Metric projections versus non-positive curvature

Alexandru Kristály [a,*], Dušan Repovš [b]

[a] Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca, Romania
[b] Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana, P.O. Box 2964, 1001 Ljubljana, Slovenia

ABSTRACT

In this paper two metric properties on geodesic length spaces are introduced by means of the metric projection, studying their validity on Alexandrov and Busemann NPC spaces. In particular, we prove that both properties characterize the non-positivity of the sectional curvature on Riemannian manifolds. Further results are also established on reversible/non-reversible Finsler–Minkowski spaces.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

The curvature notions on geodesic length spaces are formulated in terms of the metric distance. Most of them refer to non-positively curved spaces (shortly, NPC spaces) defined by means of certain metric inequalities. Here, we recall (non-rigorously) three such notions:

(a) *Alexandrov NPC spaces* (see [1]): small geodesic triangles are thinner than their Euclidean comparison triangles;
(b) *Busemann NPC spaces* (see [5]): in small geodesic triangles the geodesic segment connecting the midpoints of two sides is at most half as long as the third side;
(c) *Pedersen NPC spaces* (see [14]): small capsules (i.e., the loci equidistant to geodesic segments) are geodesic convex.

It is well known that

"Alexandrov NPC spaces ⊂ Busemann NPC spaces ⊂ Pedersen NPC spaces,"

where the inclusions are proper in general. However, on Riemannian manifolds, all these curvature notions coincide, and they characterize the non-positivity of the sectional curvature. For systematic presentation of NPC spaces, we refer the reader to the monographs of Bridson and Haefliger [3], Busemann [5], and Jost [9].

* Corresponding author.
  E-mail address: alexandrukristaly@yahoo.com (A. Kristály).

The aim of our paper is to capture new features of non-positively curved geodesic length spaces by means of the *metric projection* map. Roughly speaking, on a metric space (M, d), we consider the following two properties we are dealing with in the sequel (for precise notions, see Definitions 2.1 and 2.2):

(I) *Double-projection property*: a point is the best approximation element between two small geodesic convex sets $S_1, S_2 \subset M$ if and only if it is a fixed point of the metric projection map $P_{S_1} \circ P_{S_2}$.
(II) *Projection non-expansiveness property*: the metric projection map P_S is non-expansive for small geodesic convex sets $S \subset M$.

Our results can be summarized as follows (for precise statements and detailed comments, see Section 2). Although Busemann NPC spaces do not satisfy in general the above properties (see Remark 2.1), Alexandrov NPC spaces satisfy both of them (see Theorem 2.1, and Bridson and Haefliger [3, Proposition 2.4]). Furthermore, generic Finsler–Minkowski spaces satisfy the global double-projection property (see Theorem 2.2), but not the global projection non-expansiveness property. Finally, we prove that both properties (I) and (II) encapsulate the concept of non-positive curvature in the Riemannian context; namely, we prove that for Riemannian manifolds the double-projection property, the projection non-expansiveness property and the non-positivity of the sectional curvature are equivalent conditions (see Theorem 2.3).

2. Main results and remarks

Let (M, d) be a metric space and let

$$P_S(q) = \left\{ s \in S \colon d(q, s) = \inf_{z \in S} d(q, z) \right\} \tag{2.1}$$

be the usual *metric projection* of the point $q \in M$ onto the nonempty set $S \subset M$. If $S \subset U \subset M$, the set S is called *U-proximinal* if $P_S(q) \neq \emptyset$ for every $q \in U$ (w.r.t. the metric d), and *U-Chebyshev* if $P_S(q)$ is a singleton for every $q \in U$.

Definition 2.1. The metric space (M, d) satisfies the *double-projection property* if every point $p \in M$ has a neighborhood $U \subset M$ such that (U, d) is a geodesic length space, and for every two geodesic convex, U-proximinal sets $S_1, S_2 \subset U$ and for some $q \in S_1$ the following statements are equivalent:

(DP_1) $q \in (P_{S_1} \circ P_{S_2})(q)$;
(DP_2) there exists $\tilde{q} \in P_{S_2}(q)$ such that $d(q, \tilde{q}) \leqslant d(z_1, z_2)$ for all $z_1 \in S_1, z_2 \in S_2$.

If $U = M$, then (M, d) satisfies the *global double-projection property*.

The element $q \in S_1$ satisfying (DP_2) is called the *best approximation point from the set S_1 to S_2*. We notice that P_{S_i} $(i = 1, 2)$ may be set-valued maps in the Definition 2.1.

Definition 2.2. The metric space (M, d) satisfies the *projection non-expansiveness property* if every point $p \in M$ has a neighborhood $U \subset M$ such that (U, d) is a geodesic length space, and for every geodesic convex, U-proximinal set $S \subset U$, one has

$$d\big(P_S(q_1), P_S(q_2)\big) \leqslant d(q_1, q_2) \quad \text{for every } q_1, q_2 \in U. \tag{2.2}$$

If $U = M$, then (M, d) satisfies the *global projection non-expansiveness property*.

Note that if a set S satisfies (2.2), it is necessarily a U-Chebyshev set.

Remark 2.1. Let us discuss first the relationship between these properties and Busemann NPC spaces. We recall that every Minkowski space in the classical sense (i.e., normed linear space with strictly convex unit ball) is a Busemann NPC space, see Busemann [5].

