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On a class of Kirchhoff problems via local mountain pass

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Abstract. In the present work we study the multiplicity and concentration of positive solutions for the following class of Kirchhoff problems:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx)\Delta u + V(x)u = f(u) + \gamma u^5 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \qquad u > 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants, $\gamma \in \{0, 1\}$, V is a continuous positive potential with a local minimum, and f is a superlinear continuous function with subcritical growth. The main results are obtained through suitable variational and topological arguments. We also provide a multiplicity result for a supercritical version of the above problem by combining a truncation argument with a Moser-type iteration. Our theorems extend and improve in several directions the studies made in (*Adv. Nonlinear Stud.* **14** (2014), 483–510; *J. Differ. Equ.* **252** (2012), 1813–1834; *J. Differ. Equ.* **253** (2012), 2314–2351).

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1. Introduction

In this paper we focus our attention on the multiplicity and concentration of positive solutions for the following class of Kirchhoff problems:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx)\Delta u + V(x)u = f(u) + \gamma u^5 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \qquad u > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$ are constants, and $\gamma \in \{0, 1\}$. Throughout the paper we will assume that the potential $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying the following hypotheses introduced by del Pino and Felmer [14]:

(V_1) there exists $V_0 > 0$ such that $V_0 := \inf_{x \in \mathbb{R}^3} V(x)$,

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(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

We suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t) = 0$ for $t \leqslant 0$ and fulfills the following conditions:

(f_1) $f(t) = o(t^3)$ as $t \to 0$,
(f_2) if $\gamma = 0$ then there exists $q \in (4, 6)$ such that $f(t) = o(t^{q-1})$ as $t \to \infty$, whereas if $\gamma = 1$ then we suppose that there exist $q, \sigma \in (4, 6)$, $C_0 > 0$ such that

$$f(t) \geqslant C_0 t^{q-1} \quad \text{for all } t > 0, \ \lim_{t\to\infty} \frac{f(t)}{t^{\sigma-1}} = 0,$$

(f_3) there exists $\vartheta \in (4, 6)$ such that

$$0 < \vartheta F(t) \leqslant t f(t) \quad \text{for all } t > 0, \text{ where } F(t) := \int_0^t f(\tau)\, d\tau,$$

(f_4) the map $t \mapsto \frac{f(t)}{t^3}$ is increasing on $(0, \infty)$.

When $b = 0$ and $\mathbb{R}^3$ is replaced by the more general space $\mathbb{R}^N$, equation (1.1) reduces to a nonlinear Schrödinger equation of the type

$$-\varepsilon^2 \Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which has been widely investigated in the last thirty years. The main motivation for studying (1.2) arises from seeking standing wave solutions, namely functions of the form $\psi(x, t) = u(x)e^{-\frac{\imath E t}{\varepsilon}}$, with $E \in \mathbb{R}$ constant, for the time-dependent Schrödinger equation

$$\imath\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + \big(V(x) + E\big)\psi - g(\psi) \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

An interesting class of solutions of (1.2), sometimes called semi-classical states, are families of solutions $u_\varepsilon(x)$ which concentrate and develop a spike shape around one (or more) special points in $\mathbb{R}^N$, while vanishing elsewhere as $\varepsilon \to 0$. We refer the interested reader to [3,13,14,17,19,31] and their references for several existence and multiplicity results obtained by applying different variational and topological methods.

On the other hand, problem (1.1) is related to the stationary analogue of the Kirchhoff equation

$$\rho u_{tt} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2\, dx\right) u_{xx} = 0, \tag{1.3}$$

which was proposed in 1883 by Kirchhoff [24] as an extension of the classical D'Alembert wave equations for free vibration of elastic strings. The Kirchhoff model takes into account changes in the length of the string produced by transverse vibrations. In (1.3), $u = u(x, t)$ denotes the transverse string displacement at the spatial coordinate x and time t, L is the length of the string, h is the area of the cross section, E is Young's modulus of the material, ρ is the mass density, and p_0 is the initial tension. We refer to [10,30] for the early classical studies dedicated to (1.3). We also note that nonlocal boundary value problems like (1.3) model several physical and biological systems where u describes a process which depends on the average of itself, as for example, the population density (see [2,12]). However, only after the pioneering work of Lions [25], where a functional analysis approach was proposed to attack (1.3), problem (1.1) began to attract the attention of several mathematicians (see [2,4,16,18,21–23,34], and also [6–8,36–38] for some interesting results for fractional Kirchhoff problems). In particular, He and Zou [22] obtained the existence and multiplicity of concentrating solutions for small $\varepsilon > 0$ of the following perturbed Kirchhoff equation

$$-\left(a\varepsilon^2 + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

assuming that $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous potential satisfying the assumption introduced by Rabinowitz [31]:

$$V_\infty := \liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) = V_0, \quad \text{where } V_\infty \leqslant \infty, \tag{V}$$

and g is a C^1 subcritical nonlinearity. Subsequently, Wang et al. [34] investigated the multiplicity and concentration phenomenon for (1.4) when $g(u) = \lambda f(u) + u^5$, f is a continuous subcritical nonlinearity and λ is sufficiently large. Figueiredo and Santos Júnior [18] proved a multiplicity result for a subcritical Schrödinger–Kirchhoff equation via the generalized Nehari manifold method, when the potential V has a local minimum. He et al. [23] considered the existence and multiplicity of solutions for (1.4) when $g(u) = f(u) + u^5$, $f \in C^1$ is a subcritical nonlinearity which does not satisfies the Ambrosetti–Rabinowitz condition [5].

Motivated by the above works, in this paper we study the multiplicity and concentration of solutions for (1.1) under conditions (V_1)–(V_2) on the potential V, and assuming (f_1)–(f_4) for the continuous nonlinearity f. In order to state our main result more precisely, we recall that if Y is a given closed set of a topological space X, we denote by $cat_X(Y)$ the Ljusternik–Schnirelmann category of Y in X, this is the smallest number of closed contractible sets in X which cover Y (see [28,35] for more details). We are able to prove the following main result:

Theorem 1.1. *Assume that conditions (V_1)–(V_2) and (f_1)–(f_4) hold. Then for any $\delta > 0$ such that*

$$M_\delta := \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, M) \leqslant \delta\right\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1.1) admits at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^3$ is a global maximum point of u_ε, then

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0,$$

and there exist $C_1, C_2 > 0$ such that

$$0 < u_\varepsilon(x) \leqslant C_1 e^{-C_2 \frac{|x-x_\varepsilon|}{\varepsilon}} \quad \textit{for all } x \in \mathbb{R}^3.$$

Our proof of Theorem 1.1 is obtained by applying appropriate variational arguments. First, motivated by [14], we overcome the lack of information about the behavior of potential V at infinity by making a suitable modification on the nonlinearity, solve the modified problem and then check that, for $\varepsilon > 0$ small enough, the solutions of the modified problem are indeed solutions of the original one. Due to the fact that f is only continuous, the Nehari manifold associated with the modified problem is not differentiable, so we cannot apply standard variational arguments for C^1-Nehari manifolds developed, for example, in [3,13,22,23]. For this reason we use certain versions of critical point theorems due to Szulkin and Weth [33]. We also note that the presence of the Kirchhoff term creates some difficulties in getting the compactness of the modified functional $\mathcal{J}_\varepsilon$. Indeed, it is not clear that weak limits of bounded (PS) sequences are critical points of $\mathcal{J}_\varepsilon$. Moreover, when $\gamma = 1$, problem (1.1) presents an extra difficulty due to the presence of the critical exponent, and in order to recover some compactness properties for $\mathcal{J}_\varepsilon$, we invoke the Concentration–Compactness Lemma [27]. Since we are interested in obtaining multiple critical points, we use a technique introduced by Benci and Cerami [9], which consists in making precise comparisons between the category of some sublevel sets of $\mathcal{J}_\varepsilon$ and the category of the set M. Then we apply Ljusternik–Schnirelmann theory to deduce a multiplicity result for the modified problem. Finally, we show that the solutions of the modified problem are also solutions for (1.1), when $\varepsilon > 0$ is small enough, by using the Moser iteration technique [29].

In the last part of this paper we consider a supercritical version of problem (1.1). In this case, we deal with the sum of two homogeneous nonlinearities and add a new positive parameter μ. More precisely, we consider the following problem:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx)\Delta u + V(x)u = u^{p-1} + \mu u^{r-1} & \text{in } \mathbb{R}^3, \\ u > 0 \quad \text{in } \mathbb{R}^3, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.5}$$

where $\varepsilon, \mu > 0$ and the exponents satisfy $4 < p < 6 < r$. Our multiplicity result for the supercritical case can be stated as follows.

Theorem 1.2. *Assume that conditions* (V_1)–(V_2) *hold. Then there exists* $\mu_0 > 0$ *such that for any* $\delta > 0$ *satisfying*

$$M_\delta = \left\{ x \in \mathbb{R}^3 : \operatorname{dist}(x, M) \leqslant \delta \right\} \subset \Lambda,$$

and for any $\mu \in (0, \mu_0)$*, there exists* $\varepsilon_{\delta,\mu} > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_{\delta,\mu})$*, problem (1.5) admits at least* $\operatorname{cat}_{M_\delta}(M)$ *positive solutions. Moreover, if* u_ε *denotes one of these solutions and* $x_\varepsilon \in \mathbb{R}^3$ *is a global maximum point of* u_ε*, then*

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0.$$

The main difficulty in the study of (1.5) is due to the fact that $r > 6$ is supercritical, and we cannot directly use variational techniques because the corresponding functional is not well-defined on the Sobolev space $H^1(\mathbb{R}^3)$. In order to overcome this obstacle, we use some arguments inspired by [11,17,32] which can be summarized as follows. We first truncate in a suitable way the nonlinearity on the right hand side of (1.5), so we deal with a new problem but with subcritical growth. In the light of Theorem 1.1, we know that a multiplicity result for this truncated problem is available. Then we deduce a priori bound (independent of μ) for these solutions and by using an appropriate Moser iteration technique [29], we show that, for $\mu > 0$ sufficiently small, the solutions of the truncated problem also solve the original one.

We stress that our theorems complement and improve the main results in [22,23,34], in the sense that we are considering multiplicity results for subcritical, critical and supercritical Kirchhoff problems involving continuous nonlinearities and imposing local conditions on the potential V.

The paper is organized as follows. In Section 2 we collect some notations and basic results. We also modify the nonlinearity and prove some useful lemmas to overcome the non differentiability of the Nehari manifold. In Section 3 we provide our first existence result. In Section 4 we deal with the autonomous problems. In Section 5 we introduce some tools which are needed to establish a multiplicity result. Section 6 is devoted to the proof of Theorem 1.1. In Section 7 we study the multiplicity of positive solutions for the supercritical problem.

2. The functional setting

2.1. Notations and basic results

We start by giving some notations and collecting useful preliminary results. If $A \subset \mathbb{R}^3$ and $1 \leqslant p \leqslant \infty$, we denote by $\|u\|_{L^p(A)}$ the $L^p(A)$-norm of a function $u : \mathbb{R}^3 \to \mathbb{R}$. Let us define $D^{1,2}(\mathbb{R}^3)$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|\nabla u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$

Then we can consider the Sobolev space

$$H^1(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \|\nabla u\|^2_{L^2(\mathbb{R}^3)} < \infty\right\}$$

endowed with the norm

$$\|u\|^2_{H^1(\mathbb{R}^3)} = \|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \|u\|^2_{L^2(\mathbb{R}^3)}.$$

We have the following well-known main Sobolev embeddings.

Theorem 2.1 (see [1]). *$H^1(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for any $p \in [2, 6]$ and compactly embedded in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ for any $p \in [1, 6)$.*

We denote by S_* the best constant of the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, that is

$$S_* := \inf\left\{\frac{\|\nabla u\|^2_{L^2(\mathbb{R}^3)}}{\|u\|^2_{L^6(\mathbb{R}^3)}} : u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}\right\}.$$

We also recall the following classical lemma of Lions:

Lemma 2.1 (see [26]). *If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^3)$ and*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2\,dx=0$$

for some $R>0$, then $u_n\to 0$ in $L^p(\mathbb{R}^3)$ for all $p\in(2,6)$.

2.2. The modified problem

In order to study (1.1), we use the change of variable $x\mapsto \varepsilon x$ and we look for positive solutions to

$$\begin{cases}-(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u+V(\varepsilon x)u=f(u)+\gamma u^5 & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \quad u>0 & \text{in } \mathbb{R}^3.\end{cases} \tag{2.1}$$

In what follows, we introduce a penalized function [14] which will be useful to obtain our results. Let $K>2$ and $\alpha>0$ be such that

$$f(\alpha)+\gamma\alpha^5=\frac{V_0}{K}\alpha \tag{2.2}$$

and define

$$\tilde{f}(t):=\begin{cases}f(t)+\gamma(t^+)^5 & \text{if } t\leqslant\alpha,\\ \frac{V_0}{K}t & \text{if } t>\alpha,\end{cases}$$

and

$$g(x,t):=\chi_\Lambda(x)\big(f(t)+\gamma\big(t^+\big)^5\big)+\big(1-\chi_\Lambda(x)\big)\tilde{f}(t).$$

It is easy to check that g satisfies the following properties:

- (g_1) $\lim_{t\to 0}\frac{g(x,t)}{t^3}=0$ uniformly with respect to $x\in\mathbb{R}^3$,
- (g_2) $g(x,t)\leqslant f(t)+\gamma t^5$ for all $x\in\mathbb{R}^3$, $t>0$,
- (g_3) (i) $0<\vartheta G(x,t)\leqslant g(x,t)t$ for all $x\in\Lambda$, $t>0$,
  (ii) $0\leqslant 2G(x,t)\leqslant g(x,t)t\leqslant\frac{V_0}{K}t^2$ for all $x\in\mathbb{R}^3\setminus\Lambda$, $t>0$,
- (g_4) for each $x\in\Lambda$ the function $\frac{g(x,t)}{t^3}$ is increasing on $(0,\infty)$, and for each $x\in\mathbb{R}^3\setminus\Lambda$, the function $\frac{g(x,t)}{t^3}$ is increasing on $(0,\alpha)$.

Let us consider the following modified problem

$$\begin{cases}-(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u+V(\varepsilon x)u=g(\varepsilon x,u) & \text{in } \mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3), \quad u>0 & \text{in } \mathbb{R}^3.\end{cases} \tag{2.3}$$

It is clear that if u is a positive solution of (2.3) with $u(x)\leqslant\alpha$ for all $x\in\mathbb{R}^3\setminus\Lambda_\varepsilon$, then u is also a positive solution for (2.1), where $\Lambda_\varepsilon:=\{x\in\mathbb{R}^3:\varepsilon x\in\Lambda\}$.

The energy functional associated with (2.3) is given by

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b}{4}\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 - \int_{\mathbb{R}^3} G(\varepsilon x, u)\, dx,$$

which is well-defined on the space

$$\mathcal{H}_\varepsilon := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x) u^2\, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon^2 := a\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} V(\varepsilon x) u^2\, dx.$$

Clearly, $\mathcal{H}_\varepsilon$ is a Hilbert space with inner product

$$(u, v)_\varepsilon := \int_{\mathbb{R}^3} a\nabla u \nabla v + V(\varepsilon x) uv\, dx.$$

It is easy to check that $\mathcal{J}_\varepsilon \in C^1(\mathcal{H}_\varepsilon, \mathbb{R})$ and its differential is given by

$$\langle \mathcal{J}_\varepsilon'(u), v \rangle = (u, v)_\varepsilon + b\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} \nabla u \nabla v\, dx - \int_{\mathbb{R}^3} g(\varepsilon x, u) v\, dx$$

for any $u, v \in \mathcal{H}_\varepsilon$. Let us introduce the Nehari manifold associated with (2.3), that is,

$$\mathcal{N}_\varepsilon := \{ u \in \mathcal{H}_\varepsilon \setminus \{0\} : \langle \mathcal{J}_\varepsilon'(u), u \rangle = 0 \},$$

and we denote

$$\mathcal{H}_\varepsilon^+ := \{ u \in \mathcal{H}_\varepsilon : |\mathrm{supp}(u^+) \cap \Lambda_\varepsilon| > 0 \} \quad \text{and} \quad \mathbb{S}_\varepsilon^+ := \mathbb{S}_\varepsilon \cap \mathcal{H}_\varepsilon^+,$$

where $\mathbb{S}_\varepsilon$ is the unit sphere in $\mathcal{H}_\varepsilon$. Note that $\mathbb{S}_\varepsilon^+$ is a non-complete $C^{1,1}$-manifold of codimension one, modelled on $\mathcal{H}_\varepsilon$ and contained in the open $\mathcal{H}_\varepsilon^+$ (see [33]). Then we have that $\mathcal{H}_\varepsilon = T_u\mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$ for all $u \in \mathcal{H}_\varepsilon^+$, where $T_u\mathbb{S}_\varepsilon^+ := \{v \in \mathcal{H}_\varepsilon : (u, v)_\varepsilon = 0\}$.

