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Multiple perturbations of a singular eigenvalue problem

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ABSTRACT

We study the perturbation by a critical term and a $(p-1)$-superlinear subcritical nonlinearity of a quasilinear elliptic equation containing a singular potential. By means of variational arguments and a version of the concentration-compactness principle in the singular case, we prove the existence of solutions for positive values of the parameter under the principal eigenvalue of the associated singular eigenvalue problem.

1. Introduction

Let $\Omega \subseteq \mathbb{R}$ be an *arbitrary* open set, $1 < p < N$, and let $\mathcal{D}_0^{1,p}(\Omega)$ denote the completion of $\mathcal{D}(\Omega)$ with respect to the norm $\|u\| := (\int_\Omega |\nabla u|^p dx)^{1/p}$. Let $V \in L^1_{loc}(\Omega)$ be a function which may have strong singularities and an indefinite sign.

Smets was interested in [1] in finding nontrivial weak solutions for the following nonlinear eigenvalue problem:

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda V(x)|u|^{p-2}u & \text{in } \Omega \\ u \in \mathcal{D}_0^{1,p}(\Omega). \end{cases} \tag{1}$$

Problems of this type are in relationship with the study of the standing waves in the anisotropic Schrödinger or Klein–Gordon equations, cf. Reed and Simon [2], Strauss [3], and Wang [4]. Eq. (1) is also considered a model for several physical phenomena related to the equilibrium of anisotropic media that possibly are somewhere *perfect insulators* or *perfect conductors*, see Dautray and Lions [5, p. 79]. We point out that degenerate or singular problems have been intensively studied starting with the pioneering paper by Murthy and Stampacchia [6].

Problem (1) is in relationship with several papers dealing with nonlinear anisotropic eigenvalue problems, see Brown and Tertikas [7], Rozenblioum and Solomyak [8]. Szulkin and Willem generalize in [9] several earlier results concerning the

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existence of an infinite sequence of eigenvalues. The main hypothesis on the potential V in [9] is the following:

$$\begin{cases} V \in L^1_{loc}(\Omega), \quad V^+ = V_1 + V_2 \neq 0, \quad V_1 \in L^{N/p}(\Omega), \\ \text{for every } y \in \overline{\Omega}, \qquad \lim_{x \to y, x \in \Omega} |x-y|^p V_2(x) = 0 \quad \text{and} \\ \lim_{x \to \infty, x \in \Omega} |x|^p V_2(x) = 0. \end{cases} \tag{2}$$

Under assumption (2), the mapping $\mathcal{D}_0^{1,p}(\Omega) \ni u \longmapsto \int_\Omega V^+|u|^p dx$ is weakly continuous, so the problem is not affected by a lack of compactness. In [1] the case of indefinite potential functions V is studied for which no *a priori* compactness is assumed. The corresponding hypotheses extend condition (2), nonetheless they are not directly linked to punctual growths of V. Due to the presence of a singular potential, the classical methods cannot be applied directly, so the existence can become a delicate matter.

Consider the minimization problem

$$S_V := \inf \left\{ \int_\Omega |\nabla u|^p dx; \ u \in \mathcal{D}_0^{1,p}(\Omega), \ \int_\Omega V(x)|u|^p dx = 1 \right\}. \tag{3}$$

As established in [1] with standard constrained minimization arguments, minimizers of problem (3) correspond to weak solutions of (1), with λ appearing as a Lagrange multiplier (that is, $\lambda = S_V$). Such a parameter λ is called the *principal eigenvalue* for problem (1).

In order to have $S_V \neq 0$ and well defined, we assume that $V = V^+ - V_-$, $V^+ \neq 0$, and that there exists $c > 0$ such that for all $u \in \mathcal{D}_0^{1,p}(\Omega)$,

$$c \int_\Omega V^+|u|^p dx \leq \int_\Omega |\nabla u|^p dx. \tag{4}$$

By Hardy's inequality it follows that potentials with point singularities and decay at infinity both at most as $O(|x|^{-p})$ satisfy hypothesis (4).

Since Ω is not necessarily bounded and V can have singularities, it is not clear that the infimum in problem (3) is achieved without imposing additional conditions that allow the analysis of minimizing sequences. For all $x \in \overline{\Omega}$ and $r > 0$, we denote by $B_r(x)$ the open ball centered at x and of radius r and by B_r the closed ball centered at the origin (we can assume without any loss of generality that $0 \in \Omega$). We introduce the following quantities:

$$\begin{aligned} S_{r,V} &:= \inf \left\{ \int_\Omega |\nabla u|^p dx; \ u \in \mathcal{D}(\Omega \setminus B_r), \ \int_\Omega V^+(x)|u|^p dx = 1 \right\}; \\ S_{\infty,V} &:= \sup_{r>0} S_{r,V} = \lim_{r\to\infty} S_{r,V}; \\ S^x_{r,V} &:= \inf \left\{ \int_\Omega |\nabla u|^p dx; \ u \in \mathcal{D}(\Omega \cap B_r(x)), \ \int_\Omega V^+(x)|u|^p dx = 1 \right\}; \\ S^x_V &:= \sup_{r>0} S^x_{r,V} = \lim_{r\to 0} S^x_{r,V}; \\ S_{*,V} &:= \inf_{x\in\overline{\Omega}} S^x_V; \\ \Sigma_V &:= \{x \in \overline{\Omega}; \ S^x_V < \infty\}. \end{aligned}$$