(a) *Double-projection property fails in Busemann NPC spaces*: Let $(\mathbb{R}^3, F)$ be a Minkowski space with strictly convex unit balls. Assume that F is non-differentiable at $p \in I = \{p \in \mathbb{R}^3 \colon F(p) = 1\}$; then due to the symmetry of F, the same holds at $q = -p \in I$. On account of this assumption, we may consider supporting planes H_p and H_q at p and q to the unit ball $B = \{p \in \mathbb{R}^3 \colon F(p) \leqslant 1\}$, respectively, such that $H_p \cap H_q \neq \emptyset$. We translate H_q to the origin, denoting it by H_0. Let us finally consider an arbitrary plane H containing the origin and the point p, and $H_0 \cap H_p \cap H = \{z\}$. If $S_1 = [p, z]$ and $S_2 = [0, z]$, then by construction, one has $P_{S_1}(0) = p$ and $P_{S_2}(p) = 0$, thus $(P_{S_2} \circ P_{S_1})(0) = 0$. If the double-projection property holds (up to a scaling of the indicatrix I), then we have that $d_F(0, p) \leqslant d_F(z_1, z_2)$ for every $z_1 \in S_1$ and $z_2 \in S_2$. Let $z_1 = z_2 = z \in S_1 \cap S_2$; the latter inequality implies the contradiction $1 = F(p) = d_F(0, p) \leqslant 0$.

(b) *Global projection non-expansiveness property fails in Busemann NPC spaces*: Due to Phelps [15, Theorem 5.2], a Minkowski space (with dimension at least three) which satisfies the global projection non-expansiveness property, is necessarily Euclidean.

Next, we treat these two properties on Alexandrov NPC spaces. First, it is a well known fact that every Alexandrov NPC space satisfies the projection non-expansiveness property, see Bridson and Haefliger [3, Proposition 2.4]. Our first result reads as follows.

Theorem 2.1. *Every Alexandrov NPC space satisfies the double-projection property.*

Remark 2.2. (a) We provide two independent proofs of Theorem 2.1, each of them exploiting basic properties of Alexandrov NPC spaces: (1) Pythagorean and Ptolemaic inequalities; (2) the first variation formula and non-expansiveness of the metric projection.

(b) With respect to Remark 2.1, if we assume that a Busemann NPC space is also Ptolemy (i.e., the Ptolemaic inequality holds for every quadruple), the double-projection property holds. In fact, the latter statement is precisely Theorem 2.1, exploiting the famous characterization of CAT(0)-spaces by Foertsch, Lytchak and Schroeder [8], i.e., a metric space is a CAT(0)-space if and only if it is a Ptolemy and a Busemann NPC space.

We now present a genuinely different class of spaces where the double-projection property holds.

Theorem 2.2. *Every reversible Finsler–Minkowski space satisfies the global double-projection property.*

Remark 2.3. (a) Hereafter, the Finsler–Minkowski space is understood in the sense of Finsler geometry, see Bao, Chern and Shen [2]; in particular, we assume that the norm F belongs to $C^2(\mathbb{R}^n \setminus \{0\})$; see Section 3. As we already pointed out in Remark 2.1(a), the double-projection fails on Minkowski spaces with non-differentiable unit balls.

(b) We emphasize that the proof of Theorem 2.2 cannot follow any of the lines described in Remark 2.2(a). First, a rigidity result due to Schoenberg [16] shows that any Minkowski space on which the Ptolemaic inequality holds is necessarily Euclidean; see also Buckley, Falk and Wraith [4]. Second, if we want to apply the global projection non-expansiveness property, we come up against the rigidity result of Phelps [15, Theorem 5.2], see also Remark 2.1(b). However, the fundamental inequality of Finsler geometry and some results from Kristály, Rădulescu and Varga [12] provide a simple proof of Theorem 2.2, where the fact that F belongs to $C^2(\mathbb{R}^n \setminus \{0\})$ plays an indispensable role.

In spite of the above remarks, the following characterization can be proved in the Riemannian framework which entitles us to assert that the notions introduced in Definitions 2.1 and 2.2 provide new features of the non-positive curvature.

Theorem 2.3. *Let (M, g) be a smooth Riemannian manifold and d_g be the induced metric on M. Then the following assertions are equivalent*:

 (i) *(M, d_g) satisfies the double-projection property*;
(ii) *(M, d_g) satisfies the projection non-expansiveness property*;
(iii) *the sectional curvature of (M, g) is non-positive.*

The proof of Theorem 2.3 is based on the Toponogov comparison theorem and on the formula of the sectional curvature given by the Levi-Civita parallelogramoid.

In order for the paper to be self-contained, we recall in Section 3 some basic notions and results from Alexandrov NPC spaces and Finsler–Minkowski spaces. In Section 4 we present the proof of Theorems 2.1 and 2.3, while in Section 5 we prove Theorem 2.2 and also discuss some aspects of the double-projection property on non-reversible Finsler–Minkowski spaces.

3. Preliminaries

3.1. Alexandrov NPC spaces

We recall those notions and results from the theory Alexandrov NPC spaces which will be used in the proof of Theorems 2.1 and 2.3; for details, see Bridson and Haefliger [3, Chapter II], and Jost [9].

A metric space (M, d) is a *geodesic length space* if for every two points $p, q \in M$, there exists the shortest geodesic segment joining them, i.e., a continuous curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$ and $l(\gamma) = d(p, q)$, where

$$l(\gamma) = \sup \left\{ \sum_{i=1}^{m} d\big(\gamma(t_{i-1}), \gamma(t_i)\big) \colon 0 = t_0 < \cdots < t_m = 1, \ m \in \mathbb{N} \right\}.$$

We assume that geodesics are parametrized proportionally by the arc-length.