Now we prove that $\mathcal{J}_\varepsilon$ possesses a mountain-pass geometry [5]:

Lemma 2.2. *The functional $\mathcal{J}_\varepsilon$ satisfies the following properties:*

(a) *there exist $\eta, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geqslant \eta$ with $\|u\|_\varepsilon = \rho$;*
(b) *there exists $e \in \mathcal{H}_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $\mathcal{J}_\varepsilon(e) < 0$.*

Proof. (a) By assumptions (g_1) and (g_2), we deduce that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$\mathcal{J}_\varepsilon(u) \geqslant \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbb{R}^3} G(\varepsilon x, u)\, dx \geqslant \frac{1}{2}\|u\|_\varepsilon^2 - \xi C\|u\|_\varepsilon^2 - C_\xi C\|u\|_\varepsilon^6.$$

Then we can find $\eta, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geqslant \eta$ with $\|u\|_\varepsilon = \rho$.

(b) Using (g_3)-(i), we deduce that for any $u \in \mathcal{H}_\varepsilon^+$ and $t > 0$

$$\begin{aligned}\mathcal{J}_\varepsilon(tu) &= \frac{t^2}{2}\|u\|_\varepsilon^2 + b\frac{t^4}{4}\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 - \int_{\Lambda_\varepsilon} G(\varepsilon x, tu)\,dx \\ &\leqslant \frac{t^2}{2}\|u\|_\varepsilon^2 + b\frac{t^4}{4}\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 - C_1 t^\vartheta \int_{\Lambda_\varepsilon} (u^+)^\vartheta\,dx + C_2\big|\mathrm{supp}(u^+) \cap \Lambda_\varepsilon\big|, \end{aligned} \tag{2.4}$$

for some constants $C_1, C_2 > 0$. Recalling that $\vartheta \in (4, 6)$, we can conclude that $\mathcal{J}_\varepsilon(tu) \to -\infty$ as $t \to +\infty$. □

Since f is only continuous, the next results will be very useful to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}_\varepsilon^+$.

Lemma 2.3. *Assume that conditions (V_1)–(V_2) and (f_1)–(f_4) hold. Then the following assertions are true.*

- (i) *For each $u \in \mathcal{H}_\varepsilon^+$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{J}_\varepsilon(tu)$. Then, there is a unique $t_u > 0$ such that*

$$h_u'(t) > 0 \quad \text{for all } t \in (0, t_u) \quad \text{and} \quad h_u'(t) < 0 \quad \text{for all } t \in (t_u, \infty).$$

- (ii) *There exists $\tau > 0$ independent of u, such that $t_u \geqslant \tau$ for any $u \in \mathbb{S}_\varepsilon^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ there is a positive constant $C_\mathbb{K}$ such that $t_u \leqslant C_\mathbb{K}$ for any $u \in \mathbb{K}$.*
- (iii) *The map $\hat{m}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\hat{m}_\varepsilon(u) = t_u u$, is continuous and $m_\varepsilon := \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ is a homeomorphism between $\mathbb{S}_\varepsilon^+$ and $\mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$.*
- (iv) *If there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ such that $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

Proof. (i) Let us observe that $h_u \in C^1(\mathbb{R}^+, \mathbb{R})$. By Lemma 2.2, we can infer that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Then there exists $t_u > 0$ such that $h_u'(t_u) = 0$ and t_u is a global maximum for h_u. Hence we can deduce that $t_u u \in \mathcal{N}_\varepsilon$. Now we can prove the uniqueness of t_u. Assume by contradiction that there are two positive numbers t_1 and t_2 such that $t_1 > t_2$ and $h_u'(t_1) = h_u'(t_2) = 0$. Hence

$$t_1\|u\|_\varepsilon^2 + b t_1^3\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 = \int_{\mathbb{R}^3} g(\varepsilon x, t_1 u)u\,dx \tag{2.5}$$

and

$$t_2\|u\|_\varepsilon^2 + b t_2^3\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 = \int_{\mathbb{R}^3} g(\varepsilon x, t_2 u)u\,dx. \tag{2.6}$$

Exploiting (2.5), (2.6), $t_1 > t_2$ and (g_4), we can see that

$$
\begin{aligned}
\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)\|u\|_\varepsilon^2 &= \int_{\mathbb{R}^3}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx \\
&= \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx \\
&\quad + \int_{\Lambda_\varepsilon}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx \\
&\geqslant \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u>\alpha\}}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx, \\
I_2 &:= \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u\leqslant\alpha<t_1 u\}}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx
\end{aligned}
$$

and

$$
I_3 := \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_1 u<\alpha\}}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3}-\frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\,dx.
$$

Now we estimate each I_i, $i\in\{1,2,3\}$. Considering I_1, from the definition of g and using (g_3)-(ii), we have

$$
I_1 \geqslant \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u>\alpha\}}\left[\frac{V_0}{K}\frac{1}{(t_1 u)^2}-\frac{V_0}{K}\frac{1}{(t_2 u)^2}\right]u^4\,dx = \frac{1}{K}\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u>\alpha\}} V_0 u^2\,dx.
$$

From the definition of g and using (g_2), we can infer

$$
I_2 \geqslant \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u\leqslant\alpha<t_1 u\}}\left[\frac{V_0}{K}\frac{1}{(t_1 u)^2}-\frac{f(t_2 u)+\gamma(t_2 u^+)^5}{(t_2 u)^3}\right]u^4\,dx.
$$

Finally, let us observe that by (g_4) and from $t_1 > t_2$, it follow that $I_3 \geqslant 0$. Thus we have

$$
\begin{aligned}
\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)\|u\|_\varepsilon^2 &\geqslant \frac{1}{K}\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u>\alpha\}} V_0 u^2\,dx \\
&\quad + \int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2 u\leqslant\alpha<t_1 u\}}\left[\frac{V_0}{K}\frac{1}{(t_1 u)^2}-\frac{f(t_2 u)+\gamma(t_2 u^+)^5}{(t_2 u)^3}\right]u^4\,dx,
\end{aligned}
$$

from which, multiplying both sides by $\frac{t_1^2 t_2^2}{t_2^2-t_1^2} < 0$ and using assumption (f_4) and (2.2), we obtain

$$\begin{aligned}
\|u\|_\varepsilon^2 &\leqslant \frac{1}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u>\alpha\}} V_0u^2\,dx \\
&\quad + \frac{t_1^2t_2^2}{t_2^2-t_1^2}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u\leqslant\alpha<t_1u\}} \left[\frac{V_0}{K}\frac{1}{(t_1u)^2} - \frac{f(t_2u)+\gamma(t_2u^+)^5}{(t_2u)^3}\right]u^4\,dx \\
&= \frac{1}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u>\alpha\}} V_0u^2\,dx \\
&\quad - \frac{t_2^2}{t_1^2-t_2^2}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u\leqslant\alpha<t_1u\}} \frac{V_0}{K}u^2\,dx \\
&\quad + \frac{t_1^2}{t_1^2-t_2^2}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{t_2u\leqslant\alpha<t_1u\}} \frac{f(t_2u)+\gamma(t_2u^+)^5}{t_2u}u^2\,dx \\
&\leqslant \frac{1}{K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V_0u^2\,dx \leqslant \frac{1}{K}\|u\|_\varepsilon^2.
\end{aligned}$$

Since $u \neq 0$ and $K > 2$, we get a contradiction.

(ii) Let $u \in \mathbb{S}_\varepsilon^+$. By (i), there exists $t_u > 0$ such that $h_u'(t_u) = 0$, that is

$$t_u + bt_u^3\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 = \int_{\mathbb{R}^3} g(\varepsilon x, t_uu)u\,dx. \tag{2.7}$$

Using assumptions (g_1) and (g_2), (2.7) and Theorem 2.1, given $\xi > 0$, there exists a positive constant C_ξ such that

$$t_u \leqslant \int_{\mathbb{R}^3} g(\varepsilon x, t_uu)t_uu\,dx \leqslant \xi t_u^3C_1 + C_\xi t_u^5C_2.$$

This implies that there exists $\tau > 0$, independent of u, such that $t_u \geqslant \tau$. Now, let $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ be a compact set. We prove that $t_u \leqslant C_{\mathbb{K}}$ for any $u \in \mathbb{K}$. Assume to the contrary, that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ such that $t_n := t_{u_n} \to \infty$. Since $\mathbb{K}$ is compact, there exists $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathcal{H}_\varepsilon$. It follows from (2.4) that $\mathcal{J}_\varepsilon(t_nu_n) \to -\infty$. Now, fix $v \in \mathcal{N}_\varepsilon$ and using $\vartheta \in (4,6)$ and (g_3), we can deduce that

$$\begin{aligned}
\mathcal{J}_\varepsilon(v) &= \mathcal{J}_\varepsilon(v) - \frac{1}{\vartheta}\langle\mathcal{J}_\varepsilon'(v), v\rangle \\
&= \left(\frac{\vartheta-2}{2\vartheta}\right)\|v\|_\varepsilon^2 + b\left(\frac{\vartheta-4}{4\vartheta}\right)\|\nabla v\|_{L^2(\mathbb{R}^3)}^4 + \frac{1}{\vartheta}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}\big[g(\varepsilon x, v)v - \vartheta G(\varepsilon x, v)\big]\,dx \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon}\big[g(\varepsilon x, v)v - \vartheta G(\varepsilon x, v)\big]\,dx \\
&\geqslant \left(\frac{\vartheta-2}{2\vartheta}\right)\|v\|_\varepsilon^2 + \frac{1}{\vartheta}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}\big[g(\varepsilon x, v)v - \vartheta G(\varepsilon x, v)\big]\,dx
\end{aligned}$$

$$\geqslant \left(\frac{\vartheta-2}{2\vartheta}\right)\|v\|_{\varepsilon}^{2}-\left(\frac{\vartheta-2}{2\vartheta}\right)\frac{1}{K}\int_{\mathbb{R}^{3}\setminus\Lambda_{\varepsilon}}V(\varepsilon x)v^{2}\,dx$$
$$\geqslant \left(\frac{\vartheta-2}{2\vartheta}\right)\left(1-\frac{1}{K}\right)\|v\|_{\varepsilon}^{2}. \tag{2.8}$$

Taking $v=t_{u_n}u_n\in\mathcal{N}_\varepsilon$ in (2.8) and using the facts $\|v_n\|_\varepsilon=t_n$ and $K>2$, we get

$$0<\left(\frac{\vartheta-2}{2\vartheta}\right)\left(1-\frac{1}{K}\right)\leqslant\frac{\mathcal{J}_\varepsilon(t_nu_n)}{t_n^2}\leqslant 0$$

for n large, and this gives a contradiction.

(iii) First, we note that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (i), for each $u\in\mathcal{H}_\varepsilon^+$ there exists a unique $m_\varepsilon(u)\in\mathcal{N}_\varepsilon$. On the other hand, if $u\in\mathcal{N}_\varepsilon$ then $u\in\mathcal{H}_\varepsilon^+$. Otherwise, if $u\notin\mathcal{H}_\varepsilon^+$, we have

$$\left|\operatorname{supp}\left(u^+\right)\cap\Lambda_\varepsilon\right|=0,$$

which together with (g_3)-(ii) implies that

$$\begin{aligned}\|u\|_\varepsilon^2+b\|\nabla u\|_{L^2(\mathbb{R}^3)}^4&=\int_{\mathbb{R}^3}g(\varepsilon x,u)u\,dx\\&=\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}g(\varepsilon x,u)u\,dx+\int_{\Lambda_\varepsilon}g(\varepsilon x,u)u\,dx\\&=\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}g\left(\varepsilon x,u^+\right)u^+\,dx\\&\leqslant\frac{1}{K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}V(\varepsilon x)u^2\,dx\leqslant\frac{1}{K}\|u\|_\varepsilon^2\end{aligned}\tag{2.9}$$

and this yields a contradiction because $u\neq 0$ and $K>2$. As a consequence, $m_\varepsilon^{-1}(u)=\frac{u}{\|u\|_\varepsilon}\in\mathbb{S}_\varepsilon^+$, m_ε^{-1} is well defined and continuous. Moreover, for all $u\in\mathbb{S}_\varepsilon^+$ we have

$$m_\varepsilon^{-1}\left(m_\varepsilon(u)\right)=m_\varepsilon^{-1}(t_uu)=\frac{t_uu}{\|t_uu\|_\varepsilon}=\frac{u}{\|u\|_\varepsilon}=u$$

from which we deduce that m_ε is a bijection. Now we prove that $\hat{m}_\varepsilon$ is a continuous function. Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{H}_\varepsilon^+$ and $u\in\mathcal{H}_\varepsilon^+$ be such that $u_n\to u$ in $\mathcal{H}_\varepsilon^+$. Since $\hat{m}_\varepsilon(tu)=\hat{m}_\varepsilon(u)$ for any $t>0$, we may assume that $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{S}_\varepsilon^+$. Then by (ii), there exists $t_0>0$ such that $t_n=t_{u_n}\to t_0$. Since $t_nu_n\in\mathcal{N}_\varepsilon$, we obtain

$$t_n^2\|u_n\|_\varepsilon^2+bt_n^4\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4=\int_{\mathbb{R}^3}g(\varepsilon x,t_nu_n)t_nu_n\,dx,$$

and passing to the limit as $n\to\infty$, we get

$$t_0^2\|u\|_\varepsilon^2+bt_0^4\|\nabla u\|_{L^2(\mathbb{R}^3)}^4=\int_{\mathbb{R}^3}g(\varepsilon x,t_0u)t_0u\,dx$$

which yields $t_0 u \in \mathcal{N}_\varepsilon$. This shows that

$$\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u) \quad \text{in } \mathcal{H}_\varepsilon.$$

Therefore, $\hat{m}_\varepsilon$ and m_ε are continuous functions.

(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be such that $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Since for each $v \in \partial\mathbb{S}_\varepsilon^+$ and $n \in \mathbb{N}$ we have

$$u_n^+ \leqslant |u_n - v| \quad \text{a.e. in } \Lambda_\varepsilon,$$

it follows that

$$\|u_n^+\|_{L^p(\Lambda_\varepsilon)} \leqslant \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_{L^p(\Lambda_\varepsilon)} \quad \text{for all } p \in [2,6], \text{ for all } n \in \mathbb{N}.$$

Hence, by (V_1), (V_2) and Theorem 2.1, there is a constant $C_p > 0$ such that

$$\|u_n^+\|_{L^p(\Lambda_\varepsilon)} \leqslant \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_{L^p(\Lambda_\varepsilon)} \leqslant C_p \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_\varepsilon \leqslant C_p \mathrm{dist}\big(u_n, \partial\mathbb{S}_\varepsilon^+\big) \quad \text{for all } n \in \mathbb{N}.$$

Using (g_1), (g_2) and (g_3)-(ii), we can infer that, for each $t > 0$

$$\begin{aligned}\int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\,dx &= \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} G(\varepsilon x, tu_n)\,dx + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)\,dx \\ &\leqslant \frac{t^2}{K}\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V(\varepsilon x)u_n^2\,dx + \int_{\Lambda_\varepsilon} F(tu_n) + \gamma t^6\big(u_n^+\big)^6\,dx \\ &\leqslant \frac{t^2}{K}\|u_n\|_\varepsilon^2 + C_1 t^4 \int_{\Lambda_\varepsilon} \big(u_n^+\big)^4\,dx + C_2 t^6 \int_{\Lambda_\varepsilon} \big(u_n^+\big)^6\,dx \\ &\leqslant \frac{t^2}{K} + C_1' t^4\,\mathrm{dist}\big(u_n, \partial\mathbb{S}_\varepsilon^+\big)^4 + C_2'\,\mathrm{dist}\big(u_n, \partial\mathbb{S}_\varepsilon^+\big)^6\end{aligned}$$

from which,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\,dx \leqslant \frac{t^2}{K} \quad \text{for all } t > 0. \tag{2.10}$$

Recalling the definition of $m_\varepsilon(u_n)$ and using (2.10) we get

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{J}_\varepsilon\big(m_\varepsilon(u_n)\big) &\geqslant \liminf_{n\to\infty} \mathcal{J}_\varepsilon(tu_n) \\ &= \liminf_{n\to\infty}\left[\frac{t^2}{2}\|u_n\|_\varepsilon^2 + b\frac{t^4}{4}\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4 - \int_{\mathbb{R}^3} G(\varepsilon x, tu_n)\,dx\right] \\ &\geqslant \left(\frac{1}{2} - \frac{1}{K}\right)t^2\end{aligned}$$

which implies that

$$\liminf_{n\to\infty} \frac{1}{2}\|m_\varepsilon(u_n)\|_\varepsilon^2 + \frac{b}{4}\|\nabla m_\varepsilon(u_n)\|_{L^2(\mathbb{R}^3)}^4 \geqslant \liminf_{n\to\infty} \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \geqslant \left(\frac{1}{2}-\frac{1}{K}\right)t^2.$$

Since $K > 2$ and $t > 0$ is arbitrary, we obtain that $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$ and $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ as $n \to \infty$. This completes the proof of the lemma. □

Now, we define the maps

$$\hat{\psi}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}_\varepsilon^+ \to \mathbb{R},$$

by $\hat{\psi}_\varepsilon(u) := \mathcal{J}_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon := \hat{\psi}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$. The next result is a direct consequence of Lemma 2.3 and Corollary 2.3 in [33].