Applying Hardy's inequality

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{N}{N-p} \right)^p \int_{\mathbb{R}^N} |\nabla u|^p dx,$$

we observe that under assumption (2) introduced in [9], we have $S_{\infty,V} = S_{*,V} = +\infty$. As argued in [1, p. 475], the condition $S_{\infty,V} = S_{*,V} = +\infty$ is equivalent to the weak continuity of the mapping $u \longmapsto \int_\Omega V^+(x)|u|^p dx$.

We make the following hypothesis:

$$\text{the closure of } \Sigma_V \text{ is at most countable.} \tag{5}$$

In particular, condition (5) excludes the presence of strong *spikes* on a dense subset of Ω.

For $V \in L^1_{loc}(\Omega)$ satisfying assumptions (4) and (5), Smets proved in [1] that the singular eigenvalue problem (1) admits a principal eigenvalue, provided that $S_V < S_{\infty,V}$ and $S_V < S_{*,V}$. This result extends and simplifies the work of Tertikas [10], which deals with the positive linear case for $\Omega = \mathbb{R}^N$. We point out (see [1, p. 472]) that the condition $p < N$ is necessary only if Ω is unbounded, otherwise one can work in the standard Sobolev space $W_0^{1,p}(\Omega)$.

We are interested in studying what happens if problem (1) is affected by certain perturbations. This is needed in several applications and the idea of using perturbation methods in the treatment of nonlinear boundary value problems was introduced by Struwe [11]. Existence results for nonautonomous perturbations of critical singular elliptic boundary value problems were established by Rădulescu and Smets [12]; in their case, the singular weight allows for unbounded domains as cones and gives rise to a different noncompactness picture, as was first remarked by Caldiroli and Musina [13].

Let $\mathcal{M}(\mathbb{R}^N)$ denote the Banach space of finite Radon measures over $\mathbb{R}^N$ endowed with the norm

$$\|\mu\| := \sup_{\phi\in C_0(\mathbb{R}^N),|\phi|_\infty\le 1} |\mu(\phi)|.$$

By definition, a sequence $(\mu_n) \subset \mathcal{M}(\mathbb{R}^N)$ weakly converges to $\mu \in \mathcal{M}(\mathbb{R}^N)$ if $\mu_n(\phi) \to \mu(\phi)$ for all $\phi \in C_0(\mathbb{R}^N)$. The Banach–Alaoglu theorem implies that every bounded sequence $(\mu_n) \subset \mathcal{M}(\mathbb{R}^N)$ contains a weakly convergent subsequence. We denote by $\mathcal{M}^+(\mathbb{R}^N)$ the cone of positive Radon measures over $\mathbb{R}^N$ and by δ_x the Dirac mass at the point x.

2. Effects of a double perturbation

In the present paper, we are concerned with a perturbation of problem (1) and we are interested in the combined effects of a $(p-1)$-superlinear subcritical nonlinearity and a critical Sobolev term. To fix the ideas, we consider $\Omega = \mathbb{R}^N$ but the arguments can be adapted to any open set in $\mathbb{R}^N$. More precisely, we study the nonlinear problem

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \lambda\, V(x)|u|^{p-2}u + a(x)|u|^{r-2}u + b(x)|u|^{p^*-2}u & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{6}$$

where $p^* = Np/(N-p)$ stands for the critical Sobolev exponent.

This problem can be viewed as a prototype of pattern formation in biology and is related to the steady-state problem for a chemotactic aggregation model introduced by Keller and Segel [14]. Problem (6) also plays a crucial role in the analysis of activator–inhibitor systems modeling biological pattern formation, cf. Gierer and Meinhardt [15].

Problem (6) is related to the Brezis–Nirenberg problem

$$-\Delta u = \lambda u + u^{(N+2)/(N-2)} \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{7}$$

where Ω is an open bounded set with smooth boundary. Brezis and Nirenberg [16] showed that, contrary to intuition, the critical problem with *small* linear perturbation can provide solutions. More precisely, Brezis and Nirenberg proved that problem (7) admits a positive solution vanishing on $\partial\Omega$ if and only if $0 < \lambda < \lambda_1$ (if $N \ge 4$), where λ_1 is the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$. In [16], other results are also established (for instance, if $N = 3$ or when λ is replaced by $g(x, u)$ satisfying an appropriate growth condition) and pioneering techniques in nonlinear analysis are introduced.

