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On nonlinear Schrödinger equations on the hyperbolic space

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ABSTRACT

We study existence of weak solutions for certain classes of nonlinear Schrödinger equations on the Poincaré ball model $\mathbb{B}^N$, $N \geq 3$. By using the Palais principle of symmetric criticality and suitable group theoretical arguments, we establish the existence of a nontrivial (weak) solution.

1. Introduction

This paper was motivated by a large number of applications of the eigenvalues problem for the Laplace-Beltrami operator in the hyperbolic framework and in particular, by recent important work [34,38,39]. We study the following elliptic problem

$$-\Delta_H u = \lambda \alpha(\sigma) f(u) \ \text{ on } \mathbb{B}^N, \ \ u \in H^{1,2}(\mathbb{B}^N), \tag{1.1}$$

on the Poincaré ball model $\mathbb{B}^N$. Here, Δ_H is the Laplace–Beltrami operator, $\lambda > 0$ is a real parameter, $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is a nonnegative nontrivial radially symmetric potential, $N \geq 3$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following growth condition

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$$\alpha_f = \sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|^{q-1}} < \infty, \tag{1.2}$$

where $q \in [2, 2^*]$ and $2^* = 2N/(N-2)$ denotes the critical Sobolev exponent.

Problem (1.1) is an important generalization of the most widely investigated elliptic problems with subcritical nonlinearities which arise naturally in various areas of mathematics. For instance, an important incentive to study Kirchhoff–type problems comes from recent publications [1,2,18,19,31–33,37,44] in which Kirchhoff equations on $\mathbb{B}^N$ have been proposed as an interesting open problem (see also [21–24,35] for related topics).

Since $\mathbb{B}^N$ is an important model of a Hadamard manifold (i.e. a complete, simply connected Riemannian manifold with nonpositive sectional curvature), our approach can be used (as we plan to do in our forthcoming paper) to study existence of multiple solutions of elliptic problems on Hadamard manifolds in the presence of a compact topological group action.

Given $\sigma \in \mathbb{B}^N$, let $T_\sigma(\mathbb{B}^N)$ denote the tangent space and $\langle\cdot,\cdot\rangle_\sigma$ the related inner product. We investigate weak solutions of problem (1.1), i.e. for functions $u \in H^{1,2}(\mathbb{B}^N)$ such that for every $\varphi \in H^{1,2}(\mathbb{B}^N)$, the following is satisfied

$$\int_{\mathbb{B}^N} \langle \nabla_H u(\sigma), \nabla_H \varphi(\sigma)\rangle_\sigma \, d\mu = \lambda \int_{\mathbb{B}^N} \alpha(\sigma) f(u(\sigma))\varphi(\sigma) d\mu,$$

where $d\mu$ is for the Riemannian volume element on $\mathbb{B}^N$, and $\nabla_H = \left(\frac{(1-|\sigma|^2)}{2}\right)^2 \nabla$ is the covariant gradient (here, $|\cdot|$ and ∇ denote the Euclidean distance and the gradient in $\mathbb{R}^N$, respectively). Let $SO(N)$ be the special orthogonal group, $N \geq 3$.

We are now ready to state the main result of this paper.

Theorem 1.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

(a) *f satisfies the growth condition (1.2) for some $q \in (2, 2^*)$,*
(b) *f satisfies the asymptotic condition*

$$\lim_{t\to 0^+} \frac{\int_0^t f(s)ds}{t^2} = +\infty, \tag{1.3}$$

(c) *$\alpha \in L^1(\mathbb{B}^N)\cap L^\infty(\mathbb{B}^N)\setminus\{0\}$ is a nonnegative radially symmetric map with respect to the origin $\sigma_0 \in \mathbb{B}^N$.*

Then there exists $\lambda^\star > 0$ such that, for every $\lambda \in (0, \lambda^\star)$, problem (1.1) admits a $SO(N)$–invariant weak solution $u_\lambda \in H^{1,2}(\mathbb{B}^N)$ whose norm converges to zero as λ goes to zero.

We shall find solutions of problem (1.1) as critical points of the following energy functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu - \lambda \int_{\mathbb{B}^N} \alpha(\sigma) \left(\int_0^{u(\sigma)} f(t)dt \right) d\mu \tag{1.4}$$

defined on the Sobolev space $H^{1,2}(\mathbb{B}^N)$. In fact, we shall prove Theorem 1.1 by using variational methods (see [12] as a general reference for this topic) by means of a local minimum result for differentiable functionals, and the Palais principle of symmetric criticality (see Theorems 2.1 and 3.2 below, respectively).

Remark 1.1. Note that condition (1.3) in Theorem 1.1 has been used before - in order to study existence and multiplicity results for certain classes of elliptic problems on bounded domains (see e.g., [3–10], [26–28], and [40]).

The noncompact hyperbolic setting presents additional difficulties with respect to the cited work and appropriate geometric and algebraic tools are needed for the proof. A key tool is a detailed analysis of the energy level of J_λ on the Sobolev space

$$H^{1,2}_{SO(N)}(\mathbb{B}^N) = \{u \in H^{1,2}(\mathbb{B}^N) \mid g \circledast_{SO(N)} u = u, \text{ for every } g \in SO(N)\}$$

of $SO(N)$–invariant functions (see Section 3).

A simple prototype of a function in $H^{1,2}_{SO(N)}(\mathbb{B}^N)$, depending on parameters $0 < r < \rho$, defined by setting for every $\sigma \in \mathbb{B}^N$,

$$w^{1/2}_{\rho,r}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in A^{\rho}_{1/2r} \\ 0 & \text{if } \sigma \in \mathbb{B}^N \setminus A^{\rho}_{r} \\ \frac{2}{r}\left(r - \left|\log\left(\frac{1+|\sigma|}{1-|\sigma|}\right) - \rho\right|\right) & \text{if } \sigma \in A^{\rho}_{r} \setminus A^{\rho}_{1/2r} \end{cases} \tag{1.5}$$

has the support contained in the annuls A^{ρ}_{r} of $\mathbb{B}^N$ (see [34] for more details).

