4.6}$$

where $K>4$. In fact, we notice that

$$\begin{aligned} \mathbb{R}^N\times\mathbb{R}^N &= ((\mathbb{R}^N\setminus B(x_j,2\rho))\cup B(x_j,2\rho))\times((\mathbb{R}^N\setminus B(x_j,2\rho))\cup B(x_j,2\rho)) \\ &= ((\mathbb{R}^N\setminus B(x_j,2\rho))\times(\mathbb{R}^N\setminus B(x_j,2\rho)))\cup(B(x_j,2\rho)\times\mathbb{R}^N) \\ &\quad\cup((\mathbb{R}^N\setminus B(x_j,2\rho))\times B(x_j,2\rho)). \end{aligned}$$

Then we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy = \iint_{B(x_j,2\rho)\times\mathbb{R}^N} \frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy \\
&+ \iint_{(\mathbb{R}^N\setminus B(x_j,2\rho))\times B(x_j,2\rho)} \frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy \\
&\le C\rho^{-2s}\int_{B(x_j,K\rho)}|u_n(x)|^2dx + CK^{-N}\left(\int_{\mathbb{R}^N\setminus B(x_j,K\rho)}|u_n(x)|^{2_s^*}dx\right)^{2/2_s^*} \\
&\le C\rho^{-2s}\int_{B(x_j,K\rho)}|u_n(x)|^2dx + CK^{-N}.
\end{aligned}$$

Since $\{u_n\}_n$ is bounded in E, it follows from (4.5) and (4.6) that $\{u_n\phi_\rho\}_n$ is bounded in E. Then $\langle J'_\varepsilon(u_n), u_n\phi_\rho\rangle \to 0$, which implies

$$\begin{aligned}
&M\left([u_n]^2_{s,A_\varepsilon}\right)\iint_{\mathbb{R}^{2N}} \frac{\left|u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y)\right|^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy + \varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2\phi_\rho(x)dx \\
&= -\mathrm{Re}\left\{M\left([u_n]^2_{s,A_\varepsilon}\right)\iint_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_\rho(x)-\phi_\rho(y))}}{|x-y|^{N+2s}}dxdy\right\} \\
&+ \varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2_s^*}\phi_\rho dx + \varepsilon^{-2s}\int_{\mathbb{R}^N}h(x,|u_n|^2)|u_n|^2\phi_\rho(x)dx + o_n(1).
\end{aligned} \tag{4.7}$$

It follows from $\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dy \rightharpoonup \eta$ weakly * in $\mathcal{M}(\mathbb{R}^N)$ that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^2\phi_\rho(y)}{|x-y|^{N+2s}}dydx = \int_{\mathbb{R}^N}\phi_\rho d\eta.$$

By the diamagnetic inequality in Lemma 2.1, we have

$$\iint_{\mathbb{R}^{2N}}\frac{|\,|u_n(x)|-|u_n(y)|\,|^2\phi_\rho(y)}{|x-y|^{N+2s}}dxdy \le \int_{\mathbb{R}^N}\phi_\rho d\eta,$$

as $n\to\infty$ and

$$\int_{\mathbb{R}^N}\phi_\rho d\mu \to \eta(\{x_i\})$$

as $\rho\to 0$. Note that the Hölder inequality implies

$$\begin{aligned}
&\left|\mathrm{Re}\left\{M\left([u_n]^2_{s,A_\varepsilon}\right)\iint_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_\rho(x)-\phi_\rho(y))}}{|x-y|^{N+2s}}dxdy\right\}\right| \\
&\le C\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y)|\cdot|\phi_\rho(x)-\phi_\rho(y)|\cdot|u_n(x)|}{|x-y|^{N+2s}}dxdy \\
&\le C\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}.
\end{aligned}\tag{4.8}$$

Now, we claim that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy = 0. \tag{4.9}$$

Note that $u_n \rightharpoonup u$ weakly converges in E. By Proposition 2.1 we obtain that $u_n \to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$, $1\le t<2_s^*$, which implies in (4.6)

$$C\rho^{-2s}\int_{B(x_i,K\rho)}|u_n(x)|^2dx + CK^{-N} \to C\rho^{-2s}\int_{B(x_i,K\rho)}|u(x)|^2dx + CK^{-N},$$

as $n\to\infty$. Then the Hölder inequality yields

$$C\rho^{-2s}\int_{B(x_i,K\rho)}|u(x)|^2dx + CK^{-N} \le CK^{2s}\left(\int_{B(x_i,K\rho)}|u(x)|^{2_s^*}dx\right)^{2/2_s^*} + CK^{-N} \to CK^{-N}$$

as $\rho \to 0$. Furthermore, by (4.6) we have

$$\begin{aligned}&\limsup_{\rho\to 0}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy\\&=\lim_{K\to\infty}\limsup_{\rho\to 0}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_\rho(x)-\phi_\rho(y)|^2}{|x-y|^{N+2s}}dxdy=0.\end{aligned}$$

Hence the claim is proved.