Given a real number κ, let M_κ^2 be the two-dimensional space form with curvature κ, i.e., $M_0^2 = \mathbb{R}^2$ is the Euclidean plane, M_κ^2 is the sphere with radius $1/\sqrt{\kappa}$ if $\kappa > 0$, and M_κ^2 is the hyperbolic plane with the function multiplied by $1/\sqrt{-\kappa}$

if $\kappa < 0$. If $p, q, r \in M$, a *geodesic triangle* $\Delta(p, q, r)$ in (M, d) is defined by the three vertices and a choice of three sides which are geodesic segments joining them (they need not be unique). A triangle $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r}) \subset M_\kappa^2$ is a *comparison triangle* for $\Delta(p, q, r) \subset M$, if $d(p, q) = d(\bar{p}, \bar{q})$, $d(p, r) = d(\bar{p}, \bar{r})$, and $d(r, q) = d(\bar{r}, \bar{q})$. If $d(p, q) + d(p, r) + d(r, p) < 2D_\kappa$ (where $D_\kappa = \text{diam}(M_\kappa^2)$), such a comparison triangle exists and it is unique up to isometries. A point $\bar{x} \in \text{Im}(\overline{\gamma})$ is a comparison point for $x \in \text{Im}(\gamma)$ if $d(p, x) = d(\bar{p}, \bar{x})$, where $\gamma : [0, 1] \to M$ and $\overline{\gamma} : [0, 1] \to M_\kappa^2$ are geodesic segments such that $\gamma(0) = p$, $\gamma(0) = \bar{p}$, and $l(\gamma) = l(\overline{\gamma})$.

Let $\Delta(p, q, r) \subset M$ be a geodesic triangle with perimeter less than $2D_\kappa$, and let $\overline{\Delta}(\bar{p}, \bar{q}, \bar{r}) \subset M_\kappa^2$ be its comparison triangle. The triangle $\Delta(p, q, r)$ satisfies the *CAT(κ)-inequality*, if for every $x, y \in \Delta(p, q, r)$, for the comparison points $\bar{x}, \bar{y} \in \overline{\Delta}(\bar{p}, \bar{q}, \bar{r})$ one has $d(x, y) \leqslant d(\bar{x}, \bar{y})$. The geodesic length space (M, d) is a *CAT(κ)-space* if all geodesic triangles in M with perimeter less than $2D_\kappa$ satisfy the CAT(κ)-inequality. The metric space (M, d) is an *Alexandrov NPC space* if it is locally a CAT(0)-space, i.e., for every $p \in M$ there exists $\rho_p > 0$ such that $B(p, \rho_p) = \{q \in M : d(p, q) < \rho_p\}$ is a CAT(0)-space.

A set $S \subset M$ is *geodesic convex* if for every two points $p, q \in S$, there exists a unique geodesic segment joining p to q whose image is contained in S. The projection map $P_S : M \to 2^S$ is defined by (2.1).

Proposition 3.1. *Let* (M, d) *be a CAT(0)-space. Then the following properties hold*:

(i) (See [3, Proposition 2.2]) *The distance function* d *is convex.*

(ii) Projections (see [3, Proposition 2.4]): *If* $S \subset M$ *is a geodesic convex M-proximal set, then it is M-Chebyshev, i.e.,* $P_S(q)$ *is a singleton for every* $q \in M$. *Moreover,* P_S *is non-expansive, i.e., (2.2) holds on* M. *If* $q \notin S$ *and* $z \in S$, *then* $\angle_{P_S(q)}(q, z) \geqslant \pi/2$, *where* $\angle_p(z_1, z_2)$ *denotes the Alexandrov angle between the unique geodesic segments joining* p *to* z_1 *and* z_2, *respectively.*

(iii) First variation formula (see [3, Corollary 3.6]): *If* $\gamma : [0, 1] \to M$ *is a geodesic segment with* $\gamma(0) = p$, *and* $z \in M$ *is a distinct point from* p, *then*

$$\cos \angle_p\big(\gamma(t), z\big) = \lim_{s \to 0^+} \frac{d(p, z) - d(\gamma(s), z)}{s}, \quad t \in (0, 1].$$

(iv) Pythagorean inequality (see [9, Theorem 2.3.3]): *If* $p \in M$, $\gamma : [0, 1] \to M$ *is a geodesic segment, and* $\gamma(0) = P_{\text{Im}(\gamma)}(p)$, *then*

$$d^2\big(p, \gamma(0)\big) + d^2\big(\gamma(0), \gamma(1)\big) \leqslant d^2\big(p, \gamma(1)\big).$$

(v) Ptolemaic inequality (see [8,10]): *For every quadruple* $q_i \in M$, $i = 1, \ldots, 4$, *one has*

$$d(q_1, q_3) \cdot d(q_2, q_4) \leqslant d(q_1, q_2) \cdot d(q_3, q_4) + d(q_1, q_4) \cdot d(q_2, q_3).$$

Remark 3.1. If $P_S(q)$ is a singleton for some $q \in M$, we do not distinguish between the set and its unique point.

3.2. Finsler–Minkowski spaces

Let $F : \mathbb{R}^n \to [0, \infty)$ be a *positively homogeneous Minkowski norm*, i.e., F satisfies the properties:

(a) $F \in C^2(\mathbb{R}^n \setminus \{0\})$;
(b) $F(ty) = tF(y)$ for all $t \geqslant 0$ and $y \in \mathbb{R}^n$;
(c) The Hessian matrix $g_y = \nabla^2(F^2/2)(y)$ is positive definite for all $y \neq 0$.

The Minkowski norm F is said to be *absolutely homogeneous* if in addition, we have

(b') $F(ty) = |t|F(y)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

If (a)–(c) hold, the pair $(\mathbb{R}^n, F)$ is a *Finsler–Minkowski space*, see Bao, Chern and Shen [2, §1.2], which is the simplest (not necessarily reversible) geodesically complete Finsler manifold whose flag curvature is identically zero, the geodesics are straight lines, and the intrinsic distance between two points $p, q \in \mathbb{R}^n$ is given by

$$d_F(p, q) = F(q - p). \tag{3.1}$$

In fact, $(\mathbb{R}^n, d_F)$ is a quasi-metric space and in general, $d_F(p, q) \neq d_F(q, p)$. In particular, $g_y = g_{-y}$ for all $y \neq 0$ if and only if F is absolutely homogeneous; if so, $(\mathbb{R}^n, F)$ is a *reversible* Finsler–Minkowski space.