We conclude the introduction by describing the structure of the paper. In Section 2 we shall collect the necessary notations, definitions and facts. In Section 3 we shall present a compactness argument, based on the action of a suitable subgroup of the group of isometries of $\mathbb{B}^N$. In Section 4 we shall prove the main result (Theorem 1.1). Finally, in Section 5 we shall give an example.

Some of the abstract tools used in this paper can be found in [42]. For eigenvalue problems on the hyperbolic space we refer the reader to [11,20,30,35,45,48,49].

2. The abstract framework

Let $\mathbb{B}^N = \{\sigma = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \mid |\sigma| < 1\}$, be equipped by the Riemannian metric $g_{ij} = 4(1-|\sigma|^2)^{-2}\delta_{ij}$, where $\sigma \in \mathbb{B}^N, i, j \in \{1, ..., N\}$, and $|\cdot|$ and δ_{ij} denote the Euclidean distance and the Kronecker delta symbol, respectively. For every $i, j \in \{1, \dots, N\}$, let $g^{ij} = (g_{ij})^{-1}$ and $\mathrm{g} = \det(g_{ij})$. We locally define the Laplace–Beltrami operator Δ_H by

$$\Delta_H = \mathrm{g}^{-1/2} \sum_{\mathrm{i}=1}^{\mathrm{N}} \frac{\partial}{\partial \mathrm{x_i}} \left(\mathrm{g}^{1/2} \sum_{\mathrm{j}=1}^{\mathrm{N}} \mathrm{g}^{\mathrm{ij}} \frac{\partial}{\partial \mathrm{x_j}} \right).$$

The following is a more convenient form

$$\Delta_H = \frac{(1-|\sigma|^2)^2}{4} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{(N-2)(1-|\sigma|^2)}{2} \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i},$$

when we consider the Riemannian volume element in $\mathbb{B}^N$

$$d\mu = \sqrt{\mathrm{g}}\,dx = 2^N(1-|\sigma|^2)^{-N}dx, \tag{2.1}$$

where dx denotes the Lebesgue measure on $\mathbb{R}^N$. Finally, let

$$d_H(\sigma) = d_H(\sigma,\sigma_0) = 2\int_0^{|\sigma|} \frac{dt}{1-t^2} = \log\left(\frac{1+|\sigma|}{1-|\sigma|}\right) \tag{2.2}$$

be the geodesic distance of $\sigma \in \mathbb{B}^N$ from the origin $\sigma_0 \in \mathbb{B}^N$. Let (ϱ,θ) denote the polar geodesic coordinates of a point in $\mathbb{B}^N \setminus \{0\}$. We see that $ds^2 = d\varrho^2 + (\sinh\varrho)^2 d\theta$, on $\mathbb{B}^N \setminus \{0\}$, and

$$\Delta_H = \frac{\partial^2}{\partial\varrho^2} + (N-1)\coth\varrho\frac{\partial}{\partial\varrho} + \frac{\Delta_\theta}{(\sinh\varrho)^2},$$

where Δ_θ the Laplace–Beltrami operator on the sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. Invoking (2.2), we define the distance on $\mathbb{B}^N$ by

$$d_H(\sigma_1,\sigma_2) = \operatorname{Arccosh}\left(1 + \frac{2|\sigma_2-\sigma_1|^2}{(1-|\sigma_1|^2)(1-|\sigma_2|^2)}\right), \quad \text{for every} \quad \sigma_1,\sigma_2 \in \mathbb{B}^{\mathrm{N}}.$$

For every $r > 0$, we denote by $B(r) = \{\sigma \in \mathbb{B}^N \mid |\sigma| < r\}$ (resp. $B_H(r) = \{\sigma \in \mathbb{B}^N \mid d_H(\sigma) < r\}$) the Euclidean (resp. geodesic) ball of radius r, at the origin $\sigma_0 \in \mathbb{B}^N$. It follows by (2.2), that for every $r \in (0,1)$, $B(r) = B_H\left(\log\left(\frac{1+r}{1-r}\right)\right)$. See [45] for additional comments and related facts.

For any $\sigma \in \mathbb{R}^N$, let $T_\sigma(\mathbb{B}^N)$ be the tangent space at $\sigma \in \mathbb{B}^N$, equipped by the inner product $\langle\cdot,\cdot\rangle_\sigma$ and let $T(\mathbb{B}^N) = \bigcup_{\sigma\in\mathbb{B}^N} T_\sigma(\mathbb{B}^N)$ be the tangent bundle. Whenever possible, given $X, Y \in T_\sigma(\mathbb{B}^N)$, we write $|X|$ and $\langle X,Y\rangle$ instead of $|X|_\sigma$ and $g_\sigma(X,Y) = \langle X,Y\rangle_\sigma$, respectively.

Recall that $C_0^\infty(\mathbb{B}^N)$ denotes the space of real-valued smooth functions compactly supported on $\mathbb{B}^N$. Let

$$\|u\| = \sqrt{\int_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu}, \quad \text{for every} \quad u \in C_0^\infty(\mathbb{B}^N), \tag{2.3}$$

where $d\mu$ denotes the Riemannian measure on $\mathbb{B}^N$ from (2.1) and we get the following

$$\nabla_H = \left(\frac{(1-|\sigma|^2)}{2}\right)^2 \nabla \quad \text{and} \quad |\nabla_H u(\sigma)| = \left(\frac{(1-|\sigma|^2)}{2}\right)^2 \sqrt{\langle\nabla u(\sigma),\nabla u(\sigma)\rangle}.$$

Then $H^{1,2}(\mathbb{B}^N)$ is the completion of $C_0^\infty(\mathbb{B}^N)$ with respect to the norm (2.3) and it is a Hilbert space with the inner product

$$\langle u,v\rangle = \int_{\mathbb{B}^N} \langle\nabla_H u(\sigma),\nabla_H v(\sigma)\rangle\, d\mu, \quad \text{for every} \quad u,v \in H^{1,2}(\mathbb{B}^N). \tag{2.4}$$

We need to find critical points of the functional J_λ from (1.4) so we shall invoke the principle of symmetric criticality, together with the following critical point theorem of Ricceri [46] which we state in a form more suitable for our purpose.