It follows from the definition of ϕ_ρ and $u_n \to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$, $1 \le t < 2^*_s$, that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} h(x,|u_n|^2)|u_n|^2\phi_\rho(x)dx=0, \tag{4.10}$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n|^2\phi_\rho(x)dx=0. \tag{4.11}$$

Since ϕ_ρ has compact support, letting $n\to\infty$ in (4.7), we can deduce from (4.8)–(4.11) and the diamagnetic inequality that

$$\alpha_0\eta(\{x_j\}) \le \varepsilon^{-2s}\nu_j.$$

Combining this fact with Lemma 4.1, we obtain $\nu_j \ge \alpha_0\varepsilon^{2s}S\nu_j^{2/2^*_s}$. This result implies that

$$\text{(I)}\quad \nu_j=0 \quad\text{or}\quad \text{(II)}\quad \nu_j \ge (\alpha_0 S)^{\frac{N}{2s}}\varepsilon^N.$$

To obtain the possible concentration of mass at infinity, we similarly define a cutoff function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x)=0$ on $|x|<R$ and $\phi_R(x)=1$ on $|x|>R+1$. We can verify that $\{u_n\phi_R\}_n$ is bounded in E, hence $\langle J'_\varepsilon(u_n), u_n\phi_R\rangle \to 0$, as $n\to\infty$, which implies

$$\begin{aligned}&M\left([u_n]^2_{s,A_\varepsilon}\right)\iint_{\mathbb{R}^{2N}}\frac{\left|u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y)\right|^2\phi_R(y)}{|x-y|^{N+2s}}dxdy+\varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2\phi_R(x)dx\\&=-\mathrm{Re}\left\{M\left([u_n]^2_{s,A_\varepsilon}\right)\iint_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{N+2s}}dxdy\right\}\\&\quad+\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2^*_s}\phi_R dx+\varepsilon^{-2s}\int_{\mathbb{R}^N}h(x,|u_n|^2)|u_n|^2\phi_R(x)dx+o_n(1).\end{aligned} \tag{4.12}$$

It is easy to verify that

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{||u_n(x)|-|u_n(y)||^2\phi_R(y)}{|x-y|^{N+2s}}dxdy=\eta_\infty$$

and

$$\begin{aligned}&\left|\mathrm{Re}\left\{M\left([u_n]^2_{s,A_\varepsilon}\right)\iint_{\mathbb{R}^{2N}}\frac{(u_n(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{N+2s}}dxdy\right\}\right|\\&\le C\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}.\end{aligned}$$

Note that

$$\begin{aligned}&\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}dxdy\\&=\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{N+2s}}dxdy.\end{aligned}$$

Similar to the proof of Lemma 3.4 in [30], we have

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{N+2s}}dxdy=0. \tag{4.13}$$

It follows from the definition of ϕ_R that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}h(x,|u_n|^2)|u_n|^2\phi_R(x)dx=0 \tag{4.14}$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n|^2\phi_R(x)dx = 0. \tag{4.15}$$

It follows from (4.13)–(4.15) that

$$\alpha_0\mu_\infty \le \varepsilon^{-2s}\nu_\infty$$

as $R\to\infty$ in (4.12). By Lemma 4.2, we obtain $\nu_\infty \ge \alpha_0\varepsilon^{2s}S\nu_\infty^{2/2_s^*}$. This result implies that

$$\text{(III)}\quad \nu_\infty = 0 \text{ or } \quad \text{(IV)}\quad \nu_\infty \ge (\alpha_0 S)^{\frac{N}{2s}}\varepsilon^N.$$

Next, we claim that (II) and (IV) cannot occur. If the case (IV) holds for some $j\in I$, then by Lemma 4.2, (M) and (H), we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_\varepsilon(u_n) - \frac{1}{\mu}\langle J_\varepsilon'(u_n), u_n\rangle\right)\\
&\ge \left(\frac{\sigma}{2}-\frac{1}{\mu}\right)M\left([u_n]_{s,A_\varepsilon}^2\right)[u_n]_{s,A_\varepsilon}^2 + \left(\frac{1}{2}-\frac{1}{\mu}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2dx\\
&\quad + \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx + \varepsilon^{-2s}\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,|u_n|^2)|u_n|^2 - \frac{1}{2}H(x,|u_n|^2)\right]dx\\
&\ge \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx \ge \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\varepsilon^{-2s}\nu_\infty \ge \sigma_0\varepsilon^{N-2s},
\end{aligned}$$

where $\sigma_0 = \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)(\alpha_0 S)^{N/(2s)}$, which contradicts the condition $c\in\left(0, \sigma_0\varepsilon^{N-2s}\right)$. Consequently, $\nu_j = 0$ for all $j\in I$. Similarly, we can prove that (II) cannot occur for any j. Thus

$$\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx \to \int_{\mathbb{R}^N}|u|^{2_s^*}dx, \tag{4.16}$$