Let $S \subset \mathbb{R}^n$ be a nonempty set. Since $(\mathbb{R}^n, F)$ is not necessarily reversible, we define the *forward* (resp. *backward*) *metric projections* of q to S as follows:

- $P_S^+(q) = \{s_f \in S : d_F(q, s_f) = \inf_{s \in S} d_F(q, s)\}$;
- $P_S^-(q) = \{s_b \in S : d_F(s_b, q) = \inf_{s \in S} d_F(s, q)\}$.

Proposition 3.2. *Let* $(\mathbb{R}^n, F)$ *be a* (*not necessarily reversible*) *Finsler–Minkowski space. Then the following properties hold*:

(i) (See [12, Theorem 15.8]) *If* $S \subset \mathbb{R}^n$ *is convex and* $\mathbb{R}^n$-*proximinal, then* S *is both forward and backward* $\mathbb{R}^n$-*Chebyshev, i.e.,* $P_S^+(q)$ *and* $P_S^-(q)$ *are singletons for every* $q \in \mathbb{R}^n$.
(ii) (See [12, Theorem 15.7]) *If* $S \subset \mathbb{R}^n$ *is closed and convex, then*
 - $s \in P_S^+(q)$ *if and only if* $g_{s-q}(s-q, z-s) \geqslant 0$ *for all* $z \in S$;
 - $s \in P_S^-(q)$ *if and only if* $g_{q-s}(q-s, z-s) \leqslant 0$ *for all* $z \in S$.
(iii) Fundamental inequality of Finsler geometry (See [2, pp. 6–10]): *For every* $y \neq 0 \neq w$, *one has*

$$\left| g_y(y, w) \right| \leqslant \sqrt{g_y(y, y)} \cdot \sqrt{g_w(w, w)} = F(y) \cdot F(w).$$

4. Proof of Theorems 2.1 and 2.3

Proof of Theorem 2.1. Let $p \in M$ be fixed. Since (M, d) is an Alexandrov NPC space, there exists $\rho_p > 0$ small enough such that $B(p, \rho_p)$ is a CAT(0)-space. We fix arbitrary two geodesic convex $B(p, \rho_p)$-proximinal sets $S_1, S_2 \subset B(p, \rho_p)$. According to Proposition 3.1(ii), S_1, S_2 are $B(p, \rho_p)$-Chebyshev sets. We will prove that (DP_1) is equivalent to (DP_2). Let $q \in S_1$.

Step 1. "$(DP_2) \Rightarrow (DP_1)$". Let us choose $z_2 = P_{S_2}(q) \in S_2$ in (DP_2). Therefore, it follows that $d(q, P_{S_2}(q)) \leqslant d(z_1, P_{S_2}(q))$ for all $z_1 \in S_1$, which implies that $q \in P_{S_1}(P_{S_2}(q))$. Since S_1 is $B(p, \rho_p)$-Chebyshev, the claim follows.

Step 2. "$(DP_1) \Rightarrow (DP_2)$". Since S_1 and S_2 are $B(p, \rho_p)$-Chebyshev sets, we may assume that $(P_{S_1} \circ P_{S_2})(q) = q$ in (DP_1). Furthermore, there exists a unique element $\tilde{q} \in S_2$ with $P_{S_2}(q) = \tilde{q}$ and $P_{S_1}(\tilde{q}) = q$. We shall assume that $d(q, \tilde{q}) > 0$; otherwise, (DP_2) trivially holds. Fix $z_1 \in S_1$ and $z_2 \in S_2$ arbitrarily. Applying the Pythagorean inequality (see Proposition 3.1(iv)) to the point $\tilde{q}$ and the geodesic segment joining q to z_1, we have

$$d^2(q, \tilde{q}) + d^2(q, z_1) \leqslant d^2(z_1, \tilde{q}). \tag{4.1}$$

In a similar way, one has

$$d^2(\tilde{q}, q) + d^2(\tilde{q}, z_2) \leqslant d^2(z_2, q). \tag{4.2}$$

Since $(B(p, \rho_p), d)$ is Ptolemaic (see Proposition 3.1(v)), for the quadruple of points $z_1, z_2, \tilde{q}, q \in B(p, \rho_p)$, we obtain

$$d(z_1, \tilde{q}) \cdot d(z_2, q) \leqslant d(z_1, z_2) \cdot d(\tilde{q}, q) + d(z_1, q) \cdot d(z_2, \tilde{q}). \tag{4.3}$$

Assume to the contrary that $d(z_1, z_2) < d(q, \tilde{q}) = d(q, P_{S_2}(q))$. Then, relation (4.3) yields

$$d(z_1, \tilde{q}) \cdot d(z_2, q) < d^2(q, \tilde{q}) + d(z_1, q) \cdot d(z_2, \tilde{q}).$$

Combining this relation with (4.1) and (4.2), we obtain

$$\left[d^2(q, \tilde{q}) + d^2(q, z_1) \right] \cdot \left[d^2(\tilde{q}, q) + d^2(\tilde{q}, z_2) \right] < \left[d^2(q, \tilde{q}) + d(z_1, q) \cdot d(z_2, \tilde{q}) \right]^2,$$

which is equivalent to $[d(q, z_1) - d(\tilde{q}, z_2)]^2 < 0$, a contradiction. Therefore, we have

$$d(q, P_{S_2}(q)) = d(q, \tilde{q}) \leqslant d(z_1, z_2),$$

which concludes the proof. $\quad\square$

Remark 4.1. For "$(DP_1) \Rightarrow (DP_2)$" we can give an alternative proof. As above, let $\tilde{q} \in S_2$ with $P_{S_2}(q) = \tilde{q}$ and $P_{S_1}(\tilde{q}) = q$, and fix $z_1 \in S_1$ and $z_2 \in S_2$ arbitrarily. Let $\gamma : [0, 1] \to M$ be the unique geodesic joining $q = \gamma(0)$ and $\tilde{q} = \gamma(1)$. We claim that

$$P_{\mathrm{Im}(\gamma)}(z_1) = q \quad \text{and} \quad P_{\mathrm{Im}(\gamma)}(z_2) = \tilde{q}. \tag{4.4}$$