Proposition 2.1. *Assume that conditions (V_1)–(V_2) and (f_1)–(f_4) hold. Then the following assertions are true.*

(a) $\hat{\psi}_\varepsilon \in C^1(\mathcal{H}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}_\varepsilon'(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon}\langle \mathcal{J}_\varepsilon'(\hat{m}_\varepsilon(u)), v\rangle$$

for every $u \in \mathcal{H}_\varepsilon^+$, $v \in \mathcal{H}_\varepsilon$.

(b) $\psi_\varepsilon \in C^1(\mathbb{S}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \psi_\varepsilon'(u), v\rangle = \|m_\varepsilon(u)\|_\varepsilon \langle \mathcal{J}_\varepsilon'(m_\varepsilon(u)), v\rangle,$$

for every $v \in T_u\mathbb{S}_\varepsilon^+$.

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_ε, *then* $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* $\mathcal{J}_\varepsilon$. *If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_d$ *sequence for* $\mathcal{J}_\varepsilon$, *then* $\{m_\varepsilon^{-1}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for the functional* ψ_ε.

(d) u *is a critical point of* ψ_ε *if, and only if,* $m_\varepsilon(u)$ *is a nontrivial critical point for* $\mathcal{J}_\varepsilon$. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_\varepsilon^+} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u).$$

Remark 2.1. As in [33], we have the following variational characterization of the infimum of $\mathcal{J}_\varepsilon$ over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u) = \inf_{u\in\mathcal{H}_\varepsilon^+} \max_{t>0} \mathcal{J}_\varepsilon(tu) = \inf_{u\in\mathbb{S}_\varepsilon^+} \max_{t>0} \mathcal{J}_\varepsilon(tu) > 0.$$

Remark 2.2. Let us note that if $u \in \mathcal{N}_\varepsilon$, it follows from (g_1)–(g_2) that

$$0 = \|u\|_\varepsilon^2 + b\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 - \int_{\mathbb{R}^3} g(\varepsilon x, u)u\,dx \geqslant \frac{1}{2}\|u\|_\varepsilon^2 - C\|u\|_\varepsilon^6$$

which implies that $\|u\|_\varepsilon \geqslant r > 0$ for some r independent of u.

3. An existence result for the modified problem

In this section we focus our attention on the existence of positive solutions to (2.3) for sufficiently small $\varepsilon > 0$. We begin by showing that the functional $\mathcal{J}_\varepsilon$ satisfies the Palais–Smale condition at any level $d > 0$ if $\gamma = 0$, and $d \in (0, c_*)$ for some suitable $c_* > 0$ depending on S_*, when $\gamma = 1$. This last fact is motivated by the following result:

Lemma 3.1. *Let $\gamma = 1$. Then*

$$c_\varepsilon < \frac{1}{4}abS_*^3 + \frac{1}{24}b^3S_*^6 + \frac{1}{24}\left(b^2S_*^4 + 4aS_*\right)^{\frac{3}{2}} =: c_*$$

for all $\varepsilon > 0$.

Proof. One can argue as in the proof of Lemma 2.1 in [23]. □

In view of Lemma 2.2, we can apply a version of the mountain-pass theorem without (PS) condition (see [35]) to obtain a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\varepsilon$ such that

$$\mathcal{J}_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad \mathcal{J}_\varepsilon'(u_n) \to 0. \tag{3.1}$$

We start with the following result:

Lemma 3.2. *Every sequence satisfying (3.1) is bounded.*

Proof. Arguing as in the proof of Lemma 2.3-(ii) (see formula (2.8) there), we can deduce that

$$\begin{aligned} C\left(1 + \|u_n\|_\varepsilon\right) &\geqslant \mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\left\langle \mathcal{J}_\varepsilon'(u_n), u_n\right\rangle \\ &\geqslant \left(\frac{\vartheta - 2}{2\vartheta}\right)\left(1 - \frac{1}{K}\right)\|u_n\|_\varepsilon^2. \end{aligned}$$

Since $\vartheta > 4$ and $K > 2$, we can conclude that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$. □

Lemma 3.3. *There is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that*

$$\int_{B_R(z_n)} u_n^2\, dx \geqslant \beta.$$

Proof. Assume to the contrary, that the conclusion of lemma is not true. By Lemma 2.1, we then have

$$u_n \to 0 \quad \text{in } L^r\left(\mathbb{R}^3\right) \text{ for any } r \in (2, 6),$$

so, in view of (f_1) and (f_2), we get

$$\int_{\mathbb{R}^3} F(u_n)\, dx = \int_{\mathbb{R}^3} f(u_n)u_n\, dx = o_n(1) \quad \text{as } n \to \infty. \tag{3.2}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, we may assume that $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$.

If $\gamma = 0$, then we can use $\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle = o_n(1)$ and (3.2) to deduce that $\|u_n\|_\varepsilon \to 0$, which in turn implies that $\mathcal{J}_\varepsilon(u_n) \to 0$, and this is impossible because $c_\varepsilon > 0$.

Now assume that $\gamma = 1$. Using the definition of g and (3.2), we can deduce that

$$\int_{\mathbb{R}^3} G(\varepsilon x, u_n)\,dx \leqslant \frac{1}{6}\int_{\Lambda_\varepsilon\cup\{u_n\leqslant\alpha\}} \left(u_n^+\right)^6 dx + \frac{V_0}{2K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{u_n>\alpha\}} u_n^2\,dx + o_n(1) \tag{3.3}$$

and

$$\int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\,dx = \int_{\Lambda_\varepsilon\cup\{u_n\leqslant\alpha\}} \left(u_n^+\right)^6 dx + \frac{V_0}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{u_n>\alpha\}} u_n^2\,dx + o_n(1). \tag{3.4}$$

From $\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle = o_n(1)$ we have

$$\|u_n\|_\varepsilon^2 - \frac{V_0}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{u_n>\alpha\}} u_n^2\,dx + b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4 = \int_{\Lambda_\varepsilon\cup\{u_n\leqslant\alpha\}} \left(u_n^+\right)^6 dx + o_n(1). \tag{3.5}$$

Let $\ell_1, \ell_2 \geqslant 0$ be such that

$$\|u_n\|_\varepsilon^2 - \frac{V_0}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{u_n>\alpha\}} u_n^2\,dx \to \ell_1 \tag{3.6}$$

and

$$b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4 \to \ell_2. \tag{3.7}$$

Note that $\ell_1 > 0$, otherwise (3.5) would yield $\|u_n\|_\varepsilon \to 0$ as $n \to \infty$ and then $\mathcal{J}_\varepsilon(u_n) \to 0$, which contradicts $c_\varepsilon > 0$. Hence, putting together (3.5), (3.6) and (3.7), we have

$$\int_{\Lambda_\varepsilon\cup\{u_n\leqslant\alpha\}} \left(u_n^+\right)^6 dx \to \ell_1 + \ell_2. \tag{3.8}$$

By (3.3), (3.6), (3.7), (3.8) and $\mathcal{J}_\varepsilon(u_n) = c_\varepsilon + o_n(1)$, it follows that

$$c_\varepsilon \geqslant \frac{1}{3}\ell_1 + \frac{1}{12}\ell_2. \tag{3.9}$$

On the other hand, from the definition of S_* we can see that

$$\|u_n\|_\varepsilon^2 - \frac{V_0}{K}\int_{(\mathbb{R}^3\setminus\Lambda_\varepsilon)\cap\{u_n>\alpha\}} u_n^2\,dx \geqslant aS_*\left(\int_{\Lambda_\varepsilon\cup\{u_n\leqslant\alpha\}} \left(u_n^+\right)^6 dx\right)^{\frac{1}{3}}$$

and

$$b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4 \geqslant bS_*^2\left(\int_{\Lambda_\varepsilon\cup\{u_n\leqslant\alpha\}} \left(u_n^+\right)^6 dx\right)^{\frac{2}{3}}.$$

This, together with (3.6), (3.7) and (3.8), implies that

$$\ell_1 \geqslant aS_*(\ell_1+\ell_2)^{\frac{1}{3}} \quad \text{and} \quad \ell_2 \geqslant bS_*^2(\ell_1+\ell_2)^{\frac{2}{3}}, \tag{3.10}$$

which yields

$$\ell_1+\ell_2 \geqslant aS_*(\ell_1+\ell_2)^{\frac{1}{3}} + bS_*^2(\ell_1+\ell_2)^{\frac{2}{3}}.$$

Consequently,

$$(\ell_1+\ell_2)^{\frac{1}{3}} \geqslant \frac{bS_*^2 + (b^2S_*^4+4aS_*)^{\frac{1}{2}}}{2}. \tag{3.11}$$

Combining (3.9), (3.10), (3.11), it follows that

$$\begin{aligned} c_\varepsilon &\geqslant \frac{1}{3}\ell_1 + \frac{1}{12}\ell_2 \geqslant \frac{1}{3}aS_*(\ell_1+\ell_2)^{\frac{1}{3}} + \frac{1}{12}bS_*^2(\ell_1+\ell_2)^{\frac{2}{3}} \\ &\geqslant \frac{1}{4}abS_*^3 + \frac{1}{24}b^3S_*^6 + \frac{1}{24}\big(b^2S_*^4+4aS_*\big)^{\frac{3}{2}} \end{aligned}$$

and by Lemma 3.1, this is a contradiction. □

Lemma 3.4. *The sequence $\{z_n\}_{n\in\mathbb{N}}$ given in Lemma 3.3 is bounded in $\mathbb{R}^3$.*

Proof. For any $\rho>0$, let $\psi_\rho \in C^\infty(\mathbb{R}^3)$ be such that $\psi_\rho = 0$ in $B_\rho(0)$ and $\psi_\rho = 1$ in $\mathbb{R}^3 \setminus B_{2\rho}(0)$, with $0 \leqslant \psi_\rho \leqslant 1$ and $|\nabla\psi_\rho| \leqslant \frac{C}{\rho}$, where C is a constant independent of ρ. Since $\{\psi_\rho u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, it follows that $\langle \mathcal{J}_\varepsilon'(u_n), \psi_\rho u_n\rangle = o_n(1)$, namely

$$\begin{aligned} &a\int_{\mathbb{R}^3} |\nabla u_n|^2\psi_\rho\,dx + a\int_{\mathbb{R}^3} \nabla u_n \nabla\psi_\rho u_n\,dx \\ &\quad + b\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\psi_\rho\,dx + \int_{\mathbb{R}^3} \nabla u_n\nabla\psi_\rho u_n\,dx\right) \\ &\quad + \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2\psi_\rho\,dx = o_n(1) + \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\psi_\rho\,dx. \end{aligned}$$

Take $\rho>0$ such that $\Lambda_\varepsilon \subset B_\rho(0)$. Then, using (g_3)-(ii) and Lemma 3.2, we get

$$\begin{aligned} &\left(1-\frac{1}{K}\right)V_0\int_{\{|x|\geqslant 2\rho\}} u_n^2\,dx \\ &\quad \leqslant -a\int_{\mathbb{R}^3} \nabla u_n\nabla\psi_\rho u_n\,dx - b\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3} \nabla u_n\nabla\psi_\rho u_n\,dx\right) + o_n(1) \end{aligned}$$

$$\leqslant \frac{C}{\rho}\int_{\mathbb{R}^3}|\nabla u_n||u_n|\,dx + \frac{C}{\rho}\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3}|\nabla u_n||u_n|\,dx\right) + o_n(1)$$
$$\leqslant \frac{C}{\rho} + o_n(1),$$

which implies that

$$\int_{\{|x|\geqslant 2\rho\}} u_n^2\,dx \leqslant \frac{C}{\rho} + o_n(1). \tag{3.12}$$

Now, if $\{z_n\}_{n\in\mathbb{N}}$ is unbounded, it follows by Lemma 3.3 and (3.12), that $0 < \beta \leqslant \frac{C}{\rho} \to 0$ as $\rho \to \infty$, which gives a contradiction. □

The next results will be essential for obtaining the compactness of bounded Palais–Smale sequences.

Lemma 3.5. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(PS)_{c_\varepsilon}$ sequence for $\mathcal{J}_\varepsilon$. Then for each $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that*

$$\limsup_{n\to\infty}\left[\int_{\mathbb{R}^3\setminus B_R(0)} a|\nabla u_n|^2 + V(\varepsilon x)u_n^2\,dx\right] < \zeta. \tag{3.13}$$

Proof. Let $R > 0$ be such that $\Lambda_\varepsilon \subset B_{\frac{R}{2}}(0)$, and $\eta_R \in C^\infty(\mathbb{R}^3)$ such that $\eta_R = 0$ in $B_{\frac{R}{2}}(0)$ and $\eta_R = 1$ in $\mathbb{R}^3 \setminus B_R(0)$, with $0 \leqslant \eta_R \leqslant 1$ and $|\nabla \eta_R| \leqslant \frac{C}{R}$, where C is a constant independent of R. Since $\{\eta_R u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, we have that $\langle \mathcal{J}'_\varepsilon(u_n), \eta_R u_n\rangle = o_n(1)$, and using (g_3)-(ii), we get

$$\int_{\mathbb{R}^3} a\nabla u_n \nabla(\eta_R u_n)\,dx + \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2\eta_R\,dx + b\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}\int_{\mathbb{R}^3}\nabla u_n\nabla(\eta_R u_n)\,dx$$
$$= o_n(1) + \int_{\mathbb{R}^3} g(\varepsilon x, u_n)\eta_R u_n\,dx$$
$$\leqslant o_n(1) + \frac{1}{K}\int_{\mathbb{R}^3} V(\varepsilon x)u_n^2\eta_R\,dx.$$

Accordingly,

$$a\int_{\mathbb{R}^3\setminus B_R}|\nabla u_n|^2\,dx + \left(1 - \frac{1}{K}\right)\int_{\mathbb{R}^3\setminus B_R} V(\varepsilon x)u_n^2\,dx$$
$$\leqslant \frac{C}{R}\|\nabla u_n\|_{L^2(\mathbb{R}^3)}\|u_n\|_{L^2(\mathbb{R}^3)} + \frac{C}{R}\|\nabla u_n\|^3_{L^2(\mathbb{R}^3)}\|u_n\|_{L^2(\mathbb{R}^3)} + o_n(1) \tag{3.14}$$

from which the assertion follows. □

Lemma 3.6. *The functional $\mathcal{J}_\varepsilon$ satisfies the $(PS)_d$ condition at any level $d > 0$ if $\gamma = 0$, and $d \in (0, c_*)$ if $\gamma = 1$.*

Proof. By Lemma 3.2, we know that any $(PS)_d$ sequence is bounded, so we may assume that $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$ and $u_n \to u$ in $L^q_{\rm loc}(\mathbb{R}^3)$ for all $q \in [1, 6)$. Let us start by proving that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n \, dx = \int_{\mathbb{R}^3} g(\varepsilon x, u)u \, dx. \tag{3.15}$$

Using (3.13), (f_1), (f_2), (g_2) and the Sobolev embedding, we have, for n large,

$$\int_{\mathbb{R}^3\setminus B_R(0)} g(\varepsilon x, u_n)u_n \, dx \leqslant C \int_{\mathbb{R}^3\setminus B_R(0)} u_n^4 + |u_n|^q + u_n^6 \, dx \leqslant C\big(\delta^2 + \delta^{\frac{q}{2}} + \delta^3\big). \tag{3.16}$$

On the other hand, choosing R sufficiently large, we may assume that

$$\int_{\mathbb{R}^3\setminus B_R(0)} g(\varepsilon x, u)u \, dx \leqslant \frac{\delta}{2}.$$

From this and (3.16), we get for n large enough,

$$\left|\int_{\mathbb{R}^3\setminus B_R(0)} g(\varepsilon x, u_n)u_n \, dx - \int_{\mathbb{R}^3\setminus B_R(0)} g(\varepsilon x, u)u \, dx\right| < C\delta. \tag{3.17}$$

Taking into account the definition of g, we know that

$$g(\varepsilon x, u_n)u_n \leqslant f(u_n)u_n + \alpha^6 + \frac{V_0}{K}u_n^2 \quad \text{for any } x \in \mathbb{R}^3 \setminus \Lambda_\varepsilon.$$

Since $B_R(0) \cap (\mathbb{R}^3 \setminus \Lambda_\varepsilon)$ is bounded, from (f_1), (f_2), (g_2), the strong convergence in $L^r_{\rm loc}(\mathbb{R}^3)$ for $r \in [1, 6)$, and by the dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \int_{B_R(0)\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)} g(\varepsilon x, u_n)u_n \, dx = \int_{B_R(0)\cap(\mathbb{R}^3\setminus\Lambda_\varepsilon)} g(\varepsilon x, u)u \, dx. \tag{3.18}$$

Next, we aim to prove that

$$\int_{\Lambda_\varepsilon} \big(u_n^+\big)^6 \, dx \to \int_{\Lambda_\varepsilon} \big(u^+\big)^6 \, dx. \tag{3.19}$$

Indeed, if (3.19) holds, we can infer from (g_2), (f_1), (f_2), the strong convergence in $L^r_{\rm loc}(\mathbb{R}^3)$ for $r \in [1, 6)$ and the dominated convergence theorem that

$$\lim_{n\to\infty} \int_{B_R(0)\cap\Lambda_\varepsilon} g(\varepsilon x, u_n)u_n \, dx = \int_{B_R(0)\cap\Lambda_\varepsilon} g(\varepsilon x, u)u \, dx.$$

Therefore (3.15) follows by the above limit, (3.17) and (3.18).