Our assumptions are the following:

$$p < r < p^*; \tag{8}$$

$$a \in L^s(\mathbb{R}^N) \quad \text{with } s = \frac{Np}{Np - r(N-p)}, \qquad a(x) \ge 0 \text{ a.e. } x \in \mathbb{R}^N,\ a \not\equiv 0; \tag{9}$$

$$b \in L^\infty(\mathbb{R}^N), \qquad b(0) = \|b\|_{L^\infty(\mathbb{R}^N)}, \qquad b(x) = b(0) + o(|x|^\eta) \quad \text{as } x \to 0, \tag{10}$$

where

$$\eta = \frac{N(s-1)}{(p-1)s} \quad \text{if } N < \frac{pr}{r+1-p};$$

$$\eta = \frac{N}{s} \quad \text{if } N \ge \frac{pr}{r+1-p}.$$

The asymptotic decay of the potential b described in condition (10) compensates for the critical behavior of the corresponding nonlinearity and it provides a sufficient condition for the existence of the "valley" in the mountain pass theorem.

The solutions of problem (6) correspond to nontrivial critical points of the energy functional $\mathcal{E} : \mathcal{D}^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{1}{r}\int_{\mathbb{R}^N} a(x)|u|^r dx - \frac{1}{p^*}\int_{\mathbb{R}^N} b(x)|u|^{p^*} dx.$$

Let λ_1 denote the principal eigenvalue of problem (1), namely $\lambda_1 = S_V$ in the minimization problem (3). As remarked in [1, p. 464], hypothesis (4) implies that $\lambda_1 > 0$. Our main result asserts that the perturbed problem (6) admits nontrivial solutions for all positive parameters λ less than the principal eigenvalue of problem (1).

Theorem 2.1. *Let $V \in L^1_{loc}(\mathbb{R}^N)$ satisfy $S_V < S_{\infty,V}$, $S_V < S_{*,V}$, and hypotheses (4), (5). Assume that conditions (8)–(10) are fulfilled. Then problem (6) admits at least one nontrivial solution for all positive parameters with $\lambda < \lambda_1$.*

For $c \in \mathbb{R}$, we recall that $\mathcal{E}$ satisfies the localized Palais–Smale $(\mathrm{PS})_c$-condition if every sequence $(u_n) \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ with $\mathcal{E}(u_n) \to c$ and $\mathcal{E}'(u_n) \to 0$ in $(\mathcal{D}^{1,p}(\mathbb{R}^N))'$, has a convergent subsequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

The main idea of the proof of Theorem 2.1 is to apply the mountain pass theorem. Note that p^* is the limiting Sobolev exponent for the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$. Since this embedding is not compact, the functional $\mathcal{E}$ does not satisfy the Palais–Smale condition. By using the V-dependent concentration-compactness principle of Smets [1, Lemma 2.1], we show that $\mathcal{E}$ satisfies the localized $(\mathrm{PS})_c$-condition for certain values of c. In the final part of the proof, we argue that the geometric hypotheses of the mountain pass theorem are also fulfilled.

3. The localized Palais–Smale condition

In this section we assume that the hypotheses of Theorem 2.1 are satisfied and we are interested to find a range of values for $c > 0$ such that $\mathcal{E}$ satisfies the Palais–Smale $(\mathrm{PS})_c$-condition. An important role in this choice of c is played by the Sobolev constant

$$S := \inf\left\{\int_{\mathbb{R}^N}|\nabla u|^p dx;\ u \in W^{1,p}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{p^*}dx = 1\right\}. \tag{11}$$

This corresponds to the best constant for the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$. We recall (see Brezis and Nirenberg [16, p. 443]) some basic properties of this constant:

(i) S can be defined for any open set Ω, is independent of Ω, and depends only on N.
(ii) The infimum in (11) is never achieved in the case of bounded open sets.
(iii) For the whole Euclidean space, the infimum in (11) is achieved by the function

$$u_\varepsilon(x) = C_\varepsilon\left(\varepsilon^{p/(p-1)} + |x|^{p/(p-1)}\right)^{-\frac{N-p}{p}}, \tag{12}$$

for all $\varepsilon > 0$, where C_ε is a positive constant depending on ε.