Since $P_{S_1}(\tilde{q}) = q$, due to Proposition 3.1(ii), one has that $\angle_q(\gamma(t), z_1) \geqslant \pi/2$, $t \in (0, 1]$. The first variation formula (see Proposition 3.1(iii)) yields that

$$0 \geqslant \cos \angle_q(\gamma(t), z_1) = \lim_{s \to 0^+} \frac{d(q, z_1) - d(\gamma(s), z_1)}{s}.$$

Since d is convex, the function $s \mapsto \frac{d(q, z_1) - d(\gamma(s), z_1)}{s}$ is non-increasing. Combining the latter two facts, it follows that

$$0 \geqslant \frac{d(q, z_1) - d(\gamma(s), z_1)}{s}, \quad s \in (0, 1].$$

In particular, $d(q, z_1) \leqslant d(\gamma(s), z_1)$ for every $s \in (0, 1]$, which concludes the first part of (4.4); the second relation is proved similarly. Now, from the non-expansiveness of the projection $P_{\mathrm{Im}(\gamma)}$ (see Proposition 3.1(ii)) and relation (4.4), we obtain

$$d(q, \tilde{q}) = d(P_{\mathrm{Im}(\gamma)}(z_1), P_{\mathrm{Im}(\gamma)}(z_2)) \leqslant d(z_1, z_2).$$

Proof of Theorem 2.3. "(iii) $\Rightarrow$ (i)&(ii)" If the Riemannian manifold (M, g) has non-positive sectional curvature, (M, d_g) is an Alexandrov NPC space, see Bridson and Haefliger [3, Theorem 1A.6]. Consequently, by Theorem 2.1, (M, d_g) has the double-projection property. Moreover, by Proposition 3.1(ii) it follows that the projective non-expansiveness property also holds.

"(i) $\Rightarrow$ (iii)" We assume that (M, d_g) satisfies the double-projection property, i.e., every $p \in M$ has a neighborhood $U \subset M$ such that (U, d) is a geodesic length space, and for every two geodesic convex, U-proximinal sets $S_1, S_2 \subset U$, the statements (DP_1) and (DP_2) are equivalent.

Let $p \in M$ be fixed and $B_g(p, \tilde{\rho}_p) \subset U$ be a totally normal ball of p, see do Carmo [7, Theorem 3.7]. Clearly, $B_g(p, \tilde{\rho}_p)$ inherits the above properties of U. Fix also $W_0, V_0 \in T_p M \setminus \{0\}$. We claim that the sectional curvature of the two-dimensional subspace $\mathcal{S} = \text{span}\{W_0, V_0\} \subset T_p M$ at p is non-positive. One may assume without loss of generality that V_0 and W_0 are g-perpendicular, i.e., $g(W_0, V_0) = 0$.

Let κ be an upper bound for the sectional curvature over the closed ball $B_g[p, \tilde{\rho}_p] = \{q \in M : d_g(p, q) \leqslant \tilde{\rho}_p\}$, and let $\kappa_1 = \max\{1, \kappa\}$. We fix $\delta > 0$ such that

$$\delta\big(\|W_0\|_g + \|V_0\|_g\big) < \frac{1}{2} \min\left\{\tilde{\rho}_p, \frac{\pi}{\sqrt{\kappa_1}}\right\}. \tag{4.5}$$

Let $\sigma : [0, \delta] \to M$ be the geodesic segment $\sigma(t) = \exp_p(tV_0)$ and W be the unique parallel vector field along σ with the initial data $W(0) = W_0$. For any $t \in [0, \delta]$, we define the geodesic segment $\gamma_t : [0, \delta] \to M$ by $\gamma_t(u) = \exp_{\sigma(t)}(uW(t))$. Having in our mind these notations, we claim that

$$P_{\text{Im}(\gamma_t)}(p) = \sigma(t) \quad \text{for every } t \in [0, \delta]. \tag{4.6}$$

To show this, fix $t \in [0, \delta]$. Due to (4.5), for every $u \in [0, \delta]$, the geodesic segment $\gamma_t|_{[0,u]}$ belongs to the normal ball $B_g(p, \tilde{\rho}_p)$; thus, $\gamma_t|_{[0,u]}$ is the unique minimal geodesic joining the point $\gamma_t(0) = \sigma(t)$ to $\gamma_t(u)$. Moreover, since W is the parallel transport of $W(0) = W_0$ along σ, we have $g(W(t), \dot{\sigma}(t)) = g(W(0), \dot{\sigma}(0)) = g(W_0, V_0) = 0$; therefore,

$$g\big(\dot{\gamma}_t(0), \dot{\sigma}(t)\big) = g\big(W(t), \dot{\sigma}(t)\big) = 0. \tag{4.7}$$

Since $\text{Im}(\gamma_t)$ is compact, $P_{\text{Im}(\gamma_t)}(p) \neq \emptyset$; let $q \in P_{\text{Im}(\gamma_t)}(p)$, and assume that $q \neq \sigma(t)$. It is clear that the geodesic triangle $\Delta(p, q, \sigma(t))$ is included into $B_g(p, \tilde{\rho}_p)$, and on account of (4.5), its perimeter satisfies the inequality

$$d_g(p, q) + d_g\big(q, \sigma(t)\big) + d_g\big(p, \sigma(t)\big) < \frac{\pi}{\sqrt{\kappa_1}}. \tag{4.8}$$