At this point, we show the validity of (3.19). Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, we may assume that

$$\big|\nabla u_n^+\big|^2 \rightharpoonup \mu \quad \text{and} \quad \big|u_n^+\big|^6 \rightharpoonup \nu,$$

where μ and ν are bounded nonnegative measures in $\mathbb{R}^3$. By the Concentration Compactness Principle [27], we obtain an at most countable index set I, sequence $\{x_i\}_{i\in I} \subset \mathbb{R}^3$ and $\{\mu_i\}_{i\in I}, \{\nu_i\}_{i\in I} \subset (0,\infty)$ such that

$$\mu \geqslant |\nabla u^+|^2 + \sum_{i\in I} \mu_i \delta_{x_i}, \qquad \nu = (u^+)^6 + \sum_{i\in I} \nu_i \delta_{x_i} \quad \text{and} \quad S_* \nu_i^{\frac{1}{3}} \leqslant \mu_i \quad \text{for all } i \in I. \tag{3.20}$$

Now we prove that $\{x_i\}_{i\in I} \cap \Lambda_\varepsilon = \emptyset$. Assume to the contrary, that $x_i \in \Lambda_\varepsilon$ for some $i \in I$. For any $\rho > 0$, define the function $\psi_\rho(x) := \psi(\frac{x-x_i}{\rho})$, where $\psi \in C_0^\infty(\mathbb{R}^3)$ is such that $\psi = 1$ in $B_1(0)$, $\psi = 0$ in $\mathbb{R}^3 \setminus B_2(0)$, $0 \leqslant \psi \leqslant 1$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^3)} \leqslant C$. Assume that ρ is chosen in a such way that $\operatorname{supp}\psi_\rho \subset \Lambda_\varepsilon$. Then $\langle \mathcal{J}_\varepsilon'(u_n), \psi_\rho u_n^+\rangle \to 0$ as $n \to \infty$, that is

$$\begin{aligned} &a\int_{\mathbb{R}^3} |\nabla u_n^+|^2 \psi_\rho\,dx + a\int_{\mathbb{R}^3} \nabla u_n^+ \nabla\psi_\rho u_n^+\,dx + \int_{\mathbb{R}^3} V(\varepsilon x)(u_n^+)^2\psi_\rho\,dx \\ &\quad + b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^2 \left(\int_{\mathbb{R}^3} |\nabla u_n^+|^2\psi_\rho\,dx\right) + b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^2 \left(\int_{\mathbb{R}^3} \nabla u_n^+\nabla\psi_\rho u_n^+\,dx\right) \\ &\quad - \int_{\mathbb{R}^3} g(\varepsilon x, u_n) u_n^+ \psi_\rho\,dx = o_n(1). \end{aligned} \tag{3.21}$$

Note that by the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, the Hölder inequality, and since $H^1(\mathbb{R}^3)$ is compactly contained in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \left| \int_{\mathbb{R}^3} \nabla u_n^+ \nabla\psi_\rho u_n^+\,dx \right| &\leqslant \limsup_{n\to\infty} \|\nabla u_n\|_{L^2(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} u_n^2 |\nabla\psi_\rho|^2\,dx\right)^{\frac{1}{2}} \\ &\leqslant C\left(\int_{\mathbb{R}^3} u^2|\nabla\psi_\rho|^2\,dx\right)^{\frac{1}{2}} \\ &\leqslant C\|u\|_{L^6(B_{2\rho}(x_i))} \|\nabla\psi_\rho\|_{L^3(B_{2\rho}(x_i))} \\ &\leqslant C\|u\|_{L^6(B_{2\rho}(x_i))} \to 0 \quad \text{as } \rho \to 0, \end{aligned}$$

and we have the following relations of limits

$$\begin{aligned} &\limsup_{n\to\infty} a\int_{\mathbb{R}^3} |\nabla u_n^+|^2\psi_\rho\,dx \geqslant a\int_{\mathbb{R}^3} |\nabla u^+|^2 \psi_\rho\,dx + a\mu_i \to a\mu_i \quad \text{as } \rho \to 0, \\ &\limsup_{n\to\infty} \int_{\mathbb{R}^3} V(\varepsilon x)(u_n^+)^2\psi_\rho\,dx = \int_{\mathbb{R}^3} V(\varepsilon x)(u^+)^2\psi_\rho\,dx \to 0 \quad \text{as } \rho \to 0, \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^2 \left(\int_{\mathbb{R}^3} |\nabla u_n^+|^2\psi_\rho\,dx\right) &\geqslant \limsup_{n\to\infty} b\left(\int_{\mathbb{R}^3} |\nabla u_n^+|^2\psi_\rho\,dx\right)^2 \\ &\geqslant b\left(\int_{\mathbb{R}^3} |\nabla u^+|^2\psi_\rho\,dx + \mu_i\right)^2 \to b\mu_i^2 \quad \text{as } \rho \to 0. \end{aligned}$$

On the other hand, since ψ_ρ has compact support and f has subcritical growth, we obtain

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)u_n^+\psi_\rho\,dx = \lim_{\rho\to 0}\int_{B_\rho(x_i)} f(u)u^+\psi_\rho\,dx = 0,$$

which gives

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n^+\psi_\rho\,dx = \limsup_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)u_n^+\psi_\rho\,dx + \int_{\mathbb{R}^3}\big(u_n^+\big)^6\psi_\rho\,dx \to \nu_i \quad \text{as } \rho\to 0.$$

Therefore

$$a\mu_i + b\mu_i^2 \leqslant \nu_i,$$

which together with (3.20), gives

$$\nu_i^{\frac{1}{3}} \geqslant \frac{1}{2}\Big(bS_*^2 + \sqrt{b^2S_*^4 + 4aS_*}\Big). \tag{3.22}$$

Now, using (g_3)-(ii), we obtain

$$\begin{aligned}
d &= \mathcal{J}_\varepsilon(u_n) - \frac{1}{4}\big\langle \mathcal{J}_\varepsilon'(u_n), u_n\big\rangle + o_n(1)\\
&\geqslant \frac{1}{4}\int_{\mathbb{R}^3} a\big|\nabla u_n^+\big|^2 + V(\varepsilon x)\big(u_n^+\big)^2\,dx + \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon}\frac{1}{4}g(\varepsilon x, u_n)u_n - G(\varepsilon x, u_n)\,dx\\
&\quad + \int_{\Lambda_\varepsilon}\frac{1}{4}f(u_n)u_n - F(u_n)\,dx + \frac{1}{12}\int_{\Lambda_\varepsilon}\big(u_n^+\big)^6\,dx + o_n(1)\\
&\geqslant \frac{a}{4}\int_{\Lambda_\varepsilon}\big|\nabla u_n^+\big|^2\,dx + \left(\frac{1}{4} - \frac{1}{4K}\right)\int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} V(\varepsilon x)u_n^2\,dx + \frac{1}{12}\int_{\Lambda_\varepsilon}\big(u_n^+\big)^6\,dx\\
&\geqslant \frac{a}{4}\int_{\Lambda_\varepsilon}\psi_\rho\big|\nabla u_n^+\big|^2\,dx + \frac{1}{12}\int_{\Lambda_\varepsilon}\psi_\rho\big(u_n^+\big)^6\,dx + o_n(1).
\end{aligned}$$

Taking the limit and using (3.20) and (3.22), we get

$$\begin{aligned}
d &\geqslant \frac{1}{4}a\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\mu_i + \frac{1}{12}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\nu_i\\
&\geqslant \frac{1}{4}a\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)S_*\nu_i^{\frac{1}{3}} + \frac{1}{12}\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\psi_\rho(x_i)\nu_i\\
&\geqslant \frac{1}{8}aS_*\Big(bS_*^2 + \sqrt{b^2S_*^4 + 4aS_*}\Big) + \frac{1}{96}\Big(bS_*^2 + \sqrt{b^2S_*^4 + 4aS_*}\Big)^3\\
&= \frac{1}{6}abS_*^3 + \frac{1}{24}b^3S_*^6 + \frac{1}{24}\big(b^2S_*^4 + 4aS_*\big)^{\frac{3}{2}}
\end{aligned}$$

which yields a contradiction. This completes the proof of (3.15).

At this point, we know that $\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle = o_n(1)$, that is

$$\|u_n\|_\varepsilon^2 + b\|\nabla u_n\|^4_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\, dx + o_n(1). \tag{3.23}$$

On the other hand, by the weak convergence, it is easy to see that u is a weak solution to

$$-(a+bA)\Delta u + V(\varepsilon x)u = g(\varepsilon x, u) \quad \text{in } \mathbb{R}^3,$$

where $A := \lim_{n\to\infty} \|\nabla u_n\|^2_{L^2(\mathbb{R}^3)} \geqslant \|\nabla u\|^2_{L^2(\mathbb{R}^3)}$. Hence

$$\|u\|_\varepsilon^2 + bA\|\nabla u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} g(\varepsilon x, u)u\, dx. \tag{3.24}$$

Taking into account (3.15), (3.23) and (3.24), we can infer that $u_n \to u$ in $\mathcal{H}_\varepsilon$ as $n\to\infty$. □

Corollary 3.1. *The functional ψ_ε satisfies the $(PS)_d$ condition on $\mathbb{S}_\varepsilon^+$ at any level $d>0$ if $\gamma=0$, and $d\in(0,c_*)$ if $\gamma=1$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS) sequence for ψ_ε at level d. Then we have

$$\psi_\varepsilon(u_n)\to d \quad \text{and} \quad \psi'_\varepsilon(u_n)\to 0 \quad \text{in } \left(T_{u_n}\mathbb{S}_\varepsilon^+\right)'.$$

Using Proposition 2.1-(c), we can see that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{J}_\varepsilon$ in $\mathcal{H}_\varepsilon$. Then, we can deduce from Lemma 3.6, that $\mathcal{J}_\varepsilon$ fulfills the $(PS)_d$ condition in $\mathcal{H}_\varepsilon$, so there exists $u\in\mathbb{S}_\varepsilon^+$ such that, up to a subsequence,

$$m_\varepsilon(u_n)\to m_\varepsilon(u) \quad \text{in } \mathcal{H}_\varepsilon.$$

Applying Lemma 2.3-(iii), we can infer that $u_n\to u$ in $\mathbb{S}_\varepsilon^+$. □

Now, we give the proof of the main result of this section:

Theorem 3.1. *Assume that conditions (V_1)–(V_2) and (f_1)–(f_4) hold. Then problem (2.3) admits a positive ground state for all $\varepsilon>0$.*

Proof. In view of Lemma 2.2, we can apply a version of the mountain-pass theorem without (PS) condition (see [35]) to obtain a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\mathcal{J}_\varepsilon(u_n)\to c_\varepsilon$ and $\mathcal{J}'_\varepsilon(u_n)\to 0$. By Lemma 3.2, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, so we may assume that $u_n\rightharpoonup u$ in $\mathcal{H}_\varepsilon$. Taking into account Lemma 3.3 and Lemma 3.4, we can assume that u is nontrivial. Now, we can prove that u is a critical point of $\mathcal{J}_\varepsilon$. Indeed, for all $\varphi\in\mathcal{H}_\varepsilon$ we see that

$$\int_{\mathbb{R}^3} a\nabla u\nabla\varphi + V(\varepsilon x)u\varphi\, dx + bA\left(\int_{\mathbb{R}^3}\nabla u\nabla\varphi\, dx\right) = \int_{\mathbb{R}^3} g(\varepsilon x, u)\varphi\, dx$$

where $A := \lim_{n\to\infty}\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}$. Taking $\varphi = u$ in the above identity and noting that $A\geqslant \|\nabla u\|^2_{L^2(\mathbb{R}^3)}$ (by Fatou's Lemma), we obtain that $\langle \mathcal{J}'_\varepsilon(u), u\rangle \leqslant 0$. Let us prove that $\langle \mathcal{J}'_\varepsilon(u), u\rangle = 0$. Suppose to the contrary, that $\langle \mathcal{J}_\varepsilon'(u), u\rangle < 0$. Then there exists a unique $0 < t < 1$ such that $\langle \mathcal{J}_\varepsilon'(tu), tu\rangle = 0$. Therefore, by (g_3), (g_4) and $0 < t < 1$, it follows that

$$\begin{aligned}
c_\varepsilon &\leqslant \mathcal{J}_\varepsilon(tu) - \frac{1}{\vartheta}\langle \mathcal{J}_\varepsilon'(tu), tu\rangle \\
&= t^2\left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u\|_\varepsilon^2 + t^4\left(\frac{1}{4} - \frac{1}{\vartheta}\right)\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 + \int_{\mathbb{R}^3} \frac{1}{\vartheta} g(\varepsilon x, tu)tu - G(\varepsilon x, tu)\, dx \\
&< \left(\frac{1}{2} - \frac{1}{\vartheta}\right)\|u\|_\varepsilon^2 + \left(\frac{1}{4} - \frac{1}{\vartheta}\right)\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 + \int_{\mathbb{R}^3} \frac{1}{\vartheta} g(\varepsilon x, u)u - G(\varepsilon x, u)\, dx \\
&\leqslant \liminf_{n\to\infty}\left[\mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle\right] = c_\varepsilon
\end{aligned}$$

which gives a contradiction. Hence, $\langle \mathcal{J}_\varepsilon'(u), u\rangle = 0$ and $A = \|\nabla u\|_{L^2(\mathbb{R}^3)}^2$. From the above argument we can also deduce that $t = 1$ so that $\mathcal{J}_\varepsilon(u) = c_\varepsilon$. Since $\langle \mathcal{J}_\varepsilon'(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$, and $g(x, t) = 0$ for $t \leqslant 0$, it is easy to check that $u \geqslant 0$ in $\mathbb{R}^3$. Standard arguments (see [22,23,34]) show that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$, and using the Harnack inequality [20] we deduce that $u > 0$ in $\mathbb{R}^3$. □

4. The autonomous problem

In this section we consider the limit problem associated with (2.3). More precisely, we deal with the following autonomous Kirchhoff problem:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V_0 u = f(u) + \gamma u^5 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.1}$$

The Euler–Lagrange functional associated with (4.1) is given by

$$\mathcal{J}_0(u) = \frac{1}{2}\left(a\int_{\mathbb{R}^3} |\nabla u|^2 + V_0 u^2\, dx\right) + \frac{b}{4}\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 - \int_{\mathbb{R}^3} F(u) + \frac{\gamma}{6}\left(u^+\right)^6 dx$$

which is well defined on the Hilbert space $\mathcal{H}_0 := H^1(\mathbb{R}^3)$ endowed with the inner product

$$(u, \varphi)_0 := \int_{\mathbb{R}^3} a\nabla u\nabla\varphi + V_0 u\varphi\, dx.$$

The norm induced by the inner product is

$$\|u\|_0^2 := \int_{\mathbb{R}^3} a|\nabla u|^2 + V_0 u^2\, dx.$$

The Nehari manifold associated with $\mathcal{J}_0$ is given by

$$\mathcal{N}_0 := \left\{u \in \mathcal{H}_0 \setminus \{0\} : \langle \mathcal{J}_0'(u), u\rangle = 0\right\}.$$

We denote by $\mathcal{H}_0^+$ the open subset of $\mathcal{H}_0$ defined as

$$\mathcal{H}_0^+ := \{u \in \mathcal{H}_0 : |\mathrm{supp}(u^+)| > 0\},$$

and $\mathbb{S}_0^+ := \mathbb{S}_0 \cap \mathcal{H}_0^+$, where $\mathbb{S}_0$ is the unit sphere of $\mathcal{H}_0$. We note that $\mathbb{S}_0^+$ is a incomplete $C^{1,1}$-manifold of codimension 1 modelled on $\mathcal{H}_0$ and contained in $\mathcal{H}_0^+$. Thus $\mathcal{H}_0 = T_u\mathbb{S}_0^+ \oplus \mathbb{R}u$ for each $u \in \mathbb{S}_0^+$, where $T_u\mathbb{S}_0^+ := \{u \in \mathcal{H}_0 : (u, v)_0 = 0\}$. As in Section 2, we can see that the following results hold.