Theorem 2.1. *Let X be a reflexive real Banach space and $\Phi,\Psi : X \to \mathbb{R}$ Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive, whereas Ψ is sequentially weakly upper semicontinuous. Given $r > \inf_X \Phi$, let*

$$\varphi(r) = \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup\{\Psi(v) \mid v \in \Phi^{-1}((-\infty,r))\} - \Psi(u)}{r - \Phi(u)}.$$

Then for every $r > \inf_X \Phi$ and $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, the restriction of the functional $J_\lambda = \Phi - \lambda\Psi$ on $\Phi^{-1}((-\infty, r))$ admits a global minimum which is a critical point (local minimum) of J_λ in X.

Remark 2.1. Theorem 2.1 is a direct consequence of [9, Theorem 2.1] (see also [4,5] for related topics).

Remark 2.2. Problem (1.1) is set on the entire noncompact space $\mathbb{B}^N$. Therefore we shall take a group theoretical approach in Section 3, in order to identify those symmetric subspaces of $H^{1,2}(\mathbb{B}^N)$ on which compactness of the embedding into $L^\nu(\mathbb{B}^N)$ can be regained.

3. $SO(N)$–invariant functions

Consider the special orthogonal group $SO(N)$, $N \geq 3$. Let $\cdot : SO(N) \times \mathbb{B}^N \to \mathbb{B}^N$ be the natural action of $SO(N)$ on $\mathbb{B}^N$. The action $\circledast_{SO(N)} : SO(N) \times H^{1,2}(\mathbb{B}^N) \to H^{1,2}(\mathbb{B}^N)$ of a subgroup $SO(N) \in \mathscr{F}$ on $H^{1,2}(\mathbb{B}^N)$ is given by

$$g \circledast_{SO(N)} u(\sigma) = u(g^{-1} \cdot \sigma), \quad \text{for a.e. } \sigma \in \mathbb{B}^N, \tag{3.1}$$

for every $g \in SO(N)$ and $u \in H^{1,2}(\mathbb{B}^N)$. Denote by

$$H^{1,2}_{SO(N)}(\mathbb{B}^N) = \left\{u \in H^{1,2}(\mathbb{B}^N) \mid g \circledast_{SO(N)} u = u, \text{ for every } g \in SO(N)\right\}$$

the subspace of $SO(N)$–invariant functions of $H^{1,2}(\mathbb{B}^N)$. By using a recent embedding theorem of Skrzypczak and Tintarev [47, Theorem 1.3 and Proposition 3.1], the following compactness argument can be proved (see also [17,29]).

Theorem 3.1. (See [47].) *For every $\nu \in (2, 2^*)$, the embedding $H^{1,2}_{SO(N)}(\mathbb{B}^N) \to L^\nu(\mathbb{B}^N)$ is compact.*

Next, we recall the Palais principle of symmetric criticality. The group $(SO(N), *)$ acts continuously on the Hilbert space $H^{1,2}(\mathbb{B}^N)$ by $(\tau, u) \mapsto \tau \circledast_{SO(N)} u$ from $SO(N) \times H^{1,2}(\mathbb{B}^N)$ to $H^{1,2}(\mathbb{B}^N)$, if this map itself is continuous on $SO(N) \times H^{1,2}(\mathbb{B}^N)$ and it has the following properties

(i_1) for every $\tau \in SO(N)$, $u \mapsto \tau \circledast_{SO(N)} u$ is linear;
(i_2) for every $\tau_1, \tau_2 \in SO(N)$ and $u \in H^{1,2}(\mathbb{B}^N)$,

$$(\tau_1 * \tau_2) \circledast_{SO(N)} u = \tau_1 \circledast_{SO(N)} (\tau_2 \circledast_{SO(N)} u); \quad \text{and}$$

(i_3) for every $u \in H^{1,2}(\mathbb{B}^N)$,

$$id_{SO(N)} \circledast_{SO(N)} u = u,$$

where $id_{SO(N)} \in SO(N)$ denotes the identity element of $SO(N)$.

Define

$$\mathrm{Fix}_{\mathrm{SO(N)}}(\mathrm{H}^{1,2}(\mathbb{B}^{\mathrm{N}})) = \left\{\mathrm{u} \in \mathrm{H}^{1,2}(\mathbb{B}^{\mathrm{N}}) \mid \tau \circledast_{\mathrm{SO(N)}} \mathrm{u} = \mathrm{u}, \text{ for every } \tau \in \mathrm{SO(N)}\right\}$$

and recall that the functional $\mathcal{J} : H^{1,2}(\mathbb{B}^N) \to \mathbb{R}$ is called $SO(N)$–invariant if

$$\mathcal{J}(\tau \circledast_{SO(N)} u) = \mathcal{J}(u), \quad \text{for every } \ u \in H^{1,2}(\mathbb{B}^N).$$

The following result holds.

Theorem 3.2. (See [41].) *Let $H^{1,2}(\mathbb{B}^N)$ be the Sobolev space associated to the Poincaré model $\mathbb{B}^N$, $SO(N)$ the special orthogonal group acting continuously on $H^{1,2}(\mathbb{B}^N)$ by the map*

$$\circledast_{SO(N)} : SO(N) \times H^{1,2}(\mathbb{B}^N) \to H^{1,2}(\mathbb{B}^N),$$

and $\mathcal{J} : H^{1,2}(\mathbb{B}^N) \to \mathbb{R}$ a $SO(N)$–invariant C^1–function.