Moreover, due to the fact that $q \in P_{\text{Im}(\gamma_t)}(p)$ and (4.7), the angles in the geodesic triangle $\Delta(p, q, \sigma(t))$ fulfill

$$\angle q \geqslant \frac{\pi}{2} \quad \text{and} \quad \angle \sigma(t) = \frac{\pi}{2}. \tag{4.9}$$

Now, we are in the position to apply Toponogov's comparison theorem for triangles (where the curvature is bounded from above by the number $\kappa_1 > 0$), see Klingenberg [11, Theorem 2.7.6]. Namely, if $\overline{\Delta}(\bar{p}, \bar{q}, \overline{\sigma(t)})$ is the comparison triangle for $\Delta(p, q, \sigma(t))$ on the two-dimensional sphere with radius $\frac{1}{\sqrt{\kappa_1}}$, the comparison angles in $\overline{\Delta}(\bar{p}, \bar{q}, \overline{\sigma(t)})$ are not smaller than their corresponding angles in $\Delta(p, q, \sigma(t))$. Combining this fact with (4.9), we get that

$$\angle \bar{q} \geqslant \frac{\pi}{2} \quad \text{and} \quad \angle \overline{\sigma(t)} \geqslant \frac{\pi}{2}.$$

By the cosine rule for sides of a spherical triangle, the latter inequalities yield

$$\cos d_g\big(p, \sigma(t)\big) - \cos d_g(p, q) \cos d_g\big(q, \sigma(t)\big) = \sin d_g(p, q) \sin d_g\big(q, \sigma(t)\big) \cos \bar{q} \leqslant 0;$$
$$\cos d_g(p, q) - \cos d_g\big(p, \sigma(t)\big) \cos d_g\big(q, \sigma(t)\big) = \sin d_g\big(p, \sigma(t)\big) \sin d_g\big(q, \sigma(t)\big) \cos \angle \overline{\sigma(t)} \leqslant 0.$$

Adding these inequalities and rearranging them, we obtain

$$\big[1 - \cos d_g\big(q, \sigma(t)\big)\big] \cdot \big[\cos d_g(p, q) + \cos d_g\big(p, \sigma(t)\big)\big] \leqslant 0,$$

which is equivalent to

$$\sin^2 \frac{d_g(q, \sigma(t))}{2} \cos \frac{d_g(p, q) + d_g(p, \sigma(t))}{2} \cos \frac{d_g(p, q) - d_g(p, \sigma(t))}{2} \leqslant 0.$$

Since $q \neq \sigma(t)$, the first term is positive. On account of (4.8), the third term is also positive. Thus, the second term is necessarily non-positive, i.e., $d_g(p, q) + d_g(p, \sigma(t)) \geqslant \pi$, which contradicts (4.8). Consequently, $P_{\text{Im}(\gamma_t)}(p)$ contains the unique element $\sigma(t)$, which concludes the proof of (4.6).

In the same way as in (4.6), we can prove

$$P_{\mathrm{Im}(\gamma_0)}\big(\sigma(t)\big) = p \quad \text{for every } t \in [0, \delta].$$ (4.10)

Thus, we can conclude from (4.6) and (4.10) that for every $t \in [0, \delta]$,

$$P_{\mathrm{Im}(\gamma_0)}\big(P_{\mathrm{Im}(\gamma_t)}(p)\big) = p,$$

i.e., (DP_1) holds for the point $p \in \mathrm{Im}(\gamma_0)$ and sets $S_1 = \mathrm{Im}(\gamma_0)$ and $S_2 = \mathrm{Im}(\gamma_t)$, respectively. Since these sets are geodesic convex and compact (thus, $B_g(p, \tilde{\rho}_p)$-proximinal), the validity of the double-projection property implies that (DP_2) holds too, i.e., p is the best approximation point from $\mathrm{Im}(\gamma_0)$ to $\mathrm{Im}(\gamma_t)$. Formally, we have

$$d_g\big(p, P_{\mathrm{Im}(\gamma_t)}(p)\big) \leqslant d_g(z_1, z_2) \quad \text{for all } (z_1, z_2) \in \mathrm{Im}(\gamma_0) \times \mathrm{Im}(\gamma_t) \text{ and } t \in [0, \delta].$$

In particular, for every $t, u \in [0, \delta]$, we have

$$d_g\big(p, \sigma(t)\big) \leqslant d_g\big(\gamma_0(u), \gamma_t(u)\big).$$ (4.11)

By using the parallelogramoid of Levi-Civita for calculating the sectional curvature $K_p(\mathcal{S})$ at p and for the two-dimensional subspace $\mathcal{S} = \mathrm{span}\{W_0, V_0\} \subset T_p M$, see Cartan [6, pp. 244–245], we obtain from (4.11) that

$$K_p(\mathcal{S}) = \lim_{u, t \to 0} \frac{d_g^2(p, \sigma(t)) - d_g^2(\gamma_0(u), \gamma_t(u))}{d_g(p, \gamma_0(u)) \cdot d_g(p, \sigma(t))} \leqslant 0.$$

This concludes the proof of "(i) $\Rightarrow$ (iii)".