Lemma 4.1. *Assume that conditions (f_1)–(f_4) hold. Then the following assertions are true.*

(i) *For each $u \in \mathcal{H}_0^+$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{J}_0(tu)$. Then there is a unique $t_u > 0$ such that*

$$h_u'(t) > 0 \quad \text{for } t \in (0, t_u) \quad \text{and} \quad h_u'(t) < 0 \quad \text{for } t \in (t_u, \infty).$$

(ii) *There exists $\tau > 0$ independent of u such that $t_u \geqslant \tau$ for any $u \in \mathbb{S}_0^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_0^+$ there is a positive constant $C_{\mathbb{K}}$ such that $t_u \leqslant C_{\mathbb{K}}$ for any $u \in \mathbb{K}$.*

(iii) *The map $\hat{m}_0 : \mathcal{H}_0^+ \to \mathcal{N}_0$ given by $\hat{m}_0(u) = t_u u$, is continuous and $m_0 := \hat{m}_0|_{\mathbb{S}_0^+}$ is a homeomorphism between $\mathbb{S}_0^+$ and $\mathcal{N}_0$. Moreover, $m_0^{-1}(u) = \frac{u}{\|u\|_0}$.*

(iv) *If there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_0^+$ such that $\mathrm{dist}(u_n, \partial\mathbb{S}_0^+) \to 0$, then $\|m_0(u_n)\|_0 \to \infty$ and $\mathcal{J}_0(m_0(u_n)) \to \infty$.*

Let us define the maps

$$\hat{\psi}_0 : \mathcal{H}_0^+ \to \mathbb{R} \quad \text{and} \quad \psi_0 : \mathbb{S}_0^+ \to \mathbb{R},$$

by $\hat{\psi}_0(u) := \mathcal{J}_0(\hat{m}_0(u))$ and $\psi_0 := \hat{\psi}_0|_{\mathbb{S}_0^+}$.

Proposition 4.1. *Assume that conditions (f_1)–(f_4) hold. Then the following assertions are true.*

(a) *$\hat{\psi}_0 \in C^1(\mathcal{H}_0^+, \mathbb{R})$ and*

$$\langle \hat{\psi}_0'(u), v\rangle = \frac{\|\hat{m}_0(u)\|_0}{\|u\|_0}\langle \mathcal{J}_0'(\hat{m}_0(u)), v\rangle$$

for every $u \in \mathcal{H}_0^+$ and $v \in \mathcal{H}_0$.

(b) *$\psi_0 \in C^1(\mathbb{S}_0^+, \mathbb{R})$ and*

$$\langle \psi_0'(u), v\rangle = \|m_0(u)\|_0\langle \mathcal{J}_0'(m_0(u)), v\rangle,$$

for every $v \in T_u\mathbb{S}_0^+$.

(c) *If $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for ψ_0, then $\{m_0(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{J}_0$. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ is a bounded $(PS)_d$ sequence for $\mathcal{J}_0$, then $\{m_0^{-1}(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for the functional ψ_0.*

(d) *u is a critical point of ψ_0 if, and only if, $m_0(u)$ is a nontrivial critical point for $\mathcal{J}_0$. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_0^+}\psi_0(u)=\inf_{u\in\mathcal{N}_0}\mathcal{J}_0(u).$$

Remark 4.1. As in Section 2, we have the following variational characterization of the infimum of $\mathcal{J}_0$ over $\mathcal{N}_0$:

$$c_0:=\inf_{u\in\mathcal{N}_0}\mathcal{J}_0(u)=\inf_{u\in\mathcal{H}_0^+}\max_{t>0}\mathcal{J}_0(tu)=\inf_{u\in\mathbb{S}_0^+}\max_{t>0}\mathcal{J}_0(tu)\in(0,c_*).$$

Arguing as in Lemma 3.3, we can prove that:

Lemma 4.2. *Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{H}_0$ be a $(PS)_d$ sequence for $\mathcal{J}_0$, with $d>0$ if $\gamma=0$ and $d\in(0,c_*)$ if $\gamma=1$, and $u_n\rightharpoonup 0$. Then only one of the alternatives below holds:*

(a) *$u_n\to 0$ in $\mathcal{H}_0$;*
(b) *there exist a sequence $\{y_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ and constants $R,\beta>0$ such that*

$$\liminf_{n\to\infty}\int_{B_R(y_n)}u_n^2\,dx\geqslant\beta>0.$$

Remark 4.2. Let us observe that, if $\{u_n\}_{n\in\mathbb{N}}$ is a (PS) sequence at level c_0 for the functional $\mathcal{J}_0$ such that $u_n\rightharpoonup u$, then we can assume that $u\neq 0$. Otherwise, if $u_n\rightharpoonup 0$ and, if $u_n\to 0$ does not occur, in view of Lemma 4.2 we can find $\{y_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ and $R,\beta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(y_n)}u_n^2\,dx\geqslant\beta>0.$$

Setting $v_n(x):=u_n(x+y_n)$, we can see that $\{v_n\}_{n\in\mathbb{N}}$ is a (PS) sequence for $\mathcal{J}_0$ at the level c_0, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_0$ and there exists $v\in\mathcal{H}_0$ such that $v_n\rightharpoonup v$ and $v\neq 0$.

Now, we prove the following existence result for the autonomous problem:

Theorem 4.1. *Problem (4.1) admits a positive ground state solution.*

Proof. Since $\mathcal{J}_0$ has a mountain pass geometry, we can find (see [35]) a (PS)-sequence $\{u_n\}$ for $\mathcal{J}_0$ at level c_0. It is easy to see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, so we may assume that $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^3)$. By Remark 4.2, we may suppose that u is nontrivial. Now, we prove that u is a critical point of $\mathcal{J}_0$. Indeed, for all $\varphi\in\mathcal{H}_0$ we can see that

$$\int_{\mathbb{R}^3}a\nabla u\nabla\varphi+V_0u\varphi\,dx+bA\left(\int_{\mathbb{R}^3}\nabla u\nabla\varphi\,dx\right)-\int_{\mathbb{R}^3}\left[f(u)+\gamma\left(u^+\right)^5\right]\varphi\,dx,$$

where $A=\lim_{n\to\infty}\|\nabla u_n\|^2_{L^2(\mathbb{R}^3)}\geqslant\|\nabla u\|^2_{L^2(\mathbb{R}^3)}$ (by Fatou's Lemma). Taking $\varphi=u$ we have $\langle\mathcal{J}_0'(u),u\rangle\leqslant 0$. Let us prove that $\langle\mathcal{J}_0'(u),u\rangle=0$. Suppose to the contrary that $\langle\mathcal{J}_0'(u),u\rangle<0$, then there exists a unique $0 < t < 1$ such that $\langle \mathcal{J}_0'(tu), tu\rangle = 0$. Therefore, by (f_3) and (f_4), it follows that

$$\begin{aligned} c_0 &\leqslant \mathcal{J}_0(tu) - \frac{1}{4}\langle \mathcal{J}_0'(tu), tu\rangle \\ &= \frac{t^2}{4}\|u\|_0^2 + \int_{\mathbb{R}^3} \frac{1}{4} f(tu)tu - F(tu)\,dx + \int_{\mathbb{R}^3} \frac{1}{4}\left(tu^+\right)^6 - \frac{1}{6}\left(u^+\right)^6 \\ &< \frac{1}{4}\|u\|_0^2 + \int_{\mathbb{R}^3} \frac{1}{4} f(u)u - F(u)\,dx + \int_{\mathbb{R}^3} \frac{1}{4}\left(u^+\right)^6 - \frac{1}{6}\left(u^+\right)^6 \\ &\leqslant \liminf_{n\to\infty}\left[\mathcal{J}_0(u_n) - \frac{1}{4}\langle \mathcal{J}_0'(u_n), u_n\rangle\right] = c_0 \end{aligned}$$

which gives a contradiction. Hence, $\langle \mathcal{J}_0'(u), u\rangle = 0$ and $A = \|\nabla u\|^2_{L^2(\mathbb{R}^3)}$. From the above argument we can also deduce that $t = 1$ so that $\mathcal{J}_0(u) = c_0$. Since $\langle \mathcal{J}_0'(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$, and $f(t) = 0$ for $t \leqslant 0$, it is easy to check that $u \geqslant 0$ in $\mathbb{R}^3$. Using the arguments in [22,23,34], $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$, and from Harnack inequality [20] we obtain that $u > 0$ in $\mathbb{R}^3$. Finally, we use a comparison argument to show the exponential decay of u. Since $u(x) \to 0$ as $|x| \to \infty$ and using (f_1), we can find $R > 0$ such that

$$f\big(u(x)\big) + \gamma u^5(x) \leqslant \frac{V_0}{2} \quad \text{for all } |x| \geqslant R.$$

Let $M \geqslant \|u\|_0^2$, and define $\phi(x) := Ce^{-c|x|}$ with $c^2 < \frac{V_0}{2(a+bM)}$ and $Ce^{-cR} \geqslant u(x)$ for all $|x| = R$. It is easy to check that

$$\Delta\phi \leqslant c^2\phi \quad \text{for all } x \neq 0. \tag{4.2}$$

Since $u > 0$, we have

$$\begin{aligned} -\Delta u + \frac{V_0}{2(a+bM)}u &\leqslant -\Delta u + \frac{V_0}{2(a + b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)}u \\ &= \frac{1}{(a + b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)}\left(f(u) + \gamma u^5 - \frac{V_0}{2}\right) \leqslant 0 \quad \text{for } |x| \geqslant R. \end{aligned} \tag{4.3}$$

Set $v := \phi - u$. Taking into account (4.2) and (4.3), we get

$$\begin{cases} -\Delta v + \frac{V_0}{2(a+bM)}v \geqslant 0 & \text{in } |x| \geqslant R, \\ v \geqslant 0 & \text{on } |x| = R, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

The maximum principle [20] implies that $v \geqslant 0$ in $|x| \geqslant R$ and we deduce that $u(x) \leqslant Ce^{-c|x|}$ for all $|x| \geqslant R$. This completes the proof of theorem. □

The next lemma is a compactness result for the autonomous problem which will be useful later.

Lemma 4.3. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ be a sequence such that $\mathcal{J}_0(u_n) \to c_0$. Then $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $H^1(\mathbb{R}^3)$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0$ and $\mathcal{J}_0(u_n) \to c_0$, we can apply Lemma 4.1-(iii) and Proposition 4.1-(d) and Remark 4.1 to infer that

$$v_n = m_0^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in \mathbb{S}_0^+$$

and

$$\psi_0(v_n) = \mathcal{J}_0(u_n) \to c_0 = \inf_{v\in\mathbb{S}_0^+} \psi_0(v).$$

Let us introduce the map $\mathcal{F} : \overline{\mathbb{S}}_0^+ \to \mathbb{R} \cup \{\infty\}$ defined as follows

$$\mathcal{F}(u) := \begin{cases} \psi_0(u) & \text{if } u \in \mathbb{S}_0^+, \\ \infty & \text{if } u \in \partial\mathbb{S}_0^+. \end{cases}$$

We note that

- $(\overline{\mathbb{S}}_0^+, d_0)$, where $d(u, v) = \|u - v\|_0$, is a complete metric space;
- $\mathcal{F} \in C(\overline{\mathbb{S}}_0^+, \mathbb{R} \cup \{\infty\})$, by Lemma 4.1-(iii);
- $\mathcal{F}$ is bounded from below, by Proposition 4.1-(d).

Hence, applying the Ekeland variational principle [15] to $\mathcal{F}$, we can find $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_0^+$ such that $\{\hat{v}_n\}_{n\in\mathbb{N}}$ is a $(PS)_{c_0}$ sequence for ψ_0 on $\mathbb{S}_0^+$ and $\|\hat{v}_n - v_n\|_0 = o_n(1)$. Then, using Proposition 4.1, Theorem 4.1 and arguing as in the proof of Corollary 3.1, the assertion follows. □

Finally, we prove the following useful relation between c_ε and c_0:

Lemma 4.4. $\lim_{\varepsilon\to 0} c_\varepsilon = c_0$.

Proof. Let ω be a positive ground state given by Theorem 4.1, and set $\omega_\varepsilon(x) = \psi_\varepsilon(x)\omega(x)$, where $\psi_\varepsilon(x) = \psi(\varepsilon x)$ with $\psi \in C_0^\infty(\mathbb{R}^3)$, $\psi \in [0, 1]$, $\psi = 1$ if $|x| \leqslant \frac{1}{2}$ and $\psi = 0$ if $|x| \geqslant 1$. For simplicity, we assume that $\mathrm{supp}(\psi) \subset B_1(0) \subset \Lambda$. Invoking the dominated convergence theorem, we see that

$$\omega_\varepsilon \to \omega \quad \text{in } H^1\big(\mathbb{R}^3\big) \text{ as } \varepsilon \to 0. \tag{4.4}$$

For each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\mathcal{J}_\varepsilon(t_\varepsilon\omega_\varepsilon) = \max_{t\geqslant 0} \mathcal{J}_\varepsilon(t\omega_\varepsilon). \tag{4.5}$$

Then $\frac{d}{dt}[\mathcal{J}_\varepsilon(t\omega_\varepsilon)]_{t=t_\varepsilon} = 0$ and this implies that

$$\frac{1}{t_\varepsilon^2}\int_{\mathbb{R}^3} a|\nabla\omega_\varepsilon|^2 + V(\varepsilon x)\omega_\varepsilon^2\,dx + b\bigg(\int_{\mathbb{R}^3} |\nabla\omega_\varepsilon|^2\,dx\bigg)^2 = \int_{\mathbb{R}^3} \frac{f(t_\varepsilon\omega_\varepsilon)}{(t_\varepsilon\omega_\varepsilon)^3}\omega_\varepsilon^4\,dx + t_\varepsilon^2\int_{\mathbb{R}^3} \omega_\varepsilon^6\,dx.$$

By (f_1), (f_2), (f_4) and $\omega \in \mathcal{N}_0$, it is easy to check that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. On the other hand, from the definition of c_ε and (4.5) we can see that

$$c_\varepsilon \leqslant \max_{t \geqslant 0} \mathcal{J}_\varepsilon(t\omega_\varepsilon) = \mathcal{J}_\varepsilon(t_\varepsilon \omega_\varepsilon),$$

and using the fact that $g(x,t) = f(t) + \gamma(t^+)^5$ in $\Lambda \times \mathbb{R}$ and $\operatorname{supp} \omega_\varepsilon \subset \Lambda_\varepsilon$, we have

$$\mathcal{J}_\varepsilon(t_\varepsilon \omega_\varepsilon) = \mathcal{J}_0(t_\varepsilon \omega_\varepsilon) + \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} \big(V(\varepsilon x) - V_0\big)\omega_\varepsilon^2 \, dx.$$

Taking into account that $V(\varepsilon x)$ is bounded on the support of ω_ε, $t_\varepsilon \to 1$ as $\varepsilon \to 0$ and (4.4), we can deduce that

$$\limsup_{\varepsilon \to 0} c_\varepsilon \leqslant \mathcal{J}_0(\omega) = c_0.$$

On the other hand, in view of (V_1) and (g_2), we know that $c_\varepsilon \geqslant c_0$ for all $\varepsilon > 0$, so we can conclude that $c_\varepsilon \to c_0$ as $\varepsilon \to 0$. □

5. The barycenter map and multiplicity of solutions to (1.1)

In this section, our main purpose is to apply the Ljusternik–Schnirelmann category theory to obtain a multiplicity result for problem (2.3). We begin proving the following technical results.