If $u \in \mathrm{Fix_{SO(N)}}(\mathrm{H^{1,2}}(\mathbb{B}^N))$ is a critical point of $\mathcal{J}_{|\mathrm{Fix_{SO(N)}}(\mathrm{H^{1,2}}(\mathbb{B}^N))}$, then $u \in H^{1,2}(\mathbb{B}^N)$ is also a critical point of $\mathcal{J}$.

For details and comments we refer to [13, Section 5] and [14]. See also [34,36,43] for related topics and results.

4. Proof of the main theorem

Consider the functional $\mathcal{J}_\lambda(u) = \Phi(u) - \lambda\Psi|_{SO(N)}(u)$, $u \in H^{1,2}_{SO(N)}(\mathbb{B}^N)$, where

$$\Phi(u) = \frac{1}{2}\int\limits_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu \;\text{ and }\; \Psi(\mathrm{u}) = \int\limits_{\mathbb{B}^\mathrm{N}} \alpha(\sigma)\mathrm{F}(\mathrm{u}(\sigma))\mathrm{d}\mu.$$

We shall apply Theorem 2.1 to the energy functional $\mathcal{J}_\lambda$ and use some ideas from [34,39]. On the basis of the preliminaries collected in Sections 2 and 3, the existence of one nontrivial $SO(N)$–symmetric solution of problem (1.1) follows by the Palais criticality principle (Theorem 3.2).

The space $H^{1,2}_{SO(N)}(\mathbb{B}^N)$ admits a Hilbert structure. By [25], the functionals Φ and $\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}$ satisfy all the regularity assumptions of Theorem 2.1. More precisely, the functional Φ is (strongly) continuous, coercive in the symmetric space $H^{1,2}_{SO(N)}(\mathbb{B}^N)$ and $\inf\{\Phi(u) \mid u \in H^{1,2}_{SO(N)}(\mathbb{B}^N)\} = 0$.

Since for every $\nu \in [2, 2^*]$, the Sobolev embedding $H^{1,2}(\mathbb{B}^N) \to L^\nu(\mathbb{B}^N)$ is continuous (but noncompact - see [25]), we shall make use also of the positive constant

$$c_\nu = \sup\Big\{\Big(\int\limits_{\mathbb{B}^N} |u(\sigma)|^\nu d\mu\Big)^{\frac{1}{\nu}}\Big(\int\limits_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu\Big)^{-\frac{1}{2}} \mid u \in H^{1,2}(\mathbb{B}^N) \setminus \{0\}\Big\}.$$

Set $h(\omega) = \omega(q\sqrt{2}\|\alpha\|_p + 2^{q/2}c_q^{q-1}\|\alpha\|_\infty \omega^{q-1})^{-1}$, for every $\omega > 0$, and define

$$\lambda^\star = \frac{q\max\{h(\omega) \mid \omega > 0\}}{\alpha_f c_q}, \quad \text{where } \; p = \frac{q}{q-1}. \tag{4.1}$$

Take $0 < \lambda < \lambda^\star$. By (4.1), we have

$$\lambda < \lambda^\star(\bar\omega) = \frac{qh(\bar\omega)}{\alpha_f c_q}, \quad \text{for some } \; \bar\omega > 0. \tag{4.2}$$

Let $\Theta : (0, \infty) \to [0, \infty)$ be the real function defined by

$$\Theta(r) = \frac{1}{r}\sup\{\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}(u) \mid u \in \Phi^{-1}((-\infty, r))\}, \quad \text{for every } \; r > 0.$$

Then condition (1.2) gives

$$\Psi(u) \leq \alpha_f \int\limits_{\mathbb{B}^N} \alpha(\sigma)|u(\sigma)|d\mu + \frac{\alpha_f}{q} \int\limits_{\mathbb{B}^N} \alpha(\sigma)|u(\sigma)|^q d\mu, \quad \text{for every} \ \ u \in H^{1,2}_{SO(N)}(\mathbb{B}^N),$$

so if $\Phi(u) < r$, then

$$\int\limits_{\mathbb{B}^N} |\nabla_H u(\sigma)|^2 d\mu < 2r, \quad \text{for every} \ \ u \in H^{1,2}_{SO(N)}(\mathbb{B}^N). \tag{4.3}$$

An application of (4.3) and Theorem 3.1 yields

$$\int\limits_{\mathbb{B}^N} \alpha(\sigma)F(u(\sigma))d\mu < \alpha_f c_q \left(\|\alpha\|_p \sqrt{2r} + \frac{c_q^{q-1}}{q} \|\alpha\|_\infty (2r)^{q/2} \right), \quad \text{for every} \ \ u \in H^{1,2}_{SO(N)}(\mathbb{B}^N), \Phi(u) < r.$$

Consequently,

$$\sup\{\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}(u) \mid u \in \Phi^{-1}((-\infty, r))\} \leq \alpha_f c_q \left(\|\alpha\|_p \sqrt{2r} + \frac{c_q^{q-1}}{q} \|\alpha\|_\infty (2r)^{q/2} \right).$$

Hence

$$\Theta(r) \leq \alpha_f c_q \left(\|\alpha\|_p \sqrt{\frac{2}{r}} + \frac{2^{q/2} c_q^{q-1}}{q} \|\alpha\|_\infty r^{q/2-1} \right), \quad \text{for every} \ \ r > 0. \tag{4.4}$$

Taking $r = \bar{\omega}^2$, we get

$$\Theta(\bar{\omega}^2) \leq \alpha_f c_q \left(\sqrt{2} \frac{\|\alpha\|_p}{\bar{\omega}} + \frac{2^{q/2} c_q^{q-1}}{q} \|\alpha\|_\infty \bar{\omega}^{q-2} \right). \tag{4.5}$$