Let $(u_n) \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be such that $\mathcal{E}(u_n) \to c$ and $\mathcal{E}'(u_n) \to 0$ in $(\mathcal{D}^{1,p}(\mathbb{R}^N))'$. We find an interval $(0, c_0)$ such that (u_n) contains a convergent subsequence, provided that $c \in (0, c_0)$. For this purpose we use some ideas found in the paper by Guedda and Véron [17]. We have

$$\begin{aligned}&\frac{1}{p}\int_{\mathbb{R}^N}|\nabla u_n|^p dx - \frac{\lambda}{p}\int_{\mathbb{R}^N}V(x)|u_n|^p dx - \frac{1}{r}\int_{\mathbb{R}^N}a(x)|u_n|^r dx\\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N}b(x)|u_n|^{p^*}dx = c + o(1) \quad \text{as } n \to \infty\end{aligned} \tag{13}$$

and

$$\int_{\mathbb{R}^N}|\nabla u_n|^p dx - \lambda\int_{\mathbb{R}^N}V(x)|u_n|^p dx - \int_{\mathbb{R}^N}a(x)|u_n|^r dx - \int_{\mathbb{R}^N}b(x)|u_n|^{p^*}dx = o(\|u_n\|) \quad \text{as } n \to \infty. \tag{14}$$

Relations (13) and (14) yield

$$\left(1 - \frac{p}{r}\right)\int_{\mathbb{R}^N}a(x)|u_n|^r dx + \left(1 - \frac{p}{r^*}\right)\int_{\mathbb{R}^N}b(x)|u_n|^{p^*}dx = O(1) + o(\|u_n\|) \quad \text{as } n \to \infty. \tag{15}$$

Using hypothesis (8) in conjunction with the fact that the potentials a and b are positive, relation (15) implies

$$\int_{\mathbb{R}^N}a(x)|u_n|^r dx = O(1) + o(\|u_n\|) \quad \text{as } n \to \infty \tag{16}$$

and

$$\int_{\mathbb{R}^N}b(x)|u_n|^{p^*}dx = O(1) + o(\|u_n\|) \quad \text{as } n \to \infty. \tag{17}$$

Inserting (16) and (17) in relation (14) we find

$$\int_{\mathbb{R}^N}|\nabla u_n|^p dx - \lambda\int_{\mathbb{R}^N}V(x)|u_n|^p dx = O(1) + o(\|u_n\|) \quad \text{as } n \to \infty.$$

Now, since $\lambda < \lambda_1$ and using the minimization problem (3), we deduce that (u_n) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Thus, up to a subsequence, we can assume that (u_n) weakly converges to some u in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and in $L^{p^*}(\mathbb{R}^N)$,

$$|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup T \quad \text{in } (L^{p'}(\mathbb{R}^N))^N$$

and, by hypothesis (8),

$$u_n \to u \quad \text{in } L^p_{loc}(\mathbb{R}^N) \quad \text{and} \quad L^r_{loc}(\mathbb{R}^N).$$

Moreover, T and u satisfy

$$-\operatorname{div} T = \lambda V(x)|u|^{p-2}u + a(x)|u|^{r-2}u + b(x)|u|^{p^*-2}u \quad \text{in } (\mathcal{D}^{1,p}(\mathbb{R}^N))'. \tag{18}$$

By lower semicontinuity we find

$$\lambda \int_{\mathbb{R}^N} V(x)|u_n|^p dx + \int_{\mathbb{R}^N} a(x)|u_n|^r dx \to \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx + \int_{\mathbb{R}^N} a(x)|u|^r dx =: A \quad \text{as } n \to \infty.$$

Relation (13) and our hypothesis $0 < \lambda < \lambda_1$ imply that $A \geq 0$. We claim that $A > 0$, provided that $c > 0$ is small enough. Indeed, we first observe that relation (14) yields

$$\int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} b(x)|u_n|^{p^*} dx + A + o(\|u_n\|) \quad \text{as } n \to \infty. \tag{19}$$

But relation (13) in combination with our assumption $\lambda \in (0, \lambda_1)$ imply that

$$\ell := \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx > 0.$$

Arguing by contradiction and assuming that $A = 0$, relation (19) yields

$$\int_{\mathbb{R}^N} b(x)|u_n|^{p^*} dx \to \ell \quad \text{as } n \to \infty.$$

Returning to (13) we find that $c = \ell/N$. On the other hand, using the definition of the best Sobolev constant S, we have

$$\begin{aligned} \ell &\geq S \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |u_n|^{p^*} dx \right)^{p/p^*} = S \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |u_n|^{p^*} dx \right)^{(N-p)/N} \\ &\geq S\, \|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/N} \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} b(x)|u_n|^{p^*} dx \right)^{(N-p)/N} = S\, \|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/N} \ell^{(N-p)/N}, \end{aligned}$$

hence

$$\ell \geq S^{N/p}\, \|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/p}.$$

Since $\ell = cN$, in order to yield a contradiction with our assumption $A = 0$, it suffices to choose $c \in (0, c_0)$, where

$$c_0 := \frac{S^{N/p}}{N}\, \|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/p}. \tag{20}$$

Fixing $c \in (0, c_0)$ we have $A > 0$. Thus for some $R > 0$,

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_R(z)} \left(\lambda V(x)|u_n|^p + a(x)|u_n|^r \right) dx > 0.$$