"(ii) $\Rightarrow$ (iii)" Let us keep the notations and constructions from above. A similar geometric reasoning as in the proof of (4.6) yields that

$$P_{\mathrm{Im}(\sigma)}\big(\gamma_t(u)\big) = \sigma(t) \quad \text{for every } t, u \in [0, \delta].$$ (4.12)

Since $S = \mathrm{Im}(\sigma)$ is a geodesic convex $B_g(p, \tilde{\rho}_p)$-proximinal set and the projection non-expansiveness property holds, on account of (2.2) and (4.12) we obtain for every $t, u \in [0, \delta]$ that

$$d_g\big(p, \sigma(t)\big) = d_g\big(\sigma(0), \sigma(t)\big) = d_g\big(P_{\mathrm{Im}(\sigma)}(\gamma_0(u)), P_{\mathrm{Im}(\sigma)}(\gamma_t(u))\big) \leqslant d_g\big(\gamma_0(u), \gamma_t(u)\big),$$

which is nothing but relation (4.11). It remains to follow the previous proof. □

5. Proof of Theorem 2.2 and the double-projection property on non-reversible Finsler–Minkowski spaces

Proof of Theorem 2.2. Let $S_1, S_2 \subset \mathbb{R}^n$ be two convex and $\mathbb{R}^n$-proximinal sets. Note that the implication "$(DP_2) \Rightarrow (DP_1)$" is proved analogously as in Theorem 2.1.

Let us prove "$(DP_1) \Rightarrow (DP_2)$". To do this, let $q \in S_1$ such that $q \in P_{S_1}(P_{S_2}(q))$. Due to Proposition 3.2(i), both sets S_1 and S_2 are $\mathbb{R}^n$-Chebyshev. Consequently, there exists a unique element $\tilde{q} \in S_2$ such that $P_{S_2}(q) = \tilde{q}$ and $P_{S_1}(\tilde{q}) = q$. On account of Proposition 3.2(ii), the latter relations are equivalent to

$$g_{\tilde{q}-q}(\tilde{q} - q, z_1 - q) \leqslant 0 \quad \text{for all } z_1 \in S_1;$$

$$g_{\tilde{q}-q}(q - \tilde{q}, z_2 - \tilde{q}) \leqslant 0 \quad \text{for all } z_2 \in S_2.$$

Adding these inequalities, we obtain $g_{\tilde{q}-q}(\tilde{q} - q, \tilde{q} - q - z_2 + z_1) \leqslant 0$. By applying the fundamental inequality (see Proposition 3.2(iii)) and relation (3.1), we have

$$\begin{aligned}
d_F^2(q, \tilde{q}) = F^2(\tilde{q} - q) &= g_{\tilde{q}-q}(\tilde{q} - q, \tilde{q} - q) \\
&\leqslant g_{\tilde{q}-q}(\tilde{q} - q, z_2 - z_1) \\
&\leqslant F(\tilde{q} - q) \cdot F(z_2 - z_1) \\
&= d_F(q, \tilde{q}) \cdot d_F(z_1, z_2).
\end{aligned}$$

Consequently, $d_F(q, \tilde{q}) \leqslant d_F(z_1, z_2)$ for every $z_1 \in S_1$ and $z_2 \in S_2$, which means that $q \in S_1$ is the best approximation element from S_1 to S_2. □

Remark 5.1. Let $(\mathbb{R}^n, F)$ be a not necessarily reversible Finsler–Minkowski space; the metric distance d_F is usually only a quasi-metric. Even in this case, it is possible to state a similar result as Theorem 2.2, slightly reformulating the double-projection property.

Let $S_1, S_2 \subset \mathbb{R}^n$ be two convex and $\mathbb{R}^n$-proximinal sets, and $q \in S_1$. Note that S_1, S_2 are forward and backward $\mathbb{R}^n$-Chebyshev sets, see Proposition 3.2(i). In the *forward* case, we consider the following statements:
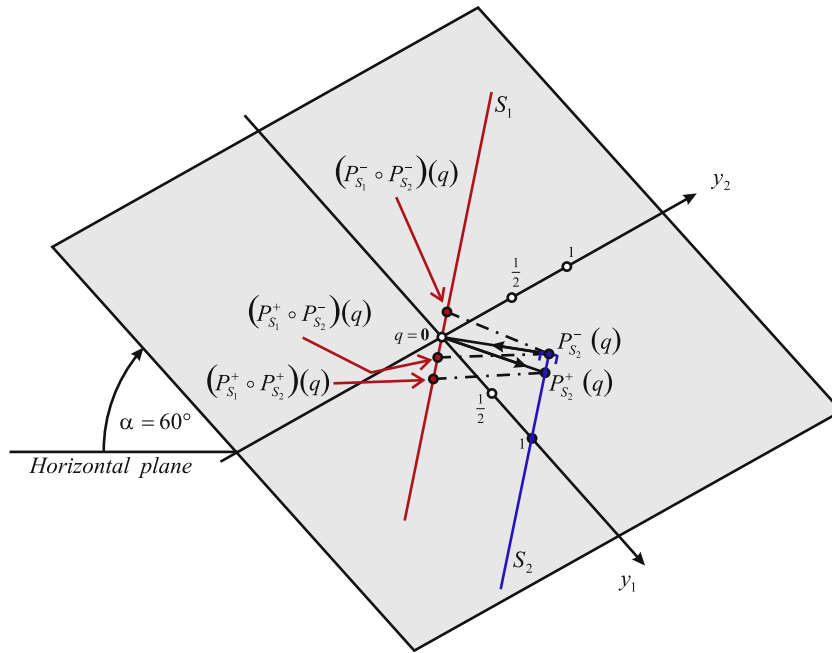
Fig. 1. Apart from the case $P^-_{S_1} \circ P^+_{S_2}$, the compositions of forward and/or backward metric projections at $q = (0,0)$ are scattered away from q.

(DP^+_1) $q = (P^-_{S_1} \circ P^+_{S_2})(q)$;
(DP^+_2) $d_F(q, P^+_{S_2}(q)) \leqslant d_F(z_1, z_2)$ for all $z_1 \in S_1$, $z_2 \in S_2$.

In the *backward* case, we consider similar statements:

(DP^-_1) $q = (P^+_{S_1} \circ P^-_{S_2})(q)$;
(DP^-_2) $d_F(P^-_{S_2}(q), q) \leqslant d_F(z_2, z_1)$ for all $z_1 \in S_1$, $z_2 \in S_2$.