as $n\to\infty$. Since $|u_n-u|^{2_s^*} \le 2^{2_s^*}(|u_n|^{2_s^*}+|u|^{2_s^*})$, it follows from the Fatou lemma that

$$\begin{aligned}
\int_{\mathbb{R}^N}2^{2_s^*+1}|u|^{2_s^*}dx &= \int_{\mathbb{R}^N}\liminf_{n\to\infty}(2^{2_s^*}|u_n|^{2_s^*} + 2^{2_s^*}|u|^{2_s^*} - |u_n-u|^{2_s^*})dx\\
&\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}(2^{2_s^*}|u_n|^{2_s^*} + 2^{2_s^*}|u|^{2_s^*} - |u_n-u|^{2_s^*})dx\\
&= \int_{\mathbb{R}^N}2^{2_s^*+1}|u|^{2_s^*}dx - \limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{2_s^*}dx,
\end{aligned}$$

which implies that $\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{2_s^*}dx = 0$. Then

$$u_n\to u \quad\text{in}\quad L^{2_s^*}(\mathbb{R}^N)\quad\text{as}\quad n\to\infty.$$

By the weak lower semicontinuity of the norm, condition (m_1) and the Brézis–Lieb lemma, we have

$$\begin{aligned}
o(1)\|u_n\| &= \langle J_\varepsilon'(u_n), u_n\rangle = M\left([u_n]_{s,A_\varepsilon}^2\right)[u_n]_{s,A_\varepsilon}^2 + \varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u_n|^2dx\\
&\quad - \varepsilon^{-2s}\int_{\mathbb{R}^N}|u_n|^{2^*(s)}dx - \varepsilon^{-2s}\int_{\mathbb{R}^N}h(x,|u_n|^2)|u_n|^2dx\\
&\ge \alpha_0\left([u_n]_{s,A_\varepsilon}^2 - [u]_{s,A_\varepsilon}^2\right) + \varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)(|u_n|^2-|u|^2)dx + M\left([u]_{s,A_\varepsilon}^2\right)[u]_{s,A_\varepsilon}^2\\
&\quad + \varepsilon^{-2s}\int_{\mathbb{R}^N}V(x)|u|^2dx - \varepsilon^{-2s}\int_{\mathbb{R}^N}|u|^{2_s^*}dx - \varepsilon^{-2s}\int_{\mathbb{R}^N}h(x,|u|^2)|u|^2dx\\
&\ge \min\{\alpha_0, 1\}\|u_n-u\|_\varepsilon^2 + o(1)\|u\|_\varepsilon.
\end{aligned}$$

Here we use the fact that $J_\varepsilon'(u) = 0$. Thus we have proved that $\{u_n\}_n$ strongly converges to u in E. Hence the proof is complete. □

5. Proof of Theorem 3.1

In the following, we will always consider $0<\varepsilon<1$. By the assumptions (V), (M) and (H), one can see that $J_\varepsilon(u)$ has the mountain pass geometry.

Lemma 5.1. *Assume that conditions* (V), (M) *and* (H) *hold. Then there exist* $\alpha_\varepsilon, \varrho_\varepsilon > 0$ *such that* $J_\varepsilon(u) > 0$ *if* $u \in B_{\varrho_\varepsilon} \setminus \{0\}$ *and* $J_\varepsilon(u) \geq \alpha_\varepsilon$ *if* $u \in \partial B_{\varrho_\varepsilon}$, *where* $B_{\varrho_\varepsilon} = \{u \in E : \|u\|_\varepsilon \leq \varrho_\varepsilon\}$.

Proof. By (H), for $0 < \xi \leq \left(2\min\left\{\frac{\sigma\alpha_0}{2}, \frac{1}{2}\right\} c_2^2\right)^{-1}\varepsilon^{2s}$, there is $C_\xi > 0$ such that

$$\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx + \frac{1}{2}\int_{\mathbb{R}^N}H(x,|u|^2)dx \leq \xi\|u\|_{L^2}^2 + C_\xi\|u\|_{L^{2_s^*}}^{2_s^*},$$

where c_2 is the embedding constant in (2.1) with $\theta = 2$. It follows from (V), (M) and (H), that

$$\begin{aligned}
J_\varepsilon(u) &= \frac{1}{2}\widetilde{M}\left([u]_{s,A_\varepsilon}^2\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N}V(x)|u|^2dx - \frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx - \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N}H(x,|u|^2)dx \\
&\geq \min\left\{\frac{\sigma\alpha_0}{2}, \frac{1}{2}\right\}\|u\|_\varepsilon^2 - \varepsilon^{-2s}\xi\|u\|_{L^2}^2 - \varepsilon^{-2s}C_\xi\|u\|_{L^{2_s^*}}^{2_s^*} \\
&\geq \frac{1}{2}\min\left\{\frac{\sigma\alpha_0}{2}, \frac{1}{2}\right\}\|u\|_\varepsilon^2 - \varepsilon^{-2s}C_\xi\|u\|_{L^{2_s^*}}^{2_s^*} \\
&\geq \frac{1}{2}\min\left\{\frac{\sigma\alpha_0}{2}, \frac{1}{2}\right\}\|u\|_\varepsilon^2 - \varepsilon^{-2s}C_\xi c_{2_s^*}^{2_s^*}\|u\|_\varepsilon^{2_s^*}.
\end{aligned}$$