On the other hand,

$$\varphi(\bar{\omega}^2) = \inf_{u \in \Phi^{-1}((-\infty, \bar{\omega}^2))} \frac{\sup\{\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}(u) \mid u \in \Phi^{-1}((-\infty, r))\} - \Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}(u)}{r - \Phi(u)} \leq \Theta(\bar{\omega}^2),$$

since $0 \in \Phi^{-1}((-\infty, \bar{\omega}^2))$ and $\Phi(0) = 0$. By virtue of (4.2) and (4.5), we have

$$\varphi(\bar{\omega}^2) \leq \Theta(\bar{\omega}^2) < \frac{1}{\lambda}, \tag{4.6}$$

hence $\lambda \in (0, 1/\varphi(\bar{\omega}^2))$. Consequently, by Theorem 2.1, there exists $u_\lambda^{SO(N)} \in \Phi^{-1}((-\infty, \bar{\omega}^2))$ such that

$$\Phi'(u_\lambda^{SO(N)}) = \lambda(\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)})'(u_\lambda^{SO(N)}).$$

Moreover, $u_\lambda^{SO(N)}$ is a global minimum of $\mathcal{J}_\lambda$ on the sublevel $\Phi^{-1}((-\infty, \bar{\omega}^2))$.

Next, we show that solution $u_\lambda^{SO(N)}$ is not the trivial (identically zero) function. If $f(0) \neq 0$, then it easily follows that $u_\lambda^{SO(N)} \not\equiv 0$ in $H^{1,2}_{SO(N)}(\mathbb{B}^N)$, since the trivial function does not solve problem (1.1).

So let us consider the case when $f(0) = 0$ and fix $\lambda \in (0, \lambda^\star(\bar{\omega}))$ for some $\bar{\omega} > 0$. Let $u_\lambda^{SO(N)}$ be such that

$$\mathcal{J}_\lambda(u_\lambda^{SO(N)}) \leq \mathcal{J}_\lambda(u), \quad \text{for every} \ \ u \in H^{1,2}_{SO(N)}(\mathbb{B}^N) \text{ such that } \Phi(u) < \bar{\omega}^2 \tag{4.7}$$

and

$$\Phi(u_\lambda^{SO(N)}) < \bar{\omega}^2 , \tag{4.8}$$

and that $u_\lambda^{SO(N)}$ is a critical point of $\mathcal{J}_\lambda$ in $H^{1,2}_{SO(N)}(\mathbb{B}^N)$.

Applying Theorem 1.1, the energy J_λ defined in (1.4) needs to be invariant with respect to the special orthogonal group $SO(N)$. To show this, fix $u \in H^{1,2}(\mathbb{B}^N)$ and $g \in SO(N)$. Since $g \in SO(N)$ is an isometry, it follows by (3.1) that

$$\nabla_H(g \circledast_{SO(N)} u)(\sigma) = Dg_{g^{-1}\cdot\sigma}\nabla_H u(g^{-1}\cdot\sigma), \quad \text{for a.e. } \ \sigma \in \mathbb{B}^N. \tag{4.9}$$

If $z = g^{-1}\cdot\sigma$, then

$$\begin{aligned}\left\|g \circledast_{SO(N)} u\right\|^2 &= \int_{\mathbb{B}^N} |\nabla_H(g \circledast_{SO(N)} u)(\sigma)|^2_\sigma d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} |\nabla_H u(g^{-1}\cdot\sigma)|^2_{g^{-1}\cdot\sigma} d\mu(\sigma) = \|u\|^2 ,\end{aligned} \tag{4.10}$$

where we have used (4.9). On the other hand, since $\alpha \in L^1(\mathbb{B}^N) \cap L^\infty(\mathbb{B}^N)$ is radially symmetric respect to the origin, it follows that

$$\begin{aligned}\int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_0^{(g\circledast_{SO(N)}u)(\sigma)} h(t)dt\right) d\mu(\sigma) &= \int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_0^{u(g^{-1}\cdot\sigma)} h(t)dt\right) d\mu(\sigma) \\ &= \int_{\mathbb{B}^N} \alpha(z)\left(\int_0^{u(z)} h(t)dt\right) d\mu(z).\end{aligned} \tag{4.11}$$

By (4.10) and (4.11), we have $J_\lambda(g \circledast_{SO(N)} u) = J_\lambda(u)$, which proves the $SO(N)$ invariance of the functional J_λ.

By Theorem 3.2, it is clear that $u_\lambda^{SO(N)}$ weakly solves problem (1.1). Proving that $u_\lambda^{SO(N)} \not\equiv 0$ in $H^{1,2}_{SO(N)}(\mathbb{B}^N)$, we show the existence of a sequence $\{w_j\}_{j\in\mathbb{N}}$ in $H^{1,2}_{SO(N)}(\mathbb{B}^N)$ such that

$$\limsup_{j\to\infty} \frac{\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}(w_j)}{\Phi(w_j)} = \infty . \tag{4.12}$$

By (1.3), there exists $\{t_j\}_{j\in\mathbb{N}} \subset (0, +\infty)$ such that $t_j \to 0^+$ when $j \to +\infty$, and

$$\lim_{j\to+\infty} \frac{F(t_j)}{t_j^2} = +\infty. \tag{4.13}$$

Therefore for every $M > 0$ and all sufficiently large j,

$$F(t_j) > Mt_j^2 . \tag{4.14}$$

Now, $\alpha \in L^\infty(\mathbb{B}^N) \setminus \{0\}$ is nonnegative in $\mathbb{B}^N$. Hence there are real numbers $\rho > r > 0$ and $\alpha_0 > 0$ such that

$$\operatorname{essinf}_{\sigma\in A_r^\rho}\alpha(\sigma) \geq \alpha_0 > 0. \tag{4.15}$$

For every $0 < a < b$, set

$$A_a^b = \left\{\sigma \in \mathbb{B}^N \mid b-a < \log\left(\frac{1+|\sigma|}{1-|\sigma|}\right) < a+b\right\}.$$

Define $w_{\rho,r}^{1/2}(\sigma) \in H^{1,2}(\mathbb{B}^N)$ by

$$w_{\rho,r}^{1/2}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in A_{1/2r}^\rho \\ 0 & \text{if } \sigma \in \mathbb{B}^N \setminus A_r^\rho \\ \frac{2}{r}\left(r - \left|\log\left(\frac{1+|\sigma|}{1-|\sigma|}\right) - \rho\right|\right) & \text{if } \sigma \in A_r^\rho \setminus A_{1/2r}^\rho, \end{cases} \tag{4.16}$$

for every $\sigma \in \mathbb{B}^N$.