Lemma 5.1. *Let $\varepsilon_n \to 0^+$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n) \to c_0$. Then there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that the translated sequence*

$$\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$$

has a subsequence which converges in $H^1(\mathbb{R}^3)$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}} := \{\varepsilon_n \tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n \to y_0 \in M$.

Proof. Since $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and $\mathcal{J}_{\varepsilon_n}(u_n) \to c_0$, we can argue as in Lemma 3.2 to show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\varepsilon_n}$. Let us observe that $\|u_n\|_{\varepsilon_n} \nrightarrow 0$ since $c_0 > 0$. Therefore, proceeding as in Lemma 3.3, we can find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 \, dx \geqslant \beta.$$

Set $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$. Then $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$ and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{weakly in } H^1\big(\mathbb{R}^3\big),$$

for some $\tilde{u} \neq 0$. Let $\{t_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ be such that $\tilde{v}_n := t_n\tilde{u}_n \in \mathcal{N}_0$ (see Lemma 4.1-(i)), and set $y_n := \varepsilon_n\tilde{y}_n$. Then, using (g_2) and $u_n \in \mathcal{N}_{\varepsilon_n}$, we can see that

$$\begin{aligned}
c_0 &\leqslant \mathcal{J}_0(\tilde{v}_n) \leqslant \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla\tilde{v}_n|^2 + V(\varepsilon_n x + y_n)\tilde{v}_n^2\,dx + \frac{b}{4}\|\nabla\tilde{v}_n\|^4_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} F(\tilde{v}_n) + \frac{\gamma}{6}\big(\tilde{v}_n^+\big)^6\,dx \\
&\leqslant \frac{t_n^2}{2}\int_{\mathbb{R}^3} a|\nabla u_n|^2 + V(\varepsilon_n z)u_n^2\,dx + \frac{b}{4}t_n^4\|\nabla u_n\|^4_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} G(\varepsilon_n z, t_n u_n)\,dx \\
&= \mathcal{J}_{\varepsilon_n}(t_n u_n) \leqslant \mathcal{J}_{\varepsilon_n}(u_n) = c_0 + o_n(1), \qquad (5.1)
\end{aligned}$$

which gives

$$\mathcal{J}_0(\tilde{v}_n) \to c_0 \quad \text{and} \quad \{\tilde{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_0. \qquad (5.2)$$

In particular, (5.2) yields that $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, so we may assume that $\tilde{v}_n \rightharpoonup \tilde{v}$. Obviously, $\{t_n\}_{n\in\mathbb{N}}$ is bounded and we may assume that $t_n \to t_0 \geqslant 0$. If $t_0 = 0$, we get from the boundedness of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$, that $\|\tilde{v}_n\|_0 = t_n\|\tilde{u}_n\|_0 \to 0$, that is $\mathcal{J}_0(\tilde{v}_n) \to 0$, in contrast with the fact that $c_0 > 0$. Hence, $t_0 > 0$. By the uniqueness of the weak limit, we have that $\tilde{v} = t_0\tilde{u}$ and $\tilde{v} \neq 0$. Using Lemma 4.3, we deduce that

$$\tilde{v}_n \to \tilde{v} \quad \text{in } H^1\big(\mathbb{R}^3\big), \qquad (5.3)$$

which implies that $\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$ and

$$\mathcal{J}_0(\tilde{v}) = c_0 \quad \text{and} \quad \big\langle \mathcal{J}_0'(\tilde{v}), \tilde{v}\big\rangle = 0.$$

Now, we show that $\{y_n\}_{n\in\mathbb{N}}$ admits a subsequence, still denoted by the same, such that $y_n \to y_0 \in M$. Assume to the contrary, that $\{y_n\}_{n\in\mathbb{N}}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}_{n\in\mathbb{N}}$, such that $|y_n| \to +\infty$. Since $u_n \in \mathcal{N}_{\varepsilon_n}$, we can see that

$$\|\tilde{u}_n\|_0^2 \leqslant a\|\nabla\tilde{u}_n\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V(\varepsilon_n x + y_n)\tilde{u}_n^2\,dx + b\|\nabla\tilde{u}_n\|^4_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} g(\varepsilon_n x + y_n, \tilde{u}_n)\tilde{u}_n\,dx.$$

Take $R > 0$ such that $\Lambda \subset B_R(0)$, and assume that $|y_n| > 2R$. Then for any $x \in B_{R/\varepsilon_n}(0)$ we get $|\varepsilon_n x + y_n| \geqslant |y_n| - |\varepsilon_n x| > R$. From the definition of g we can deduce that

$$\|v_n\|_0^2 \leqslant \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(\tilde{u}_n)\tilde{u}_n\,dx + \int_{\mathbb{R}^3\setminus B_{R/\varepsilon_n}(0)} f(\tilde{u}_n)\tilde{u}_n + \gamma\big(\tilde{u}_n^+\big)^6\,dx.$$

Since $\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$, we can apply the dominated convergence theorem to get

$$\int_{\mathbb{R}^3\setminus B_{R/\varepsilon_n}(0)} f(\tilde{u}_n)\tilde{u}_n\,dx = o_n(1).$$

Hence

$$\|\tilde{u}_n\|_0^2 \leqslant \frac{1}{K}\int_{B_{R/\varepsilon_n}(0)} V_0\tilde{u}_n^2\,dx + o_n(1),$$

which yields

$$\left(1-\frac{1}{K}\right)\|\tilde{u}_n\|_0^2 \leqslant o_n(1).$$

Since $\tilde{u}_n \to \tilde{u} \neq 0$ and $K > 2$, we get a contradiction. Thus $\{y_n\}_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, we may assume that $y_n \to y_0$. If $y_0 \notin \overline{M}$, then there exists $r > 0$ such that $y_n \in B_{r/2}(y_0) \subset \mathbb{R}^3 \setminus \overline{M}$ for any n large enough. Reasoning as before, we get a contradiction. Hence $y \in \overline{M}$. Now, we show that $V(y_0) = V_0$. Assume to the contrary, that $V(y_0) > V_0$. Taking into account (5.3), Fatou's Lemma and the invariance of $\mathbb{R}^3$ by translations, we have

$$\begin{aligned} c_0 &< \liminf_{n\to\infty}\left[\frac{1}{2}\left(\int_{\mathbb{R}^3} a|\nabla\tilde{v}_n|^2 + V(\varepsilon_n z + y_n)\tilde{v}_n^2\right) + \frac{b}{4}\|\nabla\tilde{v}_n\|_{L^2(\mathbb{R}^3)}^4 - \int_{\mathbb{R}^3}\left(F(\tilde{v}_n) + \frac{\gamma}{6}\left(\tilde{v}_n^+\right)^6\right)dx\right] \\ &\leqslant \liminf_{n\to\infty}\mathcal{J}_{\varepsilon_n}(t_n u_n) \leqslant \liminf_{n\to\infty}\mathcal{J}_{\varepsilon_n}(u_n) = c_0 \end{aligned}$$

which is impossible. □

Now, we aim to relate the number of positive solutions of (2.3) to the topology of the set Λ. For this reason, we take $\delta > 0$ such that

$$M_\delta = \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, M) \leqslant \delta\right\} \subset \Lambda,$$

and consider a smooth non increasing function η defined in $[0,\infty)$ such that $\eta(t) = 1$ if $0 \leqslant t \leqslant \frac{\delta}{2}$, $\eta(t) = 0$ if $t \geqslant \delta$, $0 \leqslant \eta \leqslant 1$, and $|\eta'(t)| \leqslant c$ for some $c > 0$.

For any $y \in \Lambda$, we define

$$\Psi_{\varepsilon,y}(x) := \eta\big(|\varepsilon x - y|\big)w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

where $w \in H^1(\mathbb{R}^3)$ is a positive ground state solution to (4.1) (such a solution exists by virtue of Theorem 4.1).

Let $t_\varepsilon > 0$ be the unique number such that

$$\mathcal{J}_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}) := \max_{t\geqslant 0}\mathcal{J}_\varepsilon(t\Psi_{\varepsilon,y}).$$

Finally, we consider $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ defined by setting

$$\Phi_\varepsilon(y) := t_\varepsilon\Psi_{\varepsilon,y}.$$

Lemma 5.2. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon\big(\Phi_\varepsilon(y)\big)=c_0 \quad \textit{uniformly in } y\in M.$$

Proof. Assume to the contrary, that there exist $\delta_0>0$, $\{y_n\}_{n\in\mathbb{N}}\subset M$ and $\varepsilon_n\to 0$ such that

$$\big|\mathcal{J}_{\varepsilon_n}\big(\Phi_{\varepsilon_n}(y_n)\big)-c_0\big|\geqslant \delta_0. \tag{5.4}$$

Let us observe that using the change of variable $z=\frac{\varepsilon_n x-y_n}{\varepsilon_n}$, if $z\in B_{\frac{\delta}{\varepsilon_n}}(0)$, it follows that $\varepsilon_n z\in B_\delta(0)$ and then $\varepsilon_n z+y_n\in B_\delta(y_n)\subset M_\delta\subset\Lambda_\varepsilon$.

Then, recalling that $G(x,t)=F(t)+\frac{\gamma}{6}(t^+)^6$ for $(x,t)\in\Lambda\times\mathbb{R}$ and $\eta(t)=0$ for $t\geqslant\delta$, we have

$$\begin{aligned}\mathcal{J}_\varepsilon\big(\Phi_{\varepsilon_n}(y_n)\big)=&\frac{t_{\varepsilon_n}^2}{2}\left(\int_{\mathbb{R}^3}a\big|\nabla\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)\big|^2\,dz+\int_{\mathbb{R}^3}V(\varepsilon_n z+y_n)\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)^2\,dz\right)\\&+\frac{bt_{\varepsilon_n}^4}{4}\left(\int_{\mathbb{R}^3}\big|\nabla\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)\big|^2\,dz\right)^2-\int_{\mathbb{R}^3}F\big(t_{\varepsilon_n}\eta\big(|\varepsilon_n z|\big)w(z)\big)\,dz\\&-\frac{t_{\varepsilon_n}^6\gamma}{6}\int_{\mathbb{R}^3}\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)^6\,dz.\end{aligned} \tag{5.5}$$

Now, we verify that the sequence $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ satisfies $t_{\varepsilon_n}\to 1$ as $\varepsilon_n\to 0$. It follows from the definition of t_{ε_n} that $\langle\mathcal{J}_{\varepsilon_n}'(\Phi_{\varepsilon_n}(y_n)),\Phi_{\varepsilon_n}(y_n)\rangle=0$, namely,

$$\begin{aligned}&t_{\varepsilon_n}^2\left(\int_{\mathbb{R}^3}a\big|\nabla\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)\big|^2+V(\varepsilon_n z+y_n)\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)^2\,dz\right)\\&\quad+bt_{\varepsilon_n}^4\left(\int_{\mathbb{R}^3}\big|\nabla\big(\eta\big(|\varepsilon_n z|\big)w(z)\big)\big|^2\,dz\right)^2\\&=\int_{\mathbb{R}^3}g\big(\varepsilon_n z+y_n,t_{\varepsilon_n}\eta\big(|\varepsilon_n z|\big)w(z)\big)t_{\varepsilon_n}\eta\big(|\varepsilon_n z|\big)w(z)\,dz.\end{aligned} \tag{5.6}$$

Since $\eta=1$ in $B_{\frac{\delta}{2}}(0)\subset B_{\frac{\delta}{\varepsilon_n}}(0)$ for all n large enough, we get from (5.6)

$$\begin{aligned}&\frac{1}{t_{\varepsilon_n}^2}\int_{\mathbb{R}^3}a|\nabla\Psi_{\varepsilon_n,y_n}|^2+V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2\,dx+b\left(\int_{\mathbb{R}^3}|\nabla\Psi_{\varepsilon_n,y_n}|^2\,dz\right)^2\\&\quad=\int_{\mathbb{R}^3}\frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})+\gamma(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^5}{(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^3}\Psi_{\varepsilon_n,y_n}^4\,dx.\end{aligned}$$

By the continuity of w we can find a vector $\hat{z}\in\mathbb{R}^3$ such that

$$w(\hat{z}):=\min_{z\in\bar{B}_{\frac{\delta}{2}}(0)}w(z)>0,$$

so that, using (f_4), we can deduce that

$$\frac{1}{t_{\varepsilon_n}^2}\int_{\mathbb{R}^3} a|\nabla\Psi_{\varepsilon_n,y_n}|^2 + V(\varepsilon_n x)\Psi_{\varepsilon_n,y_n}^2\,dx + b\left(\int_{\mathbb{R}^3}|\nabla\Psi_{\varepsilon_n,y_n}|^2\,dx\right)^2$$

$$\geqslant \begin{cases} \frac{f(t_{\varepsilon_n}w(\hat{z}))}{(f(t_{\varepsilon_n}w(\hat{z}))^3}w^4(\hat{z})|B_{\frac{\delta}{2}}(0)| & \text{if } \gamma = 0,\\ t_{\varepsilon_n}^2 w^6(\hat{z})|B_{\frac{\delta}{2}}(0)| & \text{if } \gamma = 1. \end{cases} \tag{5.7}$$

Now, assume to the contrary, that $t_{\varepsilon_n} \to \infty$. Let us observe that the dominated convergence theorem yields

$$\|\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 \to \|w\|_0^2 \in (0,\infty),$$

$$\|\Psi_{\varepsilon_n,y_n}\|_{L^6(\mathbb{R}^3)} \to \|w\|_{L^6(\mathbb{R}^3)} \quad \text{and} \quad \int_{\mathbb{R}^3}\frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})}{(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^3}\Psi_{\varepsilon_n,y_n}^4\,dx \to \int_{\mathbb{R}^3}\frac{f(t_0 w)}{(t_0 w)^3}w^4\,dx. \tag{5.8}$$

Hence, by $t_{\varepsilon_n} \to \infty$, (5.7) and (5.8), we obtain a contradiction.