Then, for all $u \in E$, with $\|u\|_\varepsilon = \rho_\varepsilon$, $\rho_\varepsilon \in (0,1)$ sufficiently small so that

$$\frac{1}{2}\min\left\{\frac{\sigma\alpha_0}{2}, \frac{1}{2}\right\} - \varepsilon^{-2s}C_\xi c_{2_s^*}^{2_s^*}\rho_\varepsilon^{2_s^*-2} > 0.$$

Thus, the lemma is proved by taking

$$\alpha_\varepsilon = \frac{1}{2}\min\left\{\frac{\sigma\alpha_0}{2}, \frac{1}{2}\right\}\rho_\varepsilon^2 - \varepsilon^{-2s}C_\xi c_{2_s^*}^{2_s^*}\rho_\varepsilon^{2_s^*}.$$

The proof is finished. □

Lemma 5.2. *Under the assumptions of Lemma 5.1, for any finite dimensional subspace* $F \subset E$,

$$J_\varepsilon(u) \to -\infty \quad \textit{as} \quad \|u\|_\varepsilon \to \infty \textit{ with } u \in F.$$

Proof. By integrating (m_2), we obtain

$$\widetilde{M}(t) \leq \frac{\widetilde{M}(t_0)}{t_0^{1/\sigma}}t^{1/\sigma} = C_0 t^{1/\sigma} \quad \text{for all } t \geq t_0 > 0. \tag{5.1}$$

Using conditions (V) and (H), we can get

$$J_\varepsilon(u) \leq \frac{C_0}{2}\|u\|_\varepsilon^{\frac{2}{\sigma}} + \frac{1}{2}\|u\|_\varepsilon^2 - \frac{\varepsilon^{-2s}}{2_s^*}\|u\|_{L^{2_s^*}}^{2_s^*} - \varepsilon^{-2s}l_0\|u\|_{L^r}^r$$

for all $u \in F$. Since all norms in a finite-dimensional space are equivalent and $2 \leq 2/\sigma < 2_s^*$, we conclude that $J_\varepsilon(u) \to -\infty$ as $\|u\|_\varepsilon \to \infty$. The proof is thus complete. □

Note that $J_\varepsilon(u)$ does not satisfy $(PS)_c$ condition for any $c > 0$. Thus, in the sequel we will find a special finite-dimensional subspace by which we will construct sufficiently small minimax levels.

Recall that the assumption (V) implies that there is $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N}V(x) = 0$. Without loss of generality we can assume from now on that $x_0 = 0$. We first notice that condition (h_3) implies

$$\frac{\varepsilon^{-2s}}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N}H(x,|u|^2)dx \geq l_0\varepsilon^{-2s}\int_{\mathbb{R}^N}|u|^r dx.$$

Define the functional $I_\varepsilon \in C^1(E,\mathbb{R})$ by

$$I_\varepsilon(u) := \frac{1}{2}M\left([u]_{s,A_\varepsilon}^2\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^N}V(x)|u|^2dx - l_0\varepsilon^{-2s}\int_{\mathbb{R}^N}|u|^r dx.$$

Then $J_\varepsilon(u) \leq I_\varepsilon(u)$ for all $u \in E$ and it suffices to construct small minimax levels for I_ε.

Note that

$$\inf\left\{\iint_{\mathbb{R}^{2N}}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}dxdy : \phi \in C_0^\infty(\mathbb{R}^N), |\phi|_q = 1\right\} = 0,$$

see [42, Theorem 3.2] for this proof. For any $0<\zeta<1$ one can choose $\phi_\zeta \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi_\zeta\|_{L^q}=1$ and $\operatorname{supp}\phi_\zeta \subset B_{r_\zeta}(0)$ so that

$$\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(x)-\phi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy \le C\zeta^{\frac{2N-(N-2s)q}{q}}.$$