Since the group $SO(N)$ is a compact connected subgroup of the isometry group $\mathrm{Isom}_g(\mathbb{B}^N)$ such that $\mathrm{Fix}_{SO(N)}(\mathbb{B}^N) = \{\sigma_0\}$, it follows that $w_{\rho,r}^{1/2} \in H^{1,2}(\mathbb{B}^N)$, given in (4.16), belongs to $H^{1,2}_{SO(N)}(\mathbb{B}^N)$. Therefore $\mathrm{supp}(w_{\rho,r}^{1/2}) \subseteq A_r^\rho(\sigma_0)$, $\|w_{\rho,r}^{1/2}\|_\infty \leq 1$, and $w_{\rho,r}^{1/2}(\sigma) = 1$, for every $\sigma \in A_{1/2r}^\rho(\sigma_0)$.

Remark 4.1. The test functions used here were introduced in [35], following [17]. We note that test functions introduced in [3,8] are different. We also emphasize that the different geometrical structure used along the proof is crucial in order to recover the $SO(N)$ invariance of the test functions.

Define $w_j = t_j w_{\rho,r}^{1/2}$ for any $j \in \mathbb{N}$. Taking into account that $w_{\rho,r}^{1/2} \in H^{1,2}_{SO(N)}(\mathbb{B}^N)$, it is easily seen that $w_j \in H^{1,2}_{SO(N)}(\mathbb{B}^N)$, for every $j \in \mathbb{N}$. Furthermore, exploiting the properties of $w_{\rho,r}^{1/2}$, by (4.14), it follows that:

$$\begin{aligned} \frac{\Psi|_{H^{1,2}_{SO(N)}(\mathbb{B}^N)}(w_j)}{\Phi(w_j)} &= \frac{\int\limits_{A_{1/2r}^\rho} \alpha(\sigma)F(w_j(x))\,d\mu + \int\limits_{A_r^\rho\setminus A_{1/2r}^\rho} \alpha(\sigma)F(w_j(\sigma))\,d\mu}{\Phi(w_j)} \\ &= \frac{\int\limits_{A_{1/2r}^\rho} \alpha(\sigma)F(t_j)\,d\mu + \int\limits_{A_r^\rho\setminus A_{1/2r}^\rho} \alpha(\sigma)F(t_j w_{\rho,r}^{1/2}(\sigma))\,d\mu}{\Phi(w_j)} \\ &\geq 2\alpha_0 \frac{M\mu(A_{1/2r}^\rho)t_j^2 + \int\limits_{A_r^\rho\setminus A_{1/2r}^\rho} F(t_j w_{\rho,r}^{1/2}(\sigma))\,d\mu}{t_j^2\|w_{\rho,r}^{1/2}\|^2}, \quad \text{for sufficiently large } j. \end{aligned} \tag{4.17}$$

Assertion (4.12) now follows by (4.17).

Now

$$\int\limits_{\mathbb{B}^N} |\nabla_H w_j(\sigma)|^2 d\mu = t_j^2 \int\limits_{\mathbb{B}^N} |\nabla_H w_{\rho,r}^{1/2}(\sigma)|^2 d\mu \to 0, \quad \text{as} \quad j \to +\infty,$$

so $\|w_j\| < \sqrt{2}\bar{\omega}$, for sufficiently large j. Hence

$$w_j \in \Phi^{-1}\big((-\infty, \bar{\omega}^2)\big), \tag{4.18}$$

provided that j is large enough. Moreover, by (4.12),

$$\mathcal{J}_\lambda(w_j) = \frac{1}{2}\int_{\mathbb{B}^N} |\nabla_H w_j(\sigma)|^2 d\mu - \lambda \int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_0^{w_j(\sigma)} f(t)dt\right) d\mu < 0, \tag{4.19}$$

for sufficiently large j and $\lambda > 0$.

Since the restriction of $\mathcal{J}_\lambda$ to $\Phi^{-1}\big((-\infty, \bar{\omega}^2)\big)$ has $u_\lambda^{SO(N)}$ as a global minimum, it follows by (4.18) and (4.19) that

$$\mathcal{J}_\lambda(u_\lambda^{SO(N)}) \leq \frac{1}{2}\int_{\mathbb{B}^N} |\nabla_H w_j(\sigma)|^2 d\mu - \lambda \int_{\mathbb{B}^N} \alpha(\sigma)\left(\int_0^{w_j(\sigma)} f(t)dt\right) d\mu < \mathcal{J}_\lambda(0)\,, \tag{4.20}$$

so $u_\lambda^{SO(N)} \not\equiv 0$ in $H^{1,2}_{SO(N)}(\mathbb{B}^N)$ as asserted.

Therefore $u_\lambda^{SO(N)}$ is a nontrivial weak solution of problem (1.1). The arbitrariness of λ implies that $u_\lambda^{SO(N)} \not\equiv 0$, for every $\lambda \in (0, \lambda^\star)$.

Finally, we show that $\lim_{\lambda\to 0^+} \|u_\lambda^{SO(N)}\| = 0$. To this end, consider $\lambda \in (0, \lambda^\star(\bar{\omega}))$ for some $\bar{\omega} > 0$. Taking into account that $\Phi(u_\lambda^{SO(N)}) < \bar{\omega}^2$, it follows that $\Phi(u_\lambda^{SO(N)}) = \frac{1}{2}\|u_\lambda^{SO(N)}\|^2 < \bar{\omega}^2$, i.e., $\|u_\lambda^{SO(N)}\| < \sqrt{2}\bar{\omega}$.