Exploiting Proposition 3.2(ii) in its full generality, we can show as in Theorem 2.2:

Theorem 5.1. *Let* $(\mathbb{R}^m, F)$ *be a Finsler–Minkowski space. Then for every two convex and* $\mathbb{R}^n$*-proximinal sets* $S_1, S_2 \subset \mathbb{R}^n$, *we have*:

(i) $(DP^+_1) \Leftrightarrow (DP^+_2)$;
(ii) $(DP^-_1) \Leftrightarrow (DP^-_2)$.

Remark 5.2. Usually, the map $P^-_{S_1} \circ P^+_{S_2}$ in (DP^+_1) *cannot* be replaced either by $P^+_{S_1} \circ P^+_{S_2}$ or by $P^+_{S_1} \circ P^-_{S_2}$ or by $P^-_{S_1} \circ P^-_{S_2}$. (The same is true for $P^+_{S_1} \circ P^-_{S_2}$ in (DP^-_1).) In order to give a concrete example, we recall the Matsumoto norm, see [13], which describes the walking-law on a mountain slope (under the action of gravity), having an angle $\alpha \in [0, \pi/2)$ with the horizontal plane. The explicit form of this norm $F : \mathbb{R}^2 \to [0, \infty)$ is

$$F(y) = \begin{cases} \dfrac{y_1^2 + y_2^2}{v\sqrt{y_1^2 + y_2^2 + \frac{g}{2} y_1 \sin \alpha}}, & y = (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}; \\ 0, & y = (y_1, y_2) = (0,0), \end{cases} \tag{5.1}$$

where v [m/s] is the constant speed on the horizontal plane, $g \approx 9.81$ [m/s^2], and $g \sin \alpha \leqslant v$. The pair $(\mathbb{R}^2, F)$ is a typical non-reversible Finsler–Minkowski space, and it becomes reversible if and only if $\alpha = 0$.

Let $v = 10$ and $\alpha = \pi/3$ in (5.1), and consider the convex and closed sets

$$S_1 = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 + y_2 = 0\},$$
$$S_2 = \{(y_1, y_2) \in \mathbb{R}^2 \colon y_1 + y_2 = 1, \ y_1 \geqslant 1/2\}.$$

Let also $q = (0,0)$. A direct calculation yields $(P^-_{S_1} \circ P^+_{S_2})(q) = q$, and $d_F(q, P^+_{S_2}(q)) \leqslant d_F(z_1, z_2)$ for all $z_1 \in S_1$, $z_2 \in S_2$. However, we have

$$\left(P_{S_1}^+ \circ P_{S_2}^+\right)(q) = (0.32338512, -0.32338512) \neq q,$$

$$\left(P_{S_1}^+ \circ P_{S_2}^-\right)(q) = P_{S_1}^+(1/2, 1/2) = (0.23349577, -0.23349577) \neq q,$$

$$\left(P_{S_1}^- \circ P_{S_2}^-\right)(q) = P_{S_1}^-(1/2, 1/2) = (-0.08988935, 0.08988935) \neq q,$$

see also Fig. 1.

Acknowledgements

References

[1] A.D. Alexandrov, A theorem on triangles in a metric space and some of its applications, Tr. Mat. Inst. Steklova 38 (1951) 5–23 (in Russian).
[2] D. Bao, S.S. Chern, Z. Shen, Introduction to Riemann–Finsler Geometry, Graduate Texts in Mathematics, vol. 200, Springer, Berlin, 2000.
[3] M. Bridson, A. Haefliger, Metric Spaces of Non-positive Curvature, Fundamental Principles of Mathematical Sciences, vol. 319, Springer, Berlin, 1999.
[4] S.M. Buckley, K. Falk, D.J. Wraith, Ptolemaic spaces and CAT(0), Glasg. Math. J. 51 (2) (2009) 301–314.
[5] H. Busemann, The Geometry of Geodesics, Academic Press, New York, 1955.
[6] E. Cartan, Geometry of Riemannian Spaces, Math. Sci. Press, Brookline, MA, 1983, translated from French: Leçons sur la Géométrie des Espaces de Riemann (Paris, 1928).
[7] M.P. do Carmo, Riemannian Geometry, Birkhäuser, Boston, 1992.
[8] T. Foertsch, A. Lytchak, V. Schroeder, Nonpositive curvature and the Ptolemy inequality, Int. Math. Res. Not. 2007 (22) (2007), Art. ID rnm100, 15 pp.
[9] J. Jost, Nonpositive Curvature: Geometric and Analytic Aspects, Birkhäuser Verlag, Basel, 1997.
[10] D.C. Kay, The Ptolemaic inequality in Hilbert geometries, Pac. J. Math. 21 (1967) 293–301.
[11] W. Klingenberg, Riemannian Geometry, second revised edition, Walter de Gruyter & Co., Berlin, 1995.
[12] A. Kristály, V. Rădulescu, Cs. Varga, Variational Principles in Mathematical Physics, Geometry, and Economics, Encyclopedia of Mathematics and Its Applications, vol. 136, Cambridge University Press, Cambridge, 2010.
[13] M. Matsumoto, A slope of a mountain is a Finsler surface with respect to a time measure, J. Math. Kyoto Univ. 29 (1989) 17–25.
[14] F.P. Pedersen, On spaces with negative curvature, Mater. Tidsskr. B (1952) 66–89.
[15] R.R. Phelps, Convex sets and nearest points, Proc. Am. Math. Soc. 8 (1957) 790–797.
[16] I.J. Schoenberg, A remark on M.M. Day's characterization of inner-product spaces and a conjecture of L.M. Blumenthal, Proc. Am. Math. Soc. 3 (1952) 961–964.