Set

$$\psi_\zeta(x)=e^{iA(0)x}\phi_\zeta(x) \tag{5.2}$$

and

$$\psi_{\varepsilon,\zeta}(x)=\psi_\zeta(\varepsilon^{-1}x). \tag{5.3}$$

By condition (5.1), we get for any $t>0$,

$$\begin{aligned} I_\varepsilon(t\psi_{\varepsilon,\zeta}) \le\ & \frac{C_0}{2}t^{\frac{2}{\sigma}}\left(\iint_{\mathbb{R}^{2N}} \frac{\left|\psi_{\varepsilon,\zeta}(x)-e^{i(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}\psi_{\varepsilon,\zeta}(y)\right|^2}{|x-y|^{N+2s}}dxdy\right)^{1/\sigma} \\ & + \frac{t^2}{2}\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|\psi_{\varepsilon,\zeta}|^2dx - t^r l_0\varepsilon^{-2s}\int_{\mathbb{R}^N}|\psi_{\varepsilon,\zeta}|^r dx \\ \le\ & \varepsilon^{N-2s}\left[\frac{C_0}{2}t^{\frac{2}{\sigma}}\left(\iint_{\mathbb{R}^{2N}} \frac{\left|\psi_\zeta(x)-e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}\psi_\zeta(y)\right|^2}{|x-y|^{N+2s}}dxdy\right)^{1/\sigma}\right. \\ & \left.+\frac{t^2}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|\psi_\zeta|^2dx - t^r l_0\int_{\mathbb{R}^N}|\psi_\zeta|^r dx\right] \\ =\ & \varepsilon^{N-2s}\Psi_\varepsilon(t\psi_\zeta), \end{aligned}$$

where $\Psi_\varepsilon \in C^1(E,\mathbb{R})$ defined by

$$\begin{aligned} \Psi_\varepsilon(u) :=\ & \frac{C_0}{2}\left(\iint_{\mathbb{R}^{2N}} \frac{\left|u(x)-e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}u(y)\right|^2}{|x-y|^{N+2s}}dxdy\right)^{1/\sigma} \\ & + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2dx - l_0\int_{\mathbb{R}^N}|u|^r dx. \end{aligned}$$

Since $r>2/\sigma$, there exists a finite number $t_0\in[0,+\infty)$ such that

$$\begin{aligned} \max_{t\ge 0}\Psi_\varepsilon(t\psi_\zeta) =\ & \frac{C_0}{2}t_0^{2/\sigma}\left(\iint_{\mathbb{R}^{2N}} \frac{\left|\psi_\zeta(x)-e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}\psi_\zeta(y)\right|^2}{|x-y|^{N+2s}}dxdy\right)^{1/\sigma} \\ & + \frac{t_0^2}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|\psi_\zeta|^2dx - t_0^r l_0\int_{\mathbb{R}^N}|\psi_\zeta|^r dx \\ \le\ & \frac{C_0}{2}t_0^{2/\sigma}\left(\iint_{\mathbb{R}^{2N}} \frac{\left|\psi_\zeta(x)-e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}\psi_\zeta(y)\right|^2}{|x-y|^{N+2s}}dxdy\right)^{1/\sigma} \\ & + \frac{t_0^2}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|\psi_\zeta|^2dx. \end{aligned}$$

Let $\psi_\zeta(x)=e^{iA(0)x}\phi_\zeta(x)$, where $\phi_\zeta(x)$ is as defined above. We have the following lemma.

Lemma 5.3. *For any $\zeta>0$ there exists $\varepsilon_0=\varepsilon_0(\zeta)>0$ such that*

$$\iint_{\mathbb{R}^{2N}} \frac{\left|\psi_\zeta(x)-e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}\psi_\zeta(y)\right|^2}{|x-y|^{N+2s}}dxdy \le C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s},$$

for all $0<\varepsilon<\varepsilon_0$ and some constant $C>0$ depending only on $[\phi_\zeta]_{s,0}$.

Proof. For any $\zeta > 0$, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\psi_\zeta(x) - e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}\psi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy \\
&\le \iint_{\mathbb{R}^{2N}} \frac{|e^{iA(0)\cdot x}\phi_\zeta(x) - e^{i(x-y)\cdot A(\frac{\varepsilon x+\varepsilon y}{2})}e^{iA(0)\cdot y}\phi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy \\
&\le 2\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(x)-\phi_\zeta(y)|^2}{|x-y|^{N+2s}}dxdy + 2\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(y)|^2|e^{i(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}-1|^2}{|x-y|^{N+2s}}dxdy.
\end{aligned}$$

Next, we will estimate the second term in the above inequality. Notice that

$$\left|e^{i(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}-1\right|^2 = 4\sin^2\left[\frac{(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}{2}\right]. \tag{5.4}$$

For any $y \in B_{r_\zeta}$, if $|x-y| \le \frac{1}{\zeta}\|\phi_\zeta\|_{L^2}^{1/\alpha}$, then $|x| \le r_\zeta + \frac{1}{\zeta}\|\phi_\zeta\|_{L^2}^{1/\alpha}$. Hence, we have

$$\left|\frac{\varepsilon x+\varepsilon y}{2}\right| \le \frac{\varepsilon}{2}\left(2r_\zeta + \frac{1}{\zeta}\|\phi_\zeta\|_{L^2}^{1/\alpha}\right).$$

Since $A : \mathbb{R}^N \to \mathbb{R}^N$ is continuous, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, we have

$$\left|A(0) - A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)\right| \le \zeta\|\phi_\zeta\|_{L^2}^{-1/\alpha} \quad \text{for } |y| \le r_\zeta \text{ and } |x| \le r_\zeta + \frac{1}{\zeta}\|\phi_\zeta\|_{L^2}^{1/\alpha},$$

which implies

$$\left|e^{i(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}-1\right|^2 \le |x-y|^2\zeta^2\|\phi_\zeta\|_{L^2}^{-2/\alpha}.$$