We have already seen that $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $u_n \to u$ almost everywhere. Passing again to a subsequence, we can assume that $|\nabla u_n - \nabla u|^p \rightharpoonup \mu$ in $\mathcal{M}^+(\mathbb{R}^N)$, $V^+|u_n - u|^p \rightharpoonup \nu$ in $\mathcal{M}^+(\mathbb{R}^N)$, $|\nabla u_n|^p \rightharpoonup \tilde{\mu}$ in $\mathcal{M}^+(\mathbb{R}^N)$, and $|u_n|^{p^*} \rightharpoonup \tilde{\nu}$ in $\mathcal{M}^+(\mathbb{R}^N)$. Set

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N \cap (|z|>R)} |\nabla u_n|^p dx$$

and

$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N \cap (|z|>R)} V|u_n|^p dx.$$

Then by Lemma 2.1 in [1],

(i) $\mu_\infty \geq S_{\infty,V} \cdot \nu_\infty$.
(ii) $\nu = \sum_{i\in I} \nu_i \delta_{x_i}$ for some $x_i \in \Sigma_V$, $\nu_i > 0$, $\mu \geq \sum_{i\in I} \nu_i S_V^{x_i} \delta_{x_i}$, $\tilde{\mu} \geq |\nabla u|^p + \sum_{i\in I} \nu_i S_V^{x_i} \delta_{x_i}$, and $\tilde{\nu} = |u|^{p^*} + \sum_{j\in J} \alpha_j \delta_{x_j}$ with $\alpha_j > 0$ (I and J are at most countable).
(iii) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^p dx = \int_{\mathbb{R}^N} V(x)|u|^p dx + \|\nu\| + \nu_\infty$.
(iv) $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} |\nabla u|^p dx + \|\mu\| + \mu_\infty$ if $p = 2$ and $\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx \geq \int_{\mathbb{R}^N} |\nabla u|^p dx + S_{*,V} \|\nu\| + \mu_\infty$ otherwise.

Returning to relations (13) and (14), we obtain

$$\begin{aligned} \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p} \sum_{i\in I} \nu_i S_V^{x_i} &\leq c + \frac{\lambda}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx + \frac{\lambda}{p} \|\nu\| + \frac{\lambda}{p} \nu_\infty \\ &\quad + \frac{1}{r} \int_{\mathbb{R}^N} a(x)|u|^r dx + \frac{1}{p^*} \int_{\mathbb{R}^N} b(x)|u|^{p^*} dx + \frac{1}{p^*} \sum_{j\in J} \alpha_j b(x_j) \end{aligned}$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^p dx + \sum_{i\in I} \nu_i S_V^{x_i} \le \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx + \lambda \,\|\nu\| + \lambda\, \nu_\infty$$
$$+ \int_{\mathbb{R}^N} a(x)|u|^r dx + \int_{\mathbb{R}^N} b(x)|u|^{p^*} dx + \sum_{j\in J} \alpha_j b(x_j). \tag{21}$$

Combining these relations, we obtain

$$\begin{aligned} c &\ge \frac{1}{N}\int_{\mathbb{R}^N} b(x)|u|^{p^*} dx + \frac{1}{N}\sum_{j\in J}\alpha_j b(x_j) + \left(\frac{1}{p} - \frac{1}{r}\right)\int_{\mathbb{R}^N} a(x)|u|^r dx \\ &\ge \frac{1}{N}\int_{\mathbb{R}^N} b(x)|u|^{p^*} dx + \frac{1}{N}\sum_{j\in J}\alpha_j b(x_j). \end{aligned} \tag{22}$$

Since $\mathcal{E}'(u_n) \to 0$ in $(\mathcal{D}^{1,p}(\mathbb{R}^N))'$ we deduce that for all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} uT \cdot \nabla\phi dx + \int_{\mathbb{R}^N} \phi d\tilde{\mu} = \int_{\mathbb{R}^N} \phi b d\tilde{\nu} + \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx.$$

Using now (18) we obtain

$$\int_{\mathbb{R}^N} (uT \cdot \nabla\phi + \phi T \cdot \nabla u) dx = \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx + \int_{\mathbb{R}^N} a(x)|u|^r \phi dx + \int_{\mathbb{R}^N} b(x)|u|^{p^*}\phi dx.$$

Combining these relations we find

$$\begin{aligned} \int_{\mathbb{R}^N} \phi d\tilde{\mu} &= \int_{\mathbb{R}^N} \phi T \cdot \nabla u dx - \int_{\mathbb{R}^N} b(x)|u|^{p^*}\phi dx + \int_{\mathbb{R}^N} \phi b d\tilde{\nu} \\ &\le \int_{\mathbb{R}^N} \phi T \cdot \nabla u dx + \int_{\mathbb{R}^N} \phi b d\tilde{\nu}. \end{aligned} \tag{23}$$