The growth condition (1.2) yields

$$\begin{aligned}\left|\Psi'(u_\lambda^{SO(N)})\right| &\leq \alpha_f\left(\int_{\mathbb{B}^N} \alpha(\sigma)|u_\lambda^{SO(N)}(\sigma)|d\mu + \int_{\mathbb{B}^N} \alpha(\sigma)|u_\lambda^{SO(N)}(\sigma)|^q d\mu\right)\\ &\leq \alpha_f\left(\|\alpha\|_p\|u_\lambda^{SO(N)}\|_q + \|\alpha\|_\infty\|u_\lambda^{SO(N)}\|_q^q\right)\\ &< c_q\alpha_f\left(\sqrt{2}\|\alpha\|_p\bar{\omega} + 2^{q/2}c_q^{q-1}\|\alpha\|_\infty\bar{\omega}^q\right) = M_{\bar{\omega}}\,.\end{aligned}$$

Since $u_\lambda^{SO(N)}$ is a critical point of $\mathcal{J}_\lambda$, it follows that $\langle \mathcal{J}_\lambda'(u_\lambda^{SO(N)}), \varphi\rangle = 0$, for every $\varphi \in H^{1,2}_{SO(N)}(\mathbb{B}^N)$ and $\lambda \in (0, \lambda^\star(\bar{\omega}))$. Hence, $\langle \mathcal{J}_\lambda'(u_\lambda^{SO(N)}), u_\lambda^{SO(N)}\rangle = 0$ and thus

$$\langle \Phi'(u_\lambda^{SO(N)}), u_\lambda^{SO(N)}\rangle = \lambda\Psi'(u_\lambda^{SO(N)}), \quad \text{for every} \quad \lambda \in (0, \lambda^\star(\bar{\omega})).$$

The relations above now ensure that

$$0 \leq \|u_\lambda^{SO(N)}\|^2 = \langle \Phi'(u_\lambda^{SO(N)}), u_\lambda^{SO(N)}\rangle = \lambda\Psi'(u_\lambda^{SO(N)}) < \lambda M_{\bar{\omega}}, \quad \text{for every} \quad \lambda \in (0, \lambda^\star(\bar{\omega})).$$

Hence $\lim_{\lambda\to 0^+} \|u_\lambda^{SO(N)}\| = 0$, as was asserted. The proof of Theorem 1.1 is now complete. □

Remark 4.2. Profile decomposition methods can be useful in order to study similar problems when a lack of compactness occurs (see, among others, the recent papers [15,16]). A further and more general investigation of this topic will be included in the forthcoming book [37].

Remark 4.3. The referee has observed that the importance of the solution as a local minimum is in that we can obtain in addition a second solution, and suggested as a further study, to attempt to apply the result contained in [7] to obtain two nonzero solutions for this type of problems.

5. An example

We conclude the paper by exhibiting the following model equation which illustrates how our main result can be applied.

Example 5.1. For any $1 < r < 2$, consider the following problem on $\mathbb{B}^4$

$$-\Delta_H u = \lambda \left(\frac{1-|\sigma|^2}{2}\right)^4 |u|^{r-2}u, \quad u \in H^{1,2}(\mathbb{B}^4). \tag{5.1}$$

By Theorem 1.1, there exists $\lambda^\star > 0$ such that for every $\lambda \in (0, \lambda^\star_{SO(N)})$, problem (5.1) admits at least one nontrivial $SO(N)$–symmetric weak solution $u_\lambda^{SO(N)} \in H^{1,2}(\mathbb{B}^4)$ such that $\lim_{\lambda\to 0^+} \|u_\lambda^{SO(N)}\| = 0$.

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References

[1] G. Autuori, F. Colasuonno, P. Pucci, On the existence of stationary solutions for higher order p-Kirchhoff problems via variational methods, Commun. Contemp. Math. 16 (2014) 1450002.
[2] G. Autuori, P. Pucci, C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, Arch. Ration. Mech. Anal. 196 (2010) 489–516.
[3] G. Bonanno, Existence of three solutions for a two point boundary value problem, Appl. Math. Lett. 13 (2000) 53–57.
[4] G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal. 75 (2012) 2992–3007.
[5] G. Bonanno, P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, J. Differ. Equ. 244 (2008) 3031–3059.
[6] G. Bonanno, A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, J. Math. Anal. Appl. 418 (2014) 812–827.
[7] G. Bonanno, G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, Z. Anal. Anwend. 35 (4) (2016) 449–464.
[8] G. Bonanno, R. Livrea, Multiplicity theorems for the Dirichlet problem involving the p-Laplacian, Nonlinear Anal. 54 (2003) 1–7.
[9] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p–Laplacian, Proc. R. Soc. Edinb., Sect. A 140 (2010) 737–752.
[10] G. Bonanno, P. Pizzimenti, Existence results for nonlinear elliptic problems, Appl. Anal. 92 (2013) 411–423.
[11] L.P. Bonorino, P.K. Klaser, Bounded λ-harmonic functions in domains of $\mathbb{H}^N$ with asymptotic boundary with fractional dimension, J. Geom. Anal. 28 (2018) 2503–2521.
[12] H. Brézis, Analyse fonctionelle. Théorie et applications, Masson, Paris, 1983.
[13] M. Caponi, P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, Ann. Mat. Pura Appl. 195 (2016) 2099–2129.
[14] D.C. de Morais Filho, M.A.S. Souto, J.M. do Ó, A compactness embedding lemma, a principle of symmetric criticality and applications to elliptic problems, Proyecciones 19 (2000) 1–17.
[15] G. Devillanova, Multiscale weak compactness in metric spaces, J. Elliptic Parabolic Equ. 2 (1–2) (2016) 131–144.
[16] G. Devillanova, S. Solimini, Some remarks on profile decomposition theorems, Adv. Nonlinear Stud. 16 (4) (2016) 795–805.
[17] Cs. Farkas, A. Krsitály, Schrödinger-Maxwell systems on non-compact Riemannian manifolds, Nonlinear Anal., Real World Appl. 31 (2016) 473–491.
[18] A. Fiscella, A fractional Kirchhoff problem involving a singular term and a critical nonlinearity, Adv. Nonlinear Anal. 8 (1) (2019) 645–660.
[19] A. Fiscella, P. Pucci, B. Zhang, p–Fractional Hardy-Schrödinger-Kirchhoff systems with critical nonlinearities, Adv. Nonlinear Anal. 8 (1) (2019) 1111–1131.
[20] D. Ganguly, K. Sandeep, Nondegeneracy of positive solutions of a semilinear elliptic problem in the hyperbolic space, Commun. Contemp. Math. 17 (1) (2015) 1450019.