For all $\zeta > 0$ and $y \in B_{r_\zeta}$, let us define

$$N_{\zeta,y} := \left\{x \in \mathbb{R}^N : |x-y| \le \frac{1}{\zeta}\|\phi_\zeta\|_{L^2}^{1/\alpha}\right\}.$$

Then together with the above facts, we have for all $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\phi_\zeta(y)|^2|e^{i(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}-1|^2}{|x-y|^{N+2s}}dxdy \\
&= \int_{B_{r_\zeta}} |\phi_\zeta(y)|^2 dy \int_{N_{\zeta,y}} \frac{\left|e^{i(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}-1\right|^2}{|x-y|^{N+2s}}dx + \\
&\qquad \int_{B_{r_\zeta}} |\phi_\zeta(y)|^2 dy \int_{\mathbb{R}^N\setminus N_{\zeta,y}} \frac{\left|e^{i(x-y)\cdot(A(0)-A(\frac{\varepsilon x+\varepsilon y}{2}))}-1\right|^2}{|x-y|^{N+2s}}dx \\
&\le \int_{B_{r_\zeta}} |\phi_\zeta(y)|^2 dy \int_{N_{\zeta,y}} \frac{|x-y|^2}{|x-y|^{N+2s}}\zeta^2|\phi_\zeta|_{L^2}^{-\frac{2}{\alpha}}dx + \int_{B_{r_\zeta}} |\phi_\zeta(y)|^2 dy \int_{\mathbb{R}^N\setminus N_{\zeta,y}} \frac{4}{|x-y|^{N+2s}}dx \\
&\le \frac{1}{2-2s}\zeta^{2s} + \frac{4}{2s}\zeta^{2s}.
\end{aligned}$$

This completes the proof. □

Next, since $V(0) = 0$ and $\operatorname{supp}\phi_\zeta \subset B_{r_\zeta}(0)$, there is $\varepsilon^* > 0$ such that

$$V(\varepsilon x) \le \frac{\zeta}{\|\phi_\zeta\|_{L^2}^2} \quad \text{for all } |x| \le r_\zeta \text{ and } 0 < \varepsilon < \varepsilon^*.$$

This together with Lemma 5.3 implies that

$$\max_{t\ge 0}\Psi_\varepsilon(t\phi_\zeta) \le \frac{C_0}{2}t_0^{2/\sigma}\left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s}\right)^{1/\sigma} + \frac{t_0^2}{2}\zeta. \tag{5.5}$$

Therefore, we have for all $0<\varepsilon<\min\{\varepsilon_0,\varepsilon^*\}$,

$$\max_{t\geq 0} J_\varepsilon(t\psi_{\varepsilon,\zeta})\leq\left[\frac{C_0}{2}t_0^{2/\sigma}\left(C\zeta^{\frac{2N-(N-2s)q}{q}}+\frac{1}{1-s}\zeta^{2s}+\frac{4}{s}\zeta^{2s}\right)^{1/\sigma}+\frac{t_0^2}{2}\zeta\right]\varepsilon^{N-2s}. \tag{5.6}$$

We are now ready to prove the following lemma.

Lemma 5.4. *Under the assumptions of Lemma 5.1, for any $\kappa>0$ there exists $\mathcal{E}_\kappa>0$ such that for each $0<\varepsilon<\mathcal{E}_\kappa$, there is $\widehat{e}_\varepsilon\in E$ with $\|\widehat{e}_\varepsilon\|_\varepsilon>\varrho_\varepsilon$, $J_\varepsilon(\widehat{e}_\varepsilon)\leq 0$, and*

$$\max_{t\in[0,1]} J_\varepsilon(t\widehat{e}_\varepsilon)\leq\kappa\varepsilon^{N-2s}. \tag{5.7}$$

Proof. Choose $\zeta>0$ so small that

$$\frac{C_0}{2}t_0^{\frac{2}{\sigma}}\left(C\zeta^{\frac{2N-(N-2s)q}{q}}+\frac{1}{1-s}\zeta^{2s}+\frac{4}{s}\zeta^{2s}\right)^{\frac{1}{\sigma}}+\frac{1}{2}t_0^2\zeta\leq\kappa.$$

Let $\psi_{\varepsilon,\zeta}\in E$ be the function defined by (5.3). Set $\mathcal{E}_\kappa=\min\{\varepsilon_0,\varepsilon^*\}$. Let $\widehat{t_\varepsilon}>0$ be such that $\widehat{t_\varepsilon}\|\psi_{\varepsilon,\zeta}\|_\varepsilon>\varrho_\varepsilon$ and $J_\varepsilon(t\psi_{\varepsilon,\zeta})\leq 0$ for all $t\geq\widehat{t_\varepsilon}$. Invoking (5.6), we let $\widehat{e}_\varepsilon=\widehat{t_\varepsilon}\psi_{\varepsilon,\zeta}$ and check that the conclusion of Lemma 5.4 holds. □