Concentrating ϕ on each x_j, relation (23) yields $\nu_j \le \alpha_j b(x_j)$. But for all j, we have $S\alpha_j^{p/p^*} \le \nu_j$. We deduce that

$$\alpha_j \ge S^{N/p}\left(b(x_j)\right)^{-N/p} \quad \text{for all } j \in J.$$

Thus if $J \neq \emptyset$, then relation (22) implies

$$c \ge \frac{1}{N}\sum_{j\in J}\alpha_j b(x_j) \ge \frac{S^{N/p}}{N}\|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/p},$$

which contradicts (20) and the choice of $c \in (0, c_0)$. This shows that J is empty, hence $\int_{\mathbb{R}^N}|u_n|^{p^*}dx \to \int_{\mathbb{R}^N}|u|^{p^*}dx$. Using Proposition 3.32 from Brezis [18] (which is a consequence of the Milman–Pettis theorem), we deduce that $u_n \to u$ strongly in $L^{p^*}(\mathbb{R}^N)$. We show that this implies the strong convergence of (u_n) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. For this purpose we employ an argument used in Filippucci, Pucci and Rădulescu [19, p. 713]. Consider the following elementary inequality (see formula (2.2) in Simon [20]): for all $\xi, \zeta \in \mathbb{R}^N$

$$|\xi - \zeta|^p \le \begin{cases} c(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta) & \text{for } p \ge 2; \\ c\langle|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle^{p/2}\left(|\xi|^p + |\eta|^p\right)^{(2-p)/2} & \text{for } 1 < p < 2, \end{cases} \tag{24}$$

where c is a positive constant.

Restricting to the case $p \ge 2$, inequality (24) implies that for all positive integers n and m,

$$\|u_n - u_m\| \le |\mathcal{E}'(u_n)(u_n - u_m)| + |\mathcal{E}'(u_m)(u_n - u_m)| + |(\mathcal{E}_0'(u_n) - \mathcal{E}_0'(u_m))(u_n - u_m)|, \tag{25}$$

where $\mathcal{E}_0 := \mathcal{E}(u) - p^{-1}\int_{\mathbb{R}^N}|\nabla u|^p dx$. Applying the strong convergence of (u_n) in $L^{p^*}(\mathbb{R}^N)$, relation (25) implies that (u_n) strongly converges in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. This concludes the proof of the Palais–Smale condition, provided that $c \in (0, c_0)$. □

Summarizing, in this section we have proved the following result.

Lemma 1. *Under the assumptions in Theorem 2.1, the functional $\mathcal{E}$ satisfies the Palais–Smale condition (PS)$_c$ for all $c \in (0, c_0)$, where $c_0 = \frac{S^{N/p}}{N}\|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/p}$.*

Assuming that $1 < p \le N^2$ and following the same arguments as in the proof of Theorem 3.5 in Guedda and Véron [17], we can show that $\mathcal{E}$ does not satisfy the localized Palais–Smale condition (PS)$_c$ if $c = \frac{kS^{N/p}}{N}\|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/p}$, for all positive integers k.

4. Proof of the main result

It remains to check the two geometric hypotheses of the mountain pass theorem. We have $\mathcal{E}(0) = 0$ and we argue the existence of a "mountain" near the origin. For this purpose we first establish that there are positive numbers d and r such that $\mathcal{E}(u) \geq d$ for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ with $\|u\| = r$. Fix $0 < \lambda < \lambda_1$. Using Theorem 3.1 from Smets [1], there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx - \lambda \int_{\mathbb{R}^N} V(x)|u|^p dx \geq \delta \int_{\mathbb{R}^N} |\nabla u|^p dx \quad \text{for all } u \in \mathcal{D}^{1,p}(\mathbb{R}^N). \tag{26}$$

Taking into account the continuous embeddings of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ and $L^{p^*}(\mathbb{R}^N)$ we obtain for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$

$$\mathcal{E}(u) \geq \frac{\delta}{p} \|u\|^p - C\left(\|u\|^r_{L^r(\mathbb{R}^N)} + \|u\|^{p^*}_{L^{p^*}(\mathbb{R}^N)}\right).$$

Using assumption (8) we deduce that $\mathcal{E}(u) \geq d$ for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ with $\|u\| = r$, for some positive numbers d and r.

The difficult part is to prove the existence of a "valley" over the mountain. This will be achieved by using hypothesis (10), which describes the decay of the potential b near its maximum point in relationship with the critical nonlinear term. Let $\phi \neq 0$ be an arbitrary function in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Then

$$\mathcal{E}(t\phi) = \frac{t^p}{p}\left(\int_{\mathbb{R}^N} |\nabla\phi|^p dx - \lambda \int_{\mathbb{R}^N} V(x)|\phi|^p dx\right) - t^p\left(\frac{t^{r-p}}{r}\int_{\mathbb{R}^N} a(x)|\phi|^r dx + \frac{t^{p^*-p}}{p^*}\int_{\mathbb{R}^N} b(x)|\phi|^{p^*} dx\right) < 0,$$

for large enough $t > 0$.