Therefore $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and, up to subsequence, we may assume that $t_{\varepsilon_n} \to t_0$ for some $t_0 \geqslant 0$. Let us prove that $t_0 > 0$. Suppose to the contrary, that $t_0 = 0$. Then, taking into account (5.8) and assumptions (g_1) and (g_2), we can see that (5.6) yields

$$\|t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 \to 0$$

which is impossible in view of Remark 2.2. Hence $t_0 > 0$. Thus, by passing to the limit as $n \to \infty$ in (5.6), we deduce from (5.8) that

$$\frac{1}{t_0^2}\|w\|_0^2 + b\|\nabla w\|_{L^2(\mathbb{R}^3)}^4 = \int_{\mathbb{R}^3}\frac{f(t_0 w) + \gamma(t_0 w)^5}{(t_0 w)^3}w^4\,dx.$$

Taking into account $w \in \mathcal{N}_0$ and using (f_4) we can infer that $t_0 = 1$. Then, letting $n \to \infty$ in (5.5) and using $t_{\varepsilon_n} \to 1$ and (5.8) we obtain

$$\lim_{n\to\infty}\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \mathcal{J}_0(w) = c_0,$$

which contradicts (5.4). □

At this point, we are in a position to define the barycenter map. For any $\delta > 0$, we take $\rho = \rho(\delta) > 0$ such that $M_\delta \subset B_\rho(0)$, and we consider $\Upsilon : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\Upsilon(x) := \begin{cases} x & \text{if } |x| < \rho\\ \frac{\rho x}{|x|} & \text{if } |x| \geqslant \rho. \end{cases}$$

We define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ as follows

$$\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^3}\Upsilon(\varepsilon x)u^2(x)\,dx}{\int_{\mathbb{R}^3}u^2(x)\,dx}.$$

Applying the dominated convergence theorem, it is easy to check that the function β_ε fulfills the following limit:

Lemma 5.3.

$$\lim_{\varepsilon\to 0}\beta_\varepsilon\big(\Phi_\varepsilon(y)\big)=y \quad \textit{uniformly in } y\in M.$$

Now, we introduce a subset $\widetilde{\mathcal{N}}_\varepsilon$ of $\mathcal{N}_\varepsilon$ by taking a function $h_1:\mathbb{R}^+\to\mathbb{R}^+$ such that $h_1(\varepsilon)\to 0$ as $\varepsilon\to 0$, and setting

$$\widetilde{\mathcal{N}}_\varepsilon=\big\{u\in\mathcal{N}_\varepsilon:\mathcal{J}_\varepsilon(u)\leqslant c_0+h_1(\varepsilon)\big\}.$$

It follows from Lemma 5.2 that $h_1(\varepsilon):=\sup_{y\in M}|\mathcal{J}_\varepsilon(\Phi_\varepsilon(y))-c_0|\to 0$ as $\varepsilon\to 0$. By the definition of $h_1(\varepsilon)$, for any $y\in M$ and $\varepsilon>0$, $\Phi_\varepsilon(y)\in\widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon\neq\emptyset$. Moreover, we can prove a very interesting relation between $\widetilde{\mathcal{N}}_\varepsilon$ and β_ε:

Lemma 5.4.

$$\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}}_\varepsilon}\operatorname{dist}\big(\beta_\varepsilon(u),M_\delta\big)=0.$$

Proof. Let $\varepsilon_n\to 0$ as $n\to\infty$. For any $n\in\mathbb{N}$ there exists $u_n\in\widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}}\inf_{z\in M_\delta}\big|\beta_{\varepsilon_n}(u)-z\big|=\inf_{z\in M_\delta}\big|\beta_{\varepsilon_n}(u_n)-z\big|+o_n(1).$$

Therefore, it is suffices to prove that there exists $\{y_n\}_{n\in\mathbb{N}}\subset M_\delta$ such that

$$\big|\beta_{\varepsilon_n}(u_n)-y_n\big|=o_n(1). \tag{5.9}$$

We note that $\{u_n\}_{n\in\mathbb{N}}\subset\widetilde{\mathcal{N}}_{\varepsilon_n}\subset\mathcal{N}_{\varepsilon_n}$, from which we deduce that

$$c_0\leqslant c_{\varepsilon_n}\leqslant\mathcal{J}_{\varepsilon_n}(u_n)\leqslant c_0+h_1(\varepsilon_n).$$

This yields $\mathcal{J}_{\varepsilon_n}(u_n)\to c_0$. Using Lemma 5.1, we can find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^3$ such that $y_n=\varepsilon_n\tilde{y}_n\in M_\delta$ for n sufficiently large. Set $\tilde{u}_n=u_n(\cdot+\tilde{y}_n)$ and we see that

$$\beta_{\varepsilon_n}(u_n)=y_n+\frac{\int_{\mathbb{R}^3}[\Upsilon(\varepsilon_n z+y_n)-y_n]\tilde{u}_n^2\,dz}{\int_{\mathbb{R}^3}\tilde{u}_n^2\,dz}.$$

Since $\varepsilon_n z+y_n\to y_0\in M_\delta$, we can deduce that $\beta_{\varepsilon_n}(u_n)=y_n+o_n(1)$, that is (5.9) indeed holds. □

In order to prove that (2.3) admits at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions, we recall the following useful abstract result whose proof can be found in [9,28].

Lemma 5.5. *Let I, I_1 and I_2 be closed sets with $I_1 \subset I_2$, and let $\pi : I \to I_2$ and $\psi : I_1 \to I$ be two continuous maps such that $\pi \circ \psi$ is homotopically equivalent to the embedding $j : I_1 \to I_2$. Then* $\mathrm{cat}_I(I) \geqslant \mathrm{cat}_{I_2}(I_1)$.

Since $\mathbb{S}_\varepsilon^+$ is not a complete metric space, we cannot directly apply standard Ljusternik–Schnirelmann theory [28,35]. However, in the light of results in Section 2, we can make use of some abstract category results contained in [33].

Theorem 5.1. *Assume that conditions (V_1)–(V_2) and (f_1)–(f_4) hold. Then, given $\delta > 0$, there exists $\bar{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_\delta)$, problem (2.3) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.*

Proof. For any $\varepsilon > 0$, we consider the map $\alpha_\varepsilon : M \to \mathbb{S}_\varepsilon^+$ defined by $\alpha_\varepsilon(y) := m_\varepsilon^{-1}(\Phi_\varepsilon(y))$.

Using Lemma 5.2, we see that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon\big(\alpha_\varepsilon(y)\big) = \lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon\big(\Phi_\varepsilon(y)\big) = c_0 \quad \text{uniformly in } y \in M. \tag{5.10}$$

Set

$$\widetilde{\mathcal{S}}_\varepsilon^+ := \big\{w \in \mathbb{S}_\varepsilon^+ : \psi_\varepsilon(w) \leqslant c_0 + h_1(\varepsilon)\big\},$$

where $h_1(\varepsilon) := \sup_{y\in M} |\psi_\varepsilon(\alpha_\varepsilon(y)) - c_0| \to 0$ as $\varepsilon \to 0^+$ by (5.10). Since $\psi_\varepsilon(\alpha_\varepsilon(y)) \in \widetilde{\mathcal{S}}_\varepsilon^+$, we have that $\widetilde{\mathcal{S}}_\varepsilon^+ \neq \emptyset$ for all $\varepsilon > 0$.

By Lemma 2.3-(iii), Lemma 5.2, Lemma 5.3 and Lemma 5.4, we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the following diagram

$$M \overset{\Phi_\varepsilon}{\to} \Phi_\varepsilon(M) \overset{m_\varepsilon^{-1}}{\to} \alpha_\varepsilon(M) \overset{m_\varepsilon}{\to} \Phi_\varepsilon(M) \overset{\beta_\varepsilon}{\to} M_\delta$$

is well defined for any $\varepsilon \in (0, \bar{\varepsilon})$.

Thanks to Lemma 5.3, and decreasing $\bar{\varepsilon}$ if necessary, we see that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for all $y \in M$, for some function $\theta(\varepsilon, y)$ such that $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$ and for all $\varepsilon \in (0, \bar{\varepsilon})$. Then it is easy to check that $H(t, y) := y + (1 - t)\theta(\varepsilon, y)$ with $(t, y) \in [0, 1] \times M$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $id : M \to M_\delta$. This fact together with Lemma 5.5 implies that

$$\mathrm{cat}_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M) \geqslant \mathrm{cat}_{M_\delta}(M). \tag{5.11}$$

Using Corollary 3.1, Lemma 4.4 and Theorem 27 in [33] with $c = c_\varepsilon \leqslant c_0 + h_1(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, we can deduce that ψ_ε has at least $\mathrm{cat}_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Taking into account Proposition 2.1-(d) and (5.11), we can infer that $\mathcal{J}_\varepsilon$ admits at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

6. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Here we prove that the solutions obtained in Section 5 are indeed solutions of the original problem (1.1) for $\varepsilon > 0$ small enough.

First, we use a Moser iteration argument [29] to prove the following useful L^∞-estimate for the solutions of the modified problem (2.3).

Lemma 6.1. *Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution to (2.3). Then, up to a subsequence, the translated sequence $v_n = u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^3)$, and there exists $C > 0$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^3)} \leqslant C \quad \text{for all } n \in \mathbb{N},$$

where $\{\tilde{y}_n\}_{n\in\mathbb{N}}$ is given in Lemma 5.1. Furthermore, $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly in $n \in \mathbb{N}$.

Proof. Since $\mathcal{J}_{\varepsilon_n}(u_n) \leqslant c_0 + h_1(\varepsilon_n)$ with $h_1(\varepsilon_n) \to 0$, we can argue as in the proof of (5.1) to prove that $\mathcal{J}_{\varepsilon_n}(u_n) \to c_0$. From Lemma 5.1 we can deduce that there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $v_n := u_n(\cdot + \tilde{y}_n) \to v$ in $H^1(\mathbb{R}^3)$, for some $v \in H^1(\mathbb{R}^3)$, $v \not\equiv 0$, and $y_n = \varepsilon_n \tilde{y}_n \to y_0 \in M$.

Note that v_n is a solution of the following problem

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx)\Delta v_n + V_n(x)v_n = g_n(v_n) & \text{in } \mathbb{R}^3, \\ v_n \in H^1(\mathbb{R}^3), \quad v_n > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{6.1}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $g_n(v_n) = g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n)$.

For any $R > 0$, $0 < r \leqslant \frac{R}{2}$, let $\eta \in C^\infty(\mathbb{R}^3)$ such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $\mathbb{R}^3 \setminus B_R(0)$, $\eta = 0$ in $\overline{B_{R-r}(0)}$ and $|\nabla\eta| \leqslant 2/r$. For each $n \in \mathbb{N}$ and for $L > 0$, let

$$z_{L,n} := \eta^2 v_n v_{L,n}^{2(\beta-1)} \quad \text{and} \quad w_{L,n} := \eta v_n v_{L,n}^{\beta-1},$$

where $v_{L,n} := \min\{v_n, L\}$, and $\beta > 1$ to be determined later. Choosing $z_{L,n}$ as a test function in (6.1) we have

$$0 = \int_{\mathbb{R}^3} a\nabla v_n \nabla z_{L,n} + V_n v_n z_{L,n}\,dx + b\|\nabla v_n\|^2_{L^2(\mathbb{R}^3)} \int_{\mathbb{R}^3} \nabla v_n \nabla z_{L,n}\,dx - \int_{\mathbb{R}^3} g_n(v_n) z_{L,n}\,dx,$$

namely

$$\begin{aligned} &\big(a + b\|\nabla v_n\|^2_{L^2(\mathbb{R}^3)}\big) \int_{\mathbb{R}^3} |\nabla v_n|^2 \eta^2 v_{L,n}^{2(\beta-1)} + 2\nabla v_n \nabla\eta\eta v_n v_{L,n}^{2(\beta-1)} + 2(\beta-1)\nabla v_n \nabla v_{L,n} v_n v_{L,n}^{2\beta-3}\eta^2\,dx \\ &\quad + \int_{\mathbb{R}^3} V_n v_n^2 \eta^2 v_{L,n}^{2(\beta-1)}\,dx = \int_{\mathbb{R}^3} g_n(v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)}\,dx. \end{aligned}$$

Set $A_n := a + b\|\nabla v_n\|^2_{L^2(\mathbb{R}^3)}$. Since $v_n \to v$ in $H^1(\mathbb{R}^3)$ with $v \neq 0$, we get $a \leqslant A_n \leqslant C$ for some positive constant C. Hence,

$$\begin{aligned} A_n \int_{\mathbb{R}^3} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2\,dx &= \int_{\mathbb{R}^3} g_n(v_n)\eta^2 v_n v_{L,n}^{2(\beta-1)}\,dx - 2A_n \int_{\mathbb{R}^3} \eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla\eta\,dx \\ &\quad - 2A_n(\beta-1) \int_{\mathbb{R}^3} \eta^2 v_{L,n}^{2\beta-3} v_n \nabla v_n \nabla v_{L,n}\,dx - \int_{\mathbb{R}^3} V_n v_n^2 \eta^2 v_{L,n}^{2(\beta-1)}\,dx. \end{aligned}$$

By assumptions (g_1) and (g_2), for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$\big|g(\varepsilon_n x, t)\big| \leqslant \xi|t| + C_\xi|t|^5 \quad \text{for all } (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Hence, using (V_1) and choosing $\xi \in (0, V_0)$, we have

$$A_n \int_{\mathbb{R}^3} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \leqslant C_\xi \int_{\mathbb{R}^3} v_n^6 \eta^2 v_{L,n}^{2(\beta-1)} \, dx - 2A_n \int_{\mathbb{R}^3} \eta v_{L,n}^{2(\beta-1)} v_n \nabla v_n \nabla \eta \, dx.$$

For each $\tau > 0$ we can use Young's inequality to obtain

$$A_n \int_{\mathbb{R}^3} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \leqslant C_\xi \int_{\mathbb{R}^3} v_n^6 \eta^2 v_{L,n}^{2(\beta-1)} \, dx + 2A_n \tau \int_{\mathbb{R}^3} |\nabla v_n|^2 v_{L,n}^{2(\beta-1)} \eta^2 \, dx + 2A_n C_\tau \int_{\mathbb{R}^3} v_n^2 |\nabla \eta|^2 v_{L,n}^{2(\beta-1)} \, dx$$

and taking $\tau \in (0, \frac{1}{2})$, we get

$$\int_{\mathbb{R}^3} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \leqslant C \int_{\mathbb{R}^3} v_n^6 \eta^2 v_{L,n}^{2(\beta-1)} \, dx + C \int_{\mathbb{R}^3} |\nabla \eta|^2 v_n^2 v_{L,n}^{2(\beta-1)} \, dx. \tag{6.2}$$

On the other hand, using the Sobolev inequality and the Hölder inequality, we can infer

$$\begin{aligned} \|w_{L,n}\|_{L^6(\mathbb{R}^3)}^2 &\leqslant C \int_{\mathbb{R}^3} |\nabla w_{L,n}|^2 \, dx = C \int_{\mathbb{R}^3} \left| \nabla \left(\eta v_{L,n}^{\beta-1} v_n \right) \right|^2 dx \\ &\leqslant C\beta^2 \left(\int_{\mathbb{R}^3} |\nabla \eta|^2 v_n^2 v_{L,n}^{2(\beta-1)} \, dx + \int_{\mathbb{R}^3} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_n|^2 \, dx \right). \end{aligned} \tag{6.3}$$

Gathering (6.2) and (6.3), we have

$$\|w_{L,n}\|_{L^6(\mathbb{R}^3)}^2 \leqslant C\beta^2 \left(\int_{\mathbb{R}^3} |\nabla \eta|^2 v_n^2 v_{L,n}^{2(\beta-1)} \, dx + \int_{\mathbb{R}^3} v_n^6 \eta^2 v_{L,n}^{2(\beta-1)} \, dx \right).$$

At this point we can argue as in Lemma 4.5 in [3] to deduce the assertion. □

Now, we are ready to give the proof of our main multiplicity result:

Proof of Theorem 1.1. Take $\delta > 0$ such that $M_\delta \subset \Lambda$. We begin by proving that there exists $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (2.3),

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3 \setminus \Lambda_\varepsilon)} < \alpha. \tag{6.4}$$

Assume to the contrary that for some subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, we can find $u_n := u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $\mathcal{J}'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n})} \geqslant \alpha. \tag{6.5}$$

Since $\mathcal{J}_{\varepsilon_n}(u_n) \leqslant c_0 + h_1(\varepsilon_n)$ and $h_1(\varepsilon_n) \to 0$, we can argue as in the proof of (5.1) to deduce that $\mathcal{J}_{\varepsilon_n}(u_n) \to c_0$. In view of Lemma 5.1, we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^3$ such that $v_n = u_n(\cdot + \tilde{y}_n) \to v \neq 0$ in

$H^1(\mathbb{R}^3)$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$. Now, if we choose $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$, we have that $B_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Then for any $y \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leqslant |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}\big(r + o_n(1)\big) < \frac{2r}{\varepsilon_n} \quad \text{for } n \text{ sufficiently large.}$$

Therefore, for any n big enough,

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n). \tag{6.6}$$

Applying Lemma 6.1, there exists $R > 0$ such that

$$v_n(x) < \alpha, \quad \text{for } |x| \geqslant R, n \in \mathbb{N},$$

from which

$$u_n(x) = v_n(x - \tilde{y}_n) < \alpha \quad \text{for } x \in B_R^c(\tilde{y}_n), n \in \mathbb{N}.$$

On the other hand, there exists $\nu \in \mathbb{N}$ such that for any $n \geqslant \nu$ we have

$$\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^3 \setminus B_R(\tilde{y}_n).$$

Consequently, $u_n(x) < \alpha$ for any $x \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n}$ and $n \geqslant \nu$. which is impossible in view of (6.5).

Now, let $\bar{\varepsilon}_\delta > 0$ be given by Theorem 5.1, and we fix $\varepsilon \in (0, \varepsilon_\delta)$ where $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. In light of Theorem 5.1, we know that problem (2.3) admits at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions. Let us denote by u_ε one of these solutions. Since $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ satisfies (6.4), it follows by definition of g that u_ε is a solution of (2.1). Then $\hat{u}(x) = u(x/\varepsilon)$ is a solution to (1.1), and we can conclude that (1.1) has at least $\operatorname{cat}_{M_\delta}(M)$ solutions.