For any $m^*\in\mathbb{N}$, one can choose m^* functions $\phi_\zeta^i\in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\zeta^i\cap\operatorname{supp}\phi_\zeta^k=\emptyset$, $i\neq k$, $\|\phi_\zeta^i\|_{L^s}=1$ and

$$\iint_{\mathbb{R}^{2N}}\frac{|\phi_\zeta^i(x)-\phi_\zeta^i(y)|^2}{|x-y|^{N+2s}}dxdy\leq C\zeta^{\frac{2N-(N-2s)q}{q}}.$$

Let $r_\zeta^{m^*}>0$ be such that $\operatorname{supp}\phi_\zeta^i\subset B_{r_\zeta}^i(0)$ for $i=1,2,\ldots,m^*$. Set

$$\psi_\zeta^i(x)=e^{iA(0)x}\phi_\zeta^i(x) \tag{5.8}$$

and

$$\psi_{\varepsilon,\zeta}^i(x)=\psi_\zeta^i(\varepsilon^{-1}x). \tag{5.9}$$

Denote

$$H_{\varepsilon\zeta}^{m^*}=\operatorname{span}\{\psi_{\varepsilon,\zeta}^1,\psi_{\varepsilon,\zeta}^2,\cdots,\psi_{\varepsilon,\zeta}^{m^*}\}.$$

Observe that for each $u=\sum_{i=1}^{m^*}c_i\psi_{\varepsilon,\zeta}^i\in H_{\varepsilon\zeta}^{m^*}$, we have

$$[u]_{s,A_\varepsilon}^2\leq C\sum_{i=1}^{m^*}|c_i|^2[\psi_{\varepsilon,\zeta}^i]_{s,A_\varepsilon}^2,$$

for some constant $C>0$,

$$\int_{\mathbb{R}^N}V(x)|u|^2dx=\sum_{i=1}^{m^*}|c_i|^2\int_{\mathbb{R}^N}V(x)|\psi_{\varepsilon,\zeta}^i|^2dx$$

and

$$\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx+\frac{1}{2}\int_{\mathbb{R}^N}H(x,|u|^2)dx=\sum_{i=1}^{m^*}\left(\frac{1}{2_s^*}\int_{\mathbb{R}^N}|c_i\psi_{\varepsilon,\zeta}^i|^{2_s^*}dx+\frac{1}{2}\int_{\mathbb{R}^N}H(x,|c_i\psi_{\varepsilon,\zeta}^i|^2)dx\right).$$

Therefore

$$J_\varepsilon(u)\leq C\sum_{i=1}^{m^*}J_\varepsilon(c_i\psi_{\varepsilon,\zeta}^i)$$

for some constant $C>0$. By a similar argument as before, we can see that

$$J_\varepsilon(c_i\psi_{\varepsilon,\zeta}^i)\leq\varepsilon^{N-2s}\Psi(|c_i|\psi_\zeta^i).$$

As before, we can obtain the following estimate:

$$\max_{u\in H_{\varepsilon\zeta}^{m^*}} J_\varepsilon(u)\leq Cm^*\left[\frac{C_0}{2}t_0^{2/\sigma}\left(C\zeta^{\frac{2N-(N-2s)q}{q}}+\frac{1}{1-s}\zeta^{2s}+\frac{4}{s}\zeta^{2s}\right)^{1/\sigma}+\frac{t_0^2}{2}\zeta\right]\varepsilon^{N-2s} \tag{5.10}$$

for all small enough ζ and some constant $C>0$. Using the estimate (5.10), we shall prove the following lemma.

Lemma 5.5. *Under the assumptions of Lemma 5.1, for any $m^* \in \mathbb{N}$ and $\kappa > 0$ there exists $\mathcal{E}_{m^*\kappa} > 0$ such that for each $0 < \varepsilon < \mathcal{E}_{m^*\kappa}$, there exists an m^*-dimensional subspace $F_{\varepsilon m^*}$ satisfying*

$$\max_{u \in F_{\varepsilon m^*}} J_\varepsilon(u) \leq \kappa \varepsilon^{N-2s}.$$

Proof. Choose $\zeta > 0$ so small that

$$Cm^* \left[\frac{C_0}{2} t_0^{\frac{2}{\sigma}} \left(C\zeta^{\frac{2N-(N-2s)q}{q}} + \frac{1}{1-s}\zeta^{2s} + \frac{4}{s}\zeta^{2s} \right)^{1/\sigma} + \frac{t_0^2}{2}\zeta \right] \leq \kappa.$$

Set $F_{\varepsilon m^*} = H_{\varepsilon\zeta}^{m^*} = \operatorname{span}\{\psi_{\varepsilon,\zeta}^1, \psi_{\varepsilon,\zeta}^2, \ldots, \psi_{\varepsilon,\zeta}^{m^*}\}$. Now the conclusion of Lemma 5.5 follows from (5.10). □

We are now ready to prove our main result which establishes the existence and multiplicity of solutions.