In order to ensure the localized Palais–Smale condition $(\mathrm{PS})_c$, it remains to show that the upper bounds of $\mathcal{E}$ are in $(0, c_0)$, where c_0 is defined in (20). More precisely, if u_ε achieves the minimum S in problem (11) (recall that u_ε is defined in (12)), then we prove that there exists $\varepsilon > 0$ small enough such that

$$\sup_{t>0} \mathcal{E}(tu_\varepsilon) < c_0 := \frac{S^{N/p}}{N} \|b\|^{(p-N)/p}_{L^\infty(\mathbb{R}^N)}. \tag{27}$$

Fix $\varepsilon > 0$. By invariance, we remark that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p dx = \int_{\mathbb{R}^N} |\nabla u_1|^p dx \quad \text{and} \quad \int_{\mathbb{R}^N} b(x) u_\varepsilon(x)^{p^*} dx = \int_{\mathbb{R}^N} b(\varepsilon x) u_1(x)^{p^*} dx. \tag{28}$$

As we have just observed, $\sup_{t>0} \mathcal{E}(tu_\varepsilon) > 0$ and this is achieved at some $t(\varepsilon) > 0$. We claim that the family $\{t(\varepsilon)\}_{\varepsilon>0}$ is bounded from below by a positive constant. Indeed, combining $\mathcal{E}'(t(\varepsilon)u_\varepsilon)(u_\varepsilon) = 0$ with relations (26) and (28), we obtain

$$t(\varepsilon)^{p^*-p}\int_{\mathbb{R}^N} b(x) u_\varepsilon^{p^*} dx + t(\varepsilon)^{r-p}\int_{\mathbb{R}^N} a(x) u_\varepsilon^r dx dx \geq \delta \int_{\mathbb{R}^N} |\nabla u_1|^p dx > 0.$$

Using (8), we deduce our claim. A straightforward computation shows that $\{t(\varepsilon)\}_{\varepsilon>0}$ is bounded from above. More precisely, our assumption (10) implies that there is some $R > 0$ such that for all $\varepsilon > 0$

$$t(\varepsilon) \leq \left(\frac{\int_{\mathbb{R}^N} |\nabla u_1|^p dx}{2^{-1} b(0) \int_{B_R(0)} u_\varepsilon(x)^{p^*} dx}\right)^{(N-p)/p^2}.$$

We control the behavior of $\mathcal{E}(t(\varepsilon)u_\varepsilon) = \sup_{t>0} \mathcal{E}(tu_\varepsilon)$ by observing that

$$\mathcal{E}(t(\varepsilon)u_\varepsilon) = \Phi_1(\varepsilon) + \Phi_2(\varepsilon) + \Phi_3(\varepsilon),$$

where

$$\Phi_1(\varepsilon) = \frac{t(\varepsilon)^p}{p}\int_{\mathbb{R}^N} |\nabla u_1|^p dx - \frac{t(\varepsilon)^{p^*}}{p^*} b(0) \int_{\mathbb{R}^N} u_1^{p^*} dx;$$
$$\Phi_2(\varepsilon) = \frac{t(\varepsilon)^{p^*}}{p^*} b(0) \int_{\mathbb{R}^N} u_1^{p^*} dx - \frac{t(\varepsilon)^{p^*}}{p^*}\int_{\mathbb{R}^N} b(\varepsilon x) u_1^{p^*} dx;$$
$$\Phi_3(x) = -\frac{\lambda t(\varepsilon)^p}{p}\int_{\mathbb{R}^N} V(x) u_\varepsilon^p dx - \frac{t(\varepsilon)^r}{r}\int_{\mathbb{R}^N} a(x) u_\varepsilon^r dx.$$

In what follows we prove that the growth of $\mathcal{E}(t(\varepsilon)u_\varepsilon)$ is given by Φ_1, while Φ_2 and Φ_3 tend to zero as $\varepsilon \to 0$.