[21] E. Hebey, Multiplicity of solutions for critical Kirchhoff type equations, Commun. Partial Differ. Equ. 41 (6) (2016) 913–924.
[22] E. Hebey, Compactness and the Palais-Smale property for critical Kirchhoff equations in closed manifolds, Pac. J. Math. 280 (1) (2016) 41–50.
[23] E. Hebey, Stationary Kirchhoff equations with powers, Adv. Calc. Var. 11 (2) (2018) 139–160.
[24] E. Hebey, P.-D. Thizy, Stationary Kirchhoff systems in closed high dimensional manifolds, Commun. Contemp. Math. 18 (2) (2016) 1550028.
[25] D. Hoffman, J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds, Commun. Pure Appl. Math. 27 (1974) 715–727.
[26] R. Kajikiya, Multiple existence of non-radial solutions with group invariance for sublinear elliptic equations, J. Differ. Equ. 186 (1) (2002) 299–343.
[27] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, J. Funct. Anal. 225 (2) (2005) 352–370.
[28] R. Kajikiya, Symmetric mountain pass lemma and sublinear elliptic equations, J. Differ. Equ. 260 (3) (2016) 2587–2610.
[29] P.-L. Lions, Symétrie et compacité dans les espaces de Sobolev, J. Funct. Anal. 49 (3) (1982) 315–334.
[30] G. Mancini, K. Sandeep, On a semilinear elliptic equation in $\mathbb{H}^N$, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 7 (4) (2008) 635–671.
[31] X. Mingqi, V.D. Rădulescu, B. Zhang, Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions, Nonlinearity 31 (7) (2018) 3228–3250.
[32] X. Mingqi, V.D. Rădulescu, B. Zhang, A critical fractional Choquard-Kirchhoff problem with magnetic field, Commun. Contemp. Math. 21 (4) (2019) 1850004.
[33] X. Mingqi, V.D. Rădulescu, B. Zhang, Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity, Calc. Var. Partial Differ. Equ. 58 (2) (2019) 57.
[34] G. Molica Bisci, A group-theoretical approach for nonlinear Schrödinger equations, Adv. Calc. Var. (2018), https://doi.org/10.1515/acv-2018-0016 (publ. online).
[35] G. Molica Bisci, Kirchhoff–type problems on a geodesic ball of the hyperbolic space, Nonlinear Anal. 186 (2019) 55–73.
[36] G. Molica Bisci, P. Pucci, Critical Dirichlet problems on $\mathcal{H}$ domains of Carnot groups, in: Two Nonlinear Days in Urbino 2017, Special Volume Dedicated to the Memory of Anna Aloe, Electron. J. Differ. Equ. Conf. 25 (2018) 179–196.
[37] G. Molica Bisci, P. Pucci, Nonlinear Problems with Lack of Compactness, De Gruyter Series in Nonlinear Analysis and Applications, 2020, in press.
[38] G. Molica Bisci, V. Rădulescu, On the nonlinear Schrödinger equation on the Poincaré ball model, Nonlinear Anal. (2020), https://doi.org/10.1016/j.na.2020.111812 (publ. online).
[39] G. Molica Bisci, S. Secchi, Elliptic problems on complete non-compact Riemannian manifolds with asymptotically non-negative Ricci curvature, Nonlinear Anal. 177 (2018) 637–672, part B.
[40] G. Molica Bisci, R. Servadei, A bifurcation result for non-local fractional equations, Anal. Appl. 13 (4) (2015) 371–394.
[41] R.S. Palais, The principle of symmetric criticality, Commun. Math. Phys. 69 (1979) 19–30.
[42] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear Analysis - Theory and Methods, Springer Monographs in Mathematics, Springer, Cham, 2019.
[43] P. Pucci, V. Rădulescu, The impact of the mountain pass theory in nonlinear analysis: a mathematical survey, Boll. Unione Mat. Ital. (9) (3) (2010) 543–584.
[44] P. Pucci, S. Saldi, Asymptotic stability for nonlinear damped Kirchhoff systems involving the fractional p-Laplacian operator, J. Differ. Equ. 263 (2017) 2375–2418.
[45] F. Punzo, On well-posedness of semilinear parabolic and elliptic problems in the hyperbolic space, J. Differ. Equ. 251 (7) (2011) 1972–1989.
[46] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410.
[47] L. Skrzypczak, C. Tintarev, A geometric criterion for compactness of invariant subspaces, Arch. Math. 101 (2013) 259–268.
[48] S. Stapelkamp, The Brézis-Nirenberg problem on $\mathbb{H}^N$. Existence and uniqueness of solutions, in: Elliptic and Parabolic Problems, Rolduc/Gaeta, 2001, World Sci. Publ., River Edge, NJ, 2002, pp. 283–290.
[49] S. Stapelkamp, Das Brezis-Nirenberg Problem im $\mathbb{H}^N$, PhD thesis, University of Basel, 2003, available at: http://edoc.unibas.ch/67.