Proof of Theorem 3.1 (1). For any $0 < \kappa < \sigma_0$, by Theorem 4.1, we can choose $\mathcal{E}_\kappa > 0$ and define for $0 < \varepsilon < \mathcal{E}_\kappa$, the minimax value

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(t\widehat{e}_\varepsilon),$$

where

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = \widehat{e}_\varepsilon\}.$$

By Lemma 5.1, we have $\alpha_\varepsilon \leq c_\varepsilon \leq \kappa\varepsilon^{N-2s}$. By virtue of Theorem 4.1, we know that J_ε satisfies the $(PS)_{c_\lambda}$ condition. In view of Lemmas 5.1 and 5.4, it follows from the mountain pass theorem that there is $u_\varepsilon \in E$ such that if $J'_\varepsilon(u_\varepsilon) = 0$ and $J_\varepsilon(u_\varepsilon) = c_\varepsilon$, then u_ε is a nontrivial mountain pass solution of problem (3.1).

Since u_ε is a critical point of J_ε, by (M) and (H) we have for $\tau \in [2, 2_s^*]$,

$$\begin{aligned}
\kappa\varepsilon^{N-2s} &\geq J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon) - \frac{1}{\tau} J'_\varepsilon(u_\varepsilon)u_\varepsilon \\
&= \frac{1}{2}\widetilde{M}\left([u_\varepsilon]_{s,A_\varepsilon}^2\right) - \frac{1}{\tau} M\left([u_\varepsilon]_{s,A_\varepsilon}^2\right)[u_\varepsilon]_{s,A_\varepsilon}^2 + \left(\frac{1}{2} - \frac{1}{\tau}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2 dx \\
&\quad + \left(\frac{1}{\tau} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} |u_\varepsilon|^{2_s^*} dx + \varepsilon^{-2s}\int_{\mathbb{R}^N}\left[\frac{1}{\tau}h(x,|u_\varepsilon|^2)u_\varepsilon - \frac{1}{2}H(x,|u_\varepsilon|^2)\right] dx \\
&\geq \left(\frac{\sigma}{2} - \frac{1}{\tau}\right)\alpha_0 [u_\varepsilon]_{s,A_\varepsilon}^2 + \left(\frac{1}{2} - \frac{1}{\tau}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} V(x)|u_\varepsilon|^2 dx \\
&\quad + \left(\frac{1}{\tau} - \frac{1}{2_s^*}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} |u_\varepsilon|^{2_s^*} dx + \left(\frac{\mu}{\tau} - \frac{1}{2}\right)\varepsilon^{-2s}\int_{\mathbb{R}^N} H(x,|u_\varepsilon|^2) dx.
\end{aligned} \tag{5.11}$$

Taking $\tau = 2/\sigma$, we obtain the estimate (3.2) and taking $\tau = \mu$ we obtain the estimate (3.3). This completes the proof of the first part of Theorem 3.1.

Proof of Theorem 3.1 (2). Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ. For each $Z \in \Sigma$, let $\operatorname{gen}(Z)$ be the Krasnoselskii genus and

$$j(Z) := \min_{\iota \in \Gamma_{m^*}} \operatorname{gen}(\iota(Z) \cap \partial B_{\varrho_\varepsilon}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $\iota \in C(E, E)$ and ϱ_ε is the number from Lemma 5.1. Then j is a version of Benci's pseudoindex [53]. Let

$$c_{\varepsilon i} := \inf_{j(Z) \geq i} \sup_{u \in Z} J_\varepsilon(u), \quad 1 \leq i \leq m^*.$$

Since $J_\varepsilon(u) \geq \alpha_\varepsilon$ for all $u \in \partial B_{\varrho_\varepsilon}^+$ and $j(F_{\varepsilon m^*}) = m^* = \dim F_{\varepsilon m^*}$, we obtain by Lemma 5.5 that

$$\alpha_\varepsilon \leq c_{\varepsilon 1} \leq \cdots \leq c_{\varepsilon m^*} \leq \sup_{u \in F_{\varepsilon m^*}} J_\varepsilon(u) \leq \kappa\varepsilon^{N-2s}.$$

It follows from Theorem 4.1 that J_ε satisfies the $(PS)_{c_\varepsilon}$ condition at all levels $c < \sigma_0\varepsilon^{N-2s}$. By the usual critical point theory, all $c_{\varepsilon i}$ are critical levels and J_ε has at least m^* pairs of nontrivial critical points satisfying

$$\alpha_\varepsilon \leq J_\varepsilon(u_\varepsilon) \leq \kappa\varepsilon^{N-2s}.$$

Hence, problem (3.1) has at least m^* pairs of solutions. Finally, as in the proof of the first of Theorem 3.1, we see that these solutions satisfy the estimates (3.2) and (3.3). This completes the proof of the second part of Theorem 3.1. □

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