Note that the mapping $(0, \infty) \ni s \longmapsto C_1 s^p - C_2 s^{p^*}$ (where C_1, C_2 are positive constants) admits a maximum for

$$s = \left(\frac{C_1(N-p)}{C_2 N}\right)^{(N-p)/p^2}.$$

Returning to Φ_1 we deduce that

$$\Phi_1(\varepsilon) \le \frac{1}{N} b(0)^{(p-N)/p} \left(\int_{\mathbb{R}^N} |\nabla u_1|^p dx\right)^{N/p} \left(\int_{\mathbb{R}^N} u_1^{p^*} dx\right)^{(p-N)/p}$$

$$= \frac{S^{N/p}}{N} \|b\|_{L^\infty(\mathbb{R}^N)}^{(p-N)/p} = c_0.$$

It remains to establish the asymptotic decay of Φ_2 and Φ_3 as $\varepsilon \to 0$. Using hypothesis (10) we obtain, for some $C > 0$ independent of ε,

$$\Phi_2(\varepsilon) \le C\varepsilon^\eta \int_{\mathbb{R}^N} |x|^\eta \left(1 + |x|^{p/(p-1)}\right)^{(p-N)/p},$$

which shows that

$$\Phi_2(\varepsilon) \le C\varepsilon^\eta \quad \text{if } N \ne \frac{pr}{r+1-p}$$

and

$$\Phi_2(\varepsilon) \le C\varepsilon^\eta \log\frac{1}{\varepsilon} \quad \text{if } N = \frac{pr}{r+1-p}.$$

A similar computation based on assumption (9) shows that

$$\Phi_3(\varepsilon) \le C\varepsilon^\eta \quad \text{if } N \ne \frac{pr}{r+1-p}$$

and

$$\Phi_3(\varepsilon) \le C\varepsilon^\eta \log\frac{1}{\varepsilon} \quad \text{if } N = \frac{pr}{r+1-p}.$$

Combining these estimates we obtain (27). This concludes the proof. □

4.1. Final remarks

Due to the singular behavior of the indefinite potential V, we cannot improve the global regularity of the weak solution u. In the special case when V is bounded (or away from its singularities, in the general case), Theorem 2.2 of Pucci and Servadei [21] implies that $u \in L^\infty_{loc}(\mathbb{R}^N)$. By the Moser iteration, with the same arguments as in the proof of Theorem 1.1 in Filippucci, Pucci and Rădulescu [19], this implies that $u \in C^{1,\alpha}(\mathbb{R}^N \cap B_R)$, for some $\alpha = \alpha(R) \in (0, 1)$. In such a case, $u \in L^m(\mathbb{R}^N)$ for all $p^* < m < \infty$ and $\lim_{|x|\to\infty} u(x) = 0$, with the same ideas as in the proof of Lemma 2 in Yu [22], which is based on Theorem 1 of Serrin [23].

We point out that an existence result in relationship with our Theorem 2.1 is proved in Theorem 3.1 of Guedda and Véron [17] in the case of *bounded* domains, with only one perturbation term, and with *constant positive* potentials. In their case, a positive solution vanishing on the boundary is found, provided that $1 < p^2 \le N$.

The result stated in Theorem 2.1 can be extended with similar arguments in the following three directions:

(i) If the nonlinearity $|u|^{r-2}u$ is replaced by a more general function $g(x, u)$ with upper and lower bounds of the type $g_1(x)u^{r_1}$ and $g_2(x)u^{r_2}$ satisfying appropriate technical conditions;

(ii) In the proof of the Palais–Smale condition $(\mathrm{PS})_c$, the fact that any bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ contains a strongly convergent subsequence can be proved under the stronger assumption that the subcritical term $|u|^{r-2}u$ is replaced by an *almost critical* nonlinearity $h(x, u)$, in the sense that $h(x, u) = o(|u|^{p^*-1})$ as $|u| \to \infty$, uniformly for $x \in \mathbb{R}^N$. Next, with similar arguments, the conclusion of Theorem 2.1 follows.

(iii) The existence result established in Theorem 2.1 remains valid if problem (6) is replaced with the following quasilinear singular problem

$$\begin{cases} -\mathrm{div}\,(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu \dfrac{|u|^{p-2}u}{|x|^{p(a+1)}} = \dfrac{|u|^{q-2}u}{|x|^{bq}} + \lambda f(x, u) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{29}$$

where $0 \in \Omega \subset \mathbb{R}^N, N \ge 3$, is a bounded domain and $1 < p < N, a < N/p, a \le b < a + 1, \lambda$ is a positive parameter, $0 \le \mu < \bar{\mu} := [(N-p)/p - a]^p$, $q = p^*(a, b) := Np/(N - pd)$ is the critical Hardy–Sobolev exponent and $d = a + 1 - b$. Note that $p^*(0, 0) = p^* = Np/(N - p)$. In this case, λ_1 is the principal eigenvalue of the differential operator $L_\mu u := -\mathrm{div}\,(|x|^{-ap}|\nabla u|^{p-2}\nabla u) - \mu\, |x|^{-p(a+1)}|u|^{p-2}u$ and the role of the concentration-compactness principle of Smets [1] is played by Lemma 2.1 in Liang and Zhang [24].

An interesting *open problem* is to study if the main result in the present paper remains true if the $(p-1)$-superlinear term $|u|^{r-2}u$ is replaced by a nonlinear term $f(u)$ such that

$$\lim_{u\to+\infty} \frac{f(u)}{u^{p-1}} = +\infty.$$

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