Finally, we study the behavior of the maximum points of solutions for problem (2.1). Take $\varepsilon_n \to 0$ and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\varepsilon_n}$ of solutions for (2.1) as above. Let us observe that (g_1) implies that we can find $\mu > 0$ such that

$$g(\varepsilon x, t)t \leqslant \frac{V_0}{K} t^2 \quad \text{for any } x \in \mathbb{R}^3, t \leqslant \mu. \tag{6.7}$$

Arguing as before, we can find $R > 0$ such that

$$\|u_n\|_{L^\infty(\mathbb{R}^3 \setminus B_R(\tilde{y}_n))} < \mu. \tag{6.8}$$

Moreover, up to extract a subsequence, we may assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geqslant \mu. \tag{6.9}$$

Indeed, if (6.9) does not hold, in view of (6.8), we see that $\|u_n\|_{L^\infty(\mathbb{R}^3)} < \mu$. Then, using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and (6.7), we can infer

$$\|u_n\|_{\varepsilon_n}^2 \leqslant \|u_n\|_{\varepsilon_n}^2 + b\|\nabla u_n\|_{L^2(\mathbb{R}^3)}^4 = \int_{\mathbb{R}^3} g(\varepsilon_n x, u_n)u_n \, dx \leqslant \frac{V_0}{K}\int_{\mathbb{R}^3} u_n^2 \, dx$$

which yields $\|u_n\|_{\varepsilon_n} = 0$, and this is impossible. Hence (6.9) holds. Taking into account (6.8) and (6.9), we can deduce that if $p_n \in \mathbb{R}^3$ is a global maximum point of u_n, then $p_n \in B_R(\tilde{y}_n)$. Therefore $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R(0)$. As a consequence, $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$ is a global maximum point of $\hat{u}_n(x) = u_n(x/\varepsilon_n)$. Since $|q_n| < R$ for any $n \in \mathbb{N}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$ (in view of Lemma 5.1), we can infer from the continuity of V that

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0.$$

In what follows we prove the exponential decay of solutions of (1.1). Since $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, and using (g_1), we can find $R > 0$ such that

$$g_n\big(v_n(x)\big) \leqslant \frac{V_0}{2} \quad \text{for all } |x| \geqslant R.$$

Let $M \geqslant \|v_n\|^2_{\varepsilon_n}$, and define $\phi(x) := Ce^{-c|x|}$ with $c^2 < \frac{V_0}{2(a+bM)}$ and $Ce^{-cR} \geqslant v_n(x)$ for all $|x| = R$. It is easy to verify that

$$\Delta\phi \leqslant c^2\phi \quad \text{for all } x \neq 0. \tag{6.10}$$

On the other hand, by (V_1), we have

$$\begin{aligned} -\Delta v_n + \frac{V_0}{2(a+bM)} v_n &\leqslant -\Delta v_n + \frac{V_0}{2(a+b\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx)} v_n \\ &= \frac{1}{(a+b\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx)}\left(g_n(v_n) - \left(V_n - \frac{V_0}{2}\right)\right) \\ &\leqslant \frac{1}{(a+b\int_{\mathbb{R}^3} |\nabla v_n|^2\,dx)}\left(g_n(v_n) - \frac{V_0}{2}\right) \leqslant 0 \quad \text{for } |x| \geqslant R. \end{aligned} \tag{6.11}$$

Set $w_n := \phi - v_n$. Putting together (6.10) and (6.11), we get

$$\begin{cases} -\Delta w_n + \frac{V_0}{2(a+bM)} w_n \geqslant 0 & \text{in } |x| \geqslant R, \\ w_n \geqslant 0 & \text{on } |x| = R, \\ w_n(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

The maximum principle [20] implies that $w_n \geqslant 0$ in $|x| \geqslant R$ and we deduce that $v_n(x) \leqslant Ce^{-c|x|}$ for all $|x| \geqslant R$ and $n \in \mathbb{N}$. Since $\hat{u}_n(x) = u_n(x/\varepsilon_n) = v_n(\frac{x}{\varepsilon_n} - \tilde{y}_n) = v_n(\frac{x+\varepsilon_n q_n - \eta_{\varepsilon_n}}{\varepsilon_n})$ solves (1.1), we obtain the desired estimate. This completes the proof of Theorem 1.1. □

7. Supercritical Kirchhoff problems

In this section we deal with the multiplicity of positive solutions for (1.5). After rescaling, we study the following Kirchhoff problem

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\Delta u + V(\varepsilon x)u = u^{q-1}u + \mu u^{r-1} & \text{in } \mathbb{R}^3, \\ u > 0 \quad \text{in } \mathbb{R}^3, \qquad u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{7.1}$$

where $\mu > 0$ and the powers are such that $4 < q < 6 < r$. In what follows, we truncate the nonlinearity $\phi(u) := |u|^{q-2}u + \mu|u|^{r-2}u$ in a suitable way.

Let $K > 0$ be a real number, whose value will be fixed later, and set

$$\phi_\mu(t) := \begin{cases} 0 & \text{if } t < 0, \\ t^{q-1} + \mu t^{r-1} & \text{if } 0 \leqslant t < K, \\ (1 + \mu K^{r-q})t^{q-1} & \text{if } t \geqslant K. \end{cases}$$

It is clear that ϕ_μ satisfies the assumptions (f_1)–(f_4) ((f_3) with $\vartheta = q > 4$). Moreover,

$$\phi_\mu(t) \leqslant \big(1 + \mu K^{r-q}\big)t^{q-1} \quad \text{for all } t \geqslant 0. \tag{7.2}$$

Therefore, we can consider the following truncated problem

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\Delta u + V(\varepsilon x)u = \phi_\mu(u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \qquad u > 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{7.3}$$

It is easy to see that weak solutions of (7.3) are critical points of the energy functional $\mathcal{J}_{\varepsilon,\mu} : \mathcal{H}_\varepsilon \to \mathbb{R}$ defined by

$$\mathcal{J}_{\varepsilon,\mu}(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b}{4}\|\nabla u\|^4_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} \Phi_\mu(u)\,dx,$$

where $\Phi_\mu(t) := \int_0^t \phi_\mu(s)ds$. We also consider the autonomous functional

$$\mathcal{J}_{0,\mu}(u) = \frac{1}{2}\|u\|_0^2 + \frac{b}{4}\|\nabla u\|^4_{L^2(\mathbb{R}^3)} - \int_{\mathbb{R}^3} \Phi_\mu(u)\,dx.$$

Using Theorem 1.1, we know that for any $\mu \geqslant 0$ and $\delta > 0$, there exists $\bar{\varepsilon}(\delta, \mu) > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}(\delta, \mu))$, problem (7.3) admits at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions $u_{\varepsilon,\mu}$. Now, we prove that it is possible to estimate the $\mathcal{H}_\varepsilon$-norm of these solutions uniformly with respect to μ. More precisely:

Lemma 7.1. *There exists $\bar{C} > 0$ such that $\|u_{\varepsilon,\mu}\|_\varepsilon \leqslant \bar{C}$ for any $\varepsilon > 0$ sufficiently small and uniformly in μ.*

Proof. A simple inspection of the proof of Theorem 1.1 shows that any solution $u_{\varepsilon,\mu}$ of (7.3) satisfies the following inequality

$$\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leqslant c_{0,\mu} + h_\mu(\varepsilon),$$

where $c_{0,\mu}$ is the mountain pass level related to the functional $\mathcal{J}_{0,\mu}$, and $h_\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, decreasing $\bar{\varepsilon}(\delta, \mu)$ if necessary, we can assume that

$$\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leqslant c_{0,\mu} + 1 \tag{7.4}$$

for any $\varepsilon \in (0, \bar{\varepsilon}(\delta, \mu))$. Using the fact that $c_{0,\mu} \leqslant c_{0,0}$ for any $\mu \geqslant 0$, we can deduce that

$$\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leqslant c_{0,0} + 1 \tag{7.5}$$

for any $\varepsilon \in (0, \bar{\varepsilon}(\delta, \mu))$. We can also note that

$$\begin{aligned}
\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) &= \mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) - \frac{1}{q}\langle \mathcal{J}'_{\varepsilon,\mu}(u_{\varepsilon,\mu}), u_{\varepsilon,\mu}\rangle \\
&= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_{\varepsilon,\mu}\|_{\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{q}\right)\|\nabla u_{\varepsilon,\mu}\|_{L^2(\mathbb{R}^3)}^4 + \int_{\mathbb{R}^3} \frac{1}{q}\phi_\mu(u_{\varepsilon,\mu})u_{\varepsilon,\mu} - \Phi_\mu(u_{\varepsilon,\mu})\,dx \\
&\geqslant \left(\frac{1}{2} - \frac{1}{q}\right)\|u_{\varepsilon,\mu}\|_{\varepsilon}^2
\end{aligned} \tag{7.6}$$

where in the last inequality we have used assumption (f_3). Putting together (7.5) and (7.6), we can infer that

$$\|u_{\varepsilon,\mu}\|_{\varepsilon} \leqslant \left[\left(\frac{2q}{q-2}\right)(c_{0,0}+1)\right]^{\frac{1}{2}}$$

for any $\varepsilon \in (0, \bar{\varepsilon}(\delta, \mu))$. $\square$

Now, our plan is to prove that $u_{\varepsilon,\mu}$ is a solution of the original problem (7.1). To this end, we will show that we can find $K_0 > 0$ such that for any $K \geqslant K_0$, there exists $\mu_0 = \mu_0(K) > 0$ such that

$$\|u_{\varepsilon,\mu}\|_{L^\infty(\mathbb{R}^3)} \leqslant K \quad \text{for all } \mu \in [0, \mu_0]. \tag{7.7}$$

In order to achieve our goal, we use a version of the Moser iteration technique [29]. For simplicity, we set $u := u_{\varepsilon,\mu}$. For any $L > 0$, we define $u_L := \min\{u, L\} \geqslant 0$, where $\beta > 1$ will be chosen later, and let $w_L := u u_L^{\beta-1}$. Taking $u_L^{2(\beta-1)}u$ in (7.3), we see that

$$\begin{aligned}
&a\int_{\mathbb{R}^3} u_L^{2(\beta-1)}|\nabla u|^2\,dx + \int_{\{u<L\}} 2(\beta-1)u_L^{2(\beta-1)}|\nabla u|^2\,dx + b\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} \nabla u \nabla\big(u_L^{2(\beta-1)}u\big)\,dx \\
&\quad = \int_{\mathbb{R}^3} \phi_\mu(u)u_L^{2(\beta-1)}u\,dx - \int_{\mathbb{R}^3} V(\varepsilon x)u^2 u_L^{2(\beta-1)}\,dx.
\end{aligned} \tag{7.8}$$

Putting together (7.8), (7.2) and (V_1), we get

$$\int_{\mathbb{R}^3} u_L^{2(\beta-1)}|\nabla u|^2\,dx \leqslant C_{\mu,K}\int_{\mathbb{R}^3} u^q u_L^{2(\beta-1)}\,dx \tag{7.9}$$

where $C_{\mu,K} := a^{-1}(1+\mu K^{r-q})$. On the other hand, by Theorem 2.1 and $\beta > 1$, we have

$$\begin{aligned}
\|w_L\|^2_{L^6(\mathbb{R}^3)} &\leqslant S_* \int_{\mathbb{R}^3} |\nabla w_L|^2\,dx \\
&= S_* \int_{\mathbb{R}^3} \left|u_L^{\beta-1}\nabla u + (\beta-1)u u_L^{\beta-2}\nabla u_L\right|^2 dx \\
&\leqslant 2S_*\left(\int_{\mathbb{R}^3} (\beta-1)^2 u_L^{2(\beta-1)}|\nabla u|^2\,dx\,dy + \int_{\mathbb{R}^3} \left|u_L^{\beta-1}\nabla u_L\right|^2 dx\right) \\
&\leqslant 2S_*\big((\beta-1)^2+1\big)\int_{\mathbb{R}^3} u_L^{2(\beta-1)}|\nabla u|^2\,dx \\
&= 2S_*\beta^2\left[\left(\frac{\beta-1}{\beta}\right)^2 + \frac{1}{\beta}^2\right]\int_{\mathbb{R}^3} u_L^{2(\beta-1)}|\nabla u|^2\,dx \\
&\leqslant 4S_*\beta^2 \int_{\mathbb{R}^3} u_L^{2(\beta-1)}|\nabla u|^2\,dx.
\end{aligned} \tag{7.10}$$

Taking into account (7.9) and (7.10), and using the Hölder inequality, we can deduce that

$$\|w_L\|^2_{L^6(\mathbb{R}^3)} \leqslant C_1\beta^2 C_{\mu,K}\|u\|^{q-2}_{L^6(\mathbb{R}^3)}\|w_L\|^2_{L^{\frac{12}{6-(q-2)}}(\mathbb{R}^3)} \tag{7.11}$$

where $2 < \frac{12}{6-(q-2)} < 6$ and $C_1 > 0$. In view of Lemma 7.1 and Lemma 2.1, we can see that

$$\|w_L\|^2_{L^6(\mathbb{R}^3)} \leqslant C_2\beta^2 C_{\mu,K}\bar{C}^{\frac{q-2}{2}}\|w_L\|^2_{L^{\alpha^*}(\mathbb{R}^3)} \tag{7.12}$$

where

$$\alpha^* := \frac{12}{6-(q-2)}.$$

Now, we observe that if $u^\beta \in L^{\alpha^*}(\mathbb{R}^3)$, we obtain from the definition of w_L, $u_L \leqslant u$, and (7.12),

$$\|w_L\|^2_{L^6(\mathbb{R}^3)} \leqslant C_3\beta^2 C_{\mu,K}\bar{C}^{\frac{q-2}{2}}\|u\|^{2\beta}_{L^{\beta\alpha^*}(\mathbb{R}^3)} < \infty. \tag{7.13}$$

Passing to the limit as $L \to +\infty$ in (7.13), the Fatou Lemma yields

$$\|u\|_{L^{6\beta}(\mathbb{R}^3)} \leqslant (C_4 C_{\mu,K})^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\|u\|_{L^{\beta\alpha^*}(\mathbb{R}^3)} \tag{7.14}$$

whenever $u^{\beta\alpha^*} \in L^1(\mathbb{R}^3)$.

Now, we set $\beta := \frac{6}{\alpha^*} > 1$, and observe that, since $v \in L^6(\mathbb{R}^3)$, the above inequality holds for this choice of β. Then, using the fact that $\beta^2\alpha^* = 6\beta$, it follows that (7.14) holds with β replaced by β^2. Therefore,

$$\|u\|_{L^{6\beta^2}(\mathbb{R}^3)} \leqslant (C_4 C_{\mu,K})^{\frac{1}{2\beta^2}}\beta^{\frac{2}{\beta^2}}\|u\|_{L^{\beta^2\alpha^*}(\mathbb{R}^3)} \leqslant (C_4 C_{\mu,K})^{\frac{1}{2}(\frac{1}{\beta}+\frac{1}{\beta^2})}\beta^{\frac{1}{\beta}+\frac{2}{\beta^2}}\|u\|_{L^{\beta\alpha^*}(\mathbb{R}^3)}.$$

Iterating this process and recalling that $\beta\alpha^* := 6$, we can infer that for every $m \in \mathbb{N}$,

$$\|u\|_{L^{6\beta^m}(\mathbb{R}^3)} \leqslant (C_4 C_{\mu,K})^{\sum_{j=1}^m \frac{1}{2\beta^j}} \beta^{\sum_{j=1}^m j\beta^{-j}} \|u\|_{L^6(\mathbb{R}^3)}. \tag{7.15}$$

Taking the limit in (7.15) as $m \to +\infty$ and using Lemma 7.1, we get

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leqslant (C_4 C_{\mu,K})^{\gamma_1} \beta^{\gamma_2} C_5 \tag{7.16}$$

where $C_5 := S_*^{-\frac{1}{2}} \bar{C}$ and

$$\gamma_1 := \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\beta^j} < \infty \quad \text{and} \quad \gamma_2 := \sum_{j=1}^{\infty} \frac{j}{\beta^j} < \infty.$$

Next, we will find some suitable values of K and μ such that the following inequality holds

$$(C_4 C_{\mu,K})^{\gamma_1} \beta^{\gamma_2} C_5 \leqslant K,$$

or equivalently,

$$1 + \mu K^{r-q} \leqslant C_4^{-1} \beta^{-\frac{\gamma_2}{\gamma_1}} \left(K C_5^{-1}\right)^{\frac{1}{\gamma_1}}.$$

Take $K > 0$ such that

$$\frac{(K C_5^{-1})^{\frac{1}{\gamma_1}}}{C_4 \beta^{\frac{\gamma_2}{\gamma_1}}} - 1 > 0,$$

and fix $\mu_0 > 0$ satisfying

$$\mu \leqslant \mu_0 \leqslant \left[\frac{(K C_5^{-1})^{\frac{1}{\gamma_1}}}{C_4 \beta^{\frac{\gamma_2}{\gamma_1}}} - 1 \right] \frac{1}{K^{r-q}}.$$

Then, thanks to (7.16), we obtain that

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leqslant K \quad \text{for all } \mu \in [0, \mu_0],$$

that is $u = u_{\varepsilon,\mu}$ is a solution of (7.1). This completes the proof of Theorem 1.